Continuous Wardrop equilibria and Mean Field Games, with congestion costs or capacity constraints

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IMS Workshop on Congestion Games
1. MFG: a coupled system of PDEs
2. Different variational problems
3. Coming back to an equilibrium
4. The analogy with continuous Wardrop equilibria
5. Some words on regularity
6. A variant: capacity constraints instead of congestion costs
What are MFG?

The theory of Mean Field Games has been introduced by Lasry and Lions to describe the evolution of a population, where each agent has to choose the strategy (i.e., a path) which best fits his preferences, but is affected by the others through a global *mean field*.

It is a differential game, with a continuum of players, all indistinguishable and all negligible. It is a typical congestion game (agents pay a congestion price, hence they try to avoid the regions with high concentrations) and we look for a Nash equilibrium, which can be translated into a system of PDEs.

P.-L. Lions, courses at Collège de France, 2006/12, videos available at [http://www.college-de-france.fr/site/pierre-louis-lions/_course.htm](http://www.college-de-france.fr/site/pierre-louis-lions/_course.htm)
P. Cardaliaguet, lecture notes, [www.ceremade.dauphine.fr/~cardalia/](http://www.ceremade.dauphine.fr/~cardalia/)
The goal behind the theory is to study the limit as $N \to \infty$ of games of $N$ player, each one choosing a trajectory $x_i(t)$ and optimizing a quantity

$$\int_0^T \left( \frac{|x_i'(t)|^2}{2} + g_i(x_1(t), \ldots, x_N(t)) \right) dt + \psi_i(x_i(T)).$$

In particular, we are interested in the case where $g_i$ penalizes points close to too many other players $x_j, j \neq i$.

Note that we consider here deterministic mean field games (no stochastic effects in the trajectories $x_i(t)$).

We will suppose that $g_i$ only depends on the position $x_i$ and on the distribution of the other player, and that all players have the same preferences. And we will not study the discrete case and pass to the limit, but directly study the continuous case.
Each agent in a population chooses his own trajectory in $\Omega$, solving

$$\min_{} \int_0^T \left( \frac{|x'(t)|^2}{2} + g(\rho_t(x(t))) \right) dt + \Psi(x(T)),$$

with given initial point $x(0)$; here $g$ is a given increasing function of the density $\rho_t$ at time $t$ (we take $g(0) = 0$ and $g \geq 0$). The agent hence tries to avoid overcrowded regions.

**Input:** the evolution of the density $\rho_t$.

A crucial tool is the value function $\varphi$ for this problem, defined as

$$\varphi(t_0, x_0) := \min \left\{ \int_{t_0}^T \left( \frac{|x'(t)|^2}{2} + g(\rho_t(x(t))) \right) dt + \Psi(x(T)), \ x(t_0) = x_0 \right\}.$$
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$$
Optimal control theory tells us that $\varphi$ solves

$$
(HJ) \quad - \partial_t \varphi(t, x) + \frac{1}{2} |\nabla \varphi(t, x)|^2 = g(\rho_t(x)), \quad \varphi(T, x) = \psi(x).
$$

Moreover, the optimal trajectories $x(t)$ follow $x'(t) = -\nabla \varphi(t, x(t))$.

Hence, given the initial $\rho_0$, we can find the density at time $t$ by solving

$$
(CE) \quad \partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0,
$$

which give as **Output:** the evolution of the density $\rho_t$.

We have an equilibrium if **Input = Output**.

This requires to solve a coupled system $(HJ)+(CE)$:

\[
\begin{cases}
  -\partial_t \varphi + \frac{|\nabla \varphi|^2}{2} = g(\rho), \\
  \partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0, \\
  \varphi(T, x) = \psi(x), \quad \rho(0, x) = \rho_0(x).
\end{cases}
\]

**Stochastic case:** we can also insert random effects $dX = \alpha dt + dB$, obtaining

\[
\begin{cases}
  -\partial_t \varphi - \Delta \varphi + \frac{|\nabla \varphi|^2}{2} - g(\rho) = 0, \\
  \partial_t \rho - \Delta \rho - \nabla \cdot (\rho \nabla \varphi) = 0.
\end{cases}
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**Stochastic case** : we can also insert random effects \( dX = \alpha dt + dB \),
obtaining \( -\partial_t \varphi - \Delta \varphi + \frac{1}{2} |\nabla \varphi|^2 - g(\rho) = 0 : \quad \partial_t \rho - \Delta \rho - \nabla \cdot (\rho \nabla \varphi) = 0 \).
Variational principle

It happens that an equilibrium is found by minimizing the (global) energy

$$A(\rho, v) := \int_0^T \int_{\Omega} \left( \frac{1}{2} \rho_t |v_t|^2 + G(\rho_t) \right) + \int_{\Omega} \psi \rho_T$$

among pairs \((\rho, v)\) such that \(\partial_t \rho + \nabla \cdot (\rho v) = 0\), with given \(\rho_0\), where \(G\) is the anti-derivative of \(g\), i.e. \(G' = g\) (in particular, \(G\) is convex).

