Probabilistic selfish routing in a network of parallel queues

Alex Wang
Joint work with: Ilze Ziedins

Department of Statistics, University of Auckland

December 18, 2015
Table of contents

1 Introduction
   • Selfish Routing

2 Model
   • Description
   • Previous Model Review

3 Our Results
   • 3-Queues Model
   • General N+1-Queues Model

4 The End
Introduction

Probabilistic selfish routing in a network of parallel queues
Which route to take?

Alex Wang  Joint work with: Ilze Ziedins  Probabilistic selfish routing in a network of parallel queues
Selfish Routing

Probabilistic selfish routing in a network of parallel queues

Alex Wang  Joint work with: Ilze Ziedins
Is it fine to be "selfish"?
Model

Alex Wang  Joint work with: Ilze Ziedins  Probabilistic selfish routing in a network of parallel queues
Two types of queue:

- $M|M|1$ FIFO queues - private transportation (e.g. cars) - served in order of arrival.
- $M|G^{(M)}|\infty$ batch-service queue - public transportation (e.g. small shuttle bus) - customers served in batch.
Two independent Poisson arrival streams

- General users at rate $\lambda$
- Loyal users to batch-service queue at rate $\lambda_0$
Model description

Single-server FIFO queue vs. Batch-service queue

- **Single-server FIFO queues**
  - delay increases as load increases
  - customers avoid the crowd

- **Infinity-server Batch service queues**
  - delay decreases as load increases
  - customers follow the crowd
Wardrop’s First Principle:

”The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.”

Wardrop, J.G. (1952)
Probabilistic routing

- Let $p_i$ denote the probability that a general customer chooses route $i$, with $p_i \geq 0$, $\sum_i p_i = 1$. And $p = (p_0, p_1, \ldots, p_N)$
- Let $W_i(p_i)$ be the expected delay via route $i$.

At a user equilibrium, $p^{EQ}$, we have:

$W_i(p^{EQ}) = W_j(p^{EQ})$, if $p_i, p_j > 0$; and
$W_i(p^{EQ}) \leq W_k(p^{EQ})$, if $p_i > 0, p_k = 0$. 

Alex Wang  Joint work with: Ilze Ziedins  Probabilistic selfish routing in a network of parallel queues
A glance at previous model (Ziedins 2005)

Expected delay:

\[ W_1 = \frac{1}{\mu_1 - \lambda(1-p_0)} \]

\[ W_0 = \frac{1}{\mu_0} + \frac{M-1}{2(\lambda_0 + \lambda p_0)} \]

Note: Both \( W_1 \) and \( W_0 \) are decreasing in \( p_0 \).
Downs-Thomson effect

Increasing road capacity can make traffic congestion worse!
Our Results
One batch service and 2 FIFO queues

Without loss of generality, we assume $\mu_1 \geq \mu_2$. 
User equilibrium

\( p = (p_0, p_1, p_2) \in [0,1]^3 \) is a user equilibrium, iff:

(a) \( p_0 = 1 \), and \( W_0(1) \leq W_1(0), W_2(0) \)

(b) \( p_0, p_1 > 0, p_2 = 0 \), and \( W_0(p) = W_1(p) \leq W_2(0) \)

(c) \( p_0, p_1, p_2 > 0 \), and \( W_0(p) = W_1(p) = W_2(p) \)

(d) \( p_0 = 0, p_1 + p_2 = 1 \), and \( W_0(0) \geq W_1(p) = W_2(p) \)

(e) \( p_0 = p_2 = 0, p_1 = 1 \), and \( W_0(0), W_2(0) \geq W_1(1) \)
Expected delay:

- $W_0(p_0) = \frac{1}{\mu_0} + \frac{N-1}{2(\lambda p_0 + \lambda_0)}$
- $W_i(p_i) = \frac{1}{\mu_i - p_i \lambda}$, for $i = 1, 2$.

**Lemma:** Given $p_0$, at a user equilibrium, $p_1^*, p_2^* \in [0, 1]$ must satisfy the conditions below, for $p = \frac{1}{2} + \frac{\mu_1 - \mu_2}{2 \lambda} - \frac{p_0}{2}$.

$$p_1^* = \begin{cases} p & \text{if } p \in [0, 1 - p_0] \\ 1 - p_0 & \text{if } p > 1 - p_0 \end{cases}$$

$$p_2^* = 1 - p_1^* - p_0$$

Let $W^*(p_0) = \min(W_1(p_1^*), W_2(p_2^*))$
Three User Equilibria

- **Stable User Equilibrium** - the system moves towards the equilibrium when it is close
- **Unstable User Equilibrium** - the system moves away from the equilibrium when it is close

\[ W_0: \quad W^*: \]

Alex Wang  Joint work with: Ilze Ziedins  Probabilistic selfish routing in a network of parallel queues
Downs-Thomson effect

\[ \mu_1 = 1, 0.8, 0.6 \text{ and } \mu_2 = 0.2 \]

Expected delay \( W_0 \):

Expected delay \( W^* \):
Multiple Equilibria

First stable user equilibrium: 
Second stable user equilibrium: 

Alex Wang  Joint work with: Ilze Ziedins  Probabilistic selfish routing in a network of parallel queues
3D plot of the first stable equilibrium
Probabilistic selfish routing in a network of parallel queues
The system is at a user equilibrium, $p^{EQ}$, if and only if one of the following holds.

(a) $p_0^{EQ} = 1$, $p_1^{EQ} = ... = p_N^{EQ} = 0$

$$W_0(1) \leq W_1(0) \leq ... \leq W_N(0)$$

(b) $p_0^{EQ}, ..., p_i^{EQ} > 0$, $p_{i+1}^{EQ} = ... = p_N^{EQ} = 0$

$$W_0(p_0^{EQ}) = W_1(p_1^{EQ}) = ... = W_i(p_i^{EQ}) \leq ... \leq W_N(0), \text{ for some } i \in \{1, ..., N-1\}$$

(c) $p_0^{EQ} > 0$, $p_1^{EQ} \geq ... \geq p_N^{EQ} > 0$

$$W_0(p_0^{EQ}) = W_1(p_1^{EQ}) = ... = W_N(p_N^{EQ})$$

(d) $p_0^{EQ} = 0$, $p_1^{EQ} \geq ... \geq p_i^{EQ} > 0$, $p_{i+1}^{EQ} = ... = p_N^{EQ} = 0$

$$W_0(0) \geq W_1(p_1^{EQ}) = ... = W_i(p_i^{EQ}) \leq ... \leq W_N(0), \text{ for some } i \in \{1, ..., N-1\}$$

(e) $p_0^{EQ} = 0$, $p_1^{EQ} \geq ... \geq p_N^{EQ} > 0$

$$W_0(0) \geq W_1(p_1^{EQ}) = ... = W_N(p_N^{EQ})$$
**Definition:** With $p_0$ fixed, we define $(p_0, p_1^*, ..., p_N^*)$ to be a $p_0$-FIFO user equilibrium if there exists a constant $c$ such that $W_i(p_i^*) = c$, if $p_i^* > 0$, and $W_i(p_i^*) \geq c$, if $p_i^* = 0$, $1 \leq i \leq N$.

**Lemma:** Given $p_0$, the $p_0$-FIFO user equilibrium is unique. For $p_0 < 1$, let $m(p_0) = \max(k : 1 \leq k \leq N, \frac{(1-p_0)\lambda - \sum_{j=1}^{k} \mu_j + k \mu_k}{k \lambda} > 0)$. A $p_0$-FIFO user equilibrium $p^*(p_0) = \{p_0, p_1^*, ..., p_N^*\}$ must satisfy:

$$p_i^* = \begin{cases} \frac{(1-p_0)\lambda - \sum_{j=1}^{m} \mu_j + m \mu_i}{m \lambda} & \in (0, 1] \quad \text{if } i \in \{1, ..., m\} \\ 0 & \quad \text{if } i \in \{m + 1, ..., N\} \end{cases}$$

Let $W^*(p_0) = \min(W_1(p_1^*), ..., W_N(p_N^*))$.

**Lemma:** $W^*(p_0)$ is continuous and decreasing in $p_0$. 

Alex Wang Joint work with: Ilze Ziedins Probabilistic selfish routing in a network of parallel queues
Lemma: In an N FIFO queue system, as \( p_0 \) increases, \( m(p_0) \) decreases.

Definition: In an N FIFO queue system, for \( 1 \leq i \leq m(0) \), we call \( x_i \in (0, 1] \) the \( i^{th} \) FIFO queue marker if there exists \( \epsilon > 0 \) such that

(i) \( m(p_0) \geq i \) for \( p_0 \in (x_i - \epsilon, x_i) \) and

(ii) \( m(p_0) \leq i - 1 \) for \( p_0 \in [x_i, x_i + \epsilon) \).
Conditions for Unique User Equilibrium

**Lemma:** There exists a unique user equilibrium for the system if and only if one of the following cases holds.

1. A unique $p_0^* \in [0, 1]$ exists such that $W_0(p_0^*) = W^*(p_0^*)$. $W_0(p_0) < W^*(p_0)$ for $p_0 \in [0, p_0^*)$ and $W_0(p_0) > W^*(p_0)$ for $p_0 \in (p_0^*, 1]$.
2. $p_0^* = 0$, $W_0(p_0) > W^*(p_0)$ for $p_0 \in [0, 1]$
3. $p_0^* = 1$, $W_0(p_0) < W^*(p_0)$ for $p_0 \in [0, 1]$

**Lemma:** For $M \geq 2N + 1$, if there exists a unique $p_0^*$ such that $W_0(p_0^*) = W^*(p_0^*)$, then case (I) above holds.
Lemma: If the batch size $M \geq 2N + 1$, there exists at most one $p_0^* \in [0, x_{m(0)})$ such that $W_0(p_0^*) = W^*(p_0^*)$.

Theorem: In a $N$ FIFO queue system, there exists a unique equilibrium, when $M \geq 2N + 1$. And when $M \leq 2N$, there exist multiple equilibria for at least some combination of parameters.

Note: For two-queues model, the equilibrium is always unique if $N \geq 3$, Afimeimounga, Solomon, Ziedins (2005).
Thank You!

Taken by Alex, @Mt Egmont, NZ 2014