**Warning:** as it often happens in congestion games, this is not the total cost for all the agents, as we put \(G(\rho)\) instead of \(\rho g(\rho)\). The equilibrium minimizes an overall energy (it’s a potential game), but not the total cost: there is a price of anarchy.

**Important:** this problem is convex in the variables \((\rho, w := \rho v)\) and it recalls Benamou-Brenier formulation for optimal transport.
This formulation can be used to do numerics!!

J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, preprint.
As all convex minimization problem, \( \min \mathcal{A} \) admits a dual problem, obtained from

\[
\min_{\rho, v} \mathcal{A}(\rho, v) + \sup_{\phi} \int_0^T \int_\Omega (\rho \partial_t \phi + \nabla \phi \cdot \rho v) + \int_\Omega \phi_0 \rho_0 - \int_\Omega \phi_T \rho_T,
\]

interchanging inf and sup. We get

\[
\sup \left\{-B(\phi, p) := \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega G^*(p_+) : \phi_T \leq \Psi, -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\},
\]

where \( G^* \) is the Legendre transform of \( G \), i.e. \( G^*(p) = \sup_q pq - G(q) \).

For optimal \((\rho, v, \phi, p)\) we have (\(\rho\)-a.e.) \( v = -\nabla \phi \), \( p = g(\rho) \) and \( \phi_T = \Psi \)
i.e., a solution to the MFG system (up to some technicalities).
Measures on possible trajectories

The same variational problem can also be written in the following way: let $C = H^1([0, T]; \Omega)$ be the space of curves valued in $\Omega$ and $e_t : C \to \Omega$ the evaluation map, $e_t(\gamma) = \gamma(t)$. Solve

$$\min \left\{ \int_C KdQ + \int_0^T \mathcal{G}((e_t)\# Q) + \int_\Omega \psi d(e_T)\# Q, \quad Q \in \mathcal{P}(C), (e_0)\# Q = \rho_0 \right\},$$

where $K : C \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\Omega) \to \mathbb{R}$ are given by

$$K(\gamma) = \frac{1}{2} \int_0^T |\gamma'|^2 \quad \text{and} \quad \mathcal{G}(\rho) = \int G(\rho(x))dx.$$

(\# denotes image measure, or push-forward).

**Existence:** by semicontinuity in the space $\mathcal{P}(C)$.

**Optimality conditions:** take $\overline{Q}$ optimal, $\widetilde{Q}$ another competitor, and $Q_\varepsilon = (1 - \varepsilon)\overline{Q} + \varepsilon \widetilde{Q}$. Setting $\rho_t = (e_t)\# \overline{Q}$ and $h(t, x) = g(\rho_t(x))$, differentiating w.r.t. $\varepsilon$ gives

$$J_h(\widetilde{Q}) \geq J_h(\overline{Q}),$$

where $J_h$ is the linear functional

$$J_h(Q) = \int KdQ + \int_0^T \int_\Omega h(t, x)(e_t)\# Q + \int_\Omega \psi d(e_T)\# Q.$$
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where $K : C \to \mathbb{R}$ and $G : \mathcal{P}(\Omega) \to \mathbb{R}$ are given by $K(\gamma) = \frac{1}{2} \int_0^T |\gamma'|^2$ and $G(\rho) = \int G(\rho(x))dx$. ($#\ $ denotes image measure, or push-forward).

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Back to an equilibrium

Look at $J_h$. It is well-defined for $h \geq 0$ measurable.
But if $h \in C^0$ we can also write $\int_0^T \int_\Omega h(t, x)(e_t) \# Q = \int_C dQ \int_0^T h(t, \gamma(t)) dt$
and hence we get that

$$Q \mapsto \int_C dQ(\gamma) \left( K(\gamma) + \int_0^T h(t, \gamma(t)) dt + \Psi(\gamma(T)) \right)$$

is minimal for $Q = \overline{Q}$. Hence $\overline{Q}$ is concentrated on curves minimizing
$K(\gamma) + \int_0^T h(t, \gamma(t)) dt + \Psi(\gamma(T))$. This means Input=Output.

A rigorous proof can also be done even for $h \notin C^0$ but one has to choose a precise representative. Techniques from incompressible fluid mechanics (incompressible Euler à la Brenier) allow to handle the case $G(\rho) \approx \rho^q$, $h \in L^{q'}$, $q, q' > 1$ using $\hat{h}(x) := \lim sup_{r \to 0} \int_{B(x, r)} h(t, y) dy$ (use of the maximal function needed to justify some convergences, which requires $h$ to be better than $L^1$).

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Continuous Wardrop equilibria

A very much related problem is the following continuous version of Wardrop equilibria: find $Q \in \mathcal{P}(C)$ such that $Q$–a.e. curve is a geodesic for the distance

$$d_k(x, y) := \inf \left\{ \int_0^1 k(\gamma)|\gamma'| : \gamma(0) = x, \gamma(1) = y \right\}$$

with $k = g(i_Q)$, where $i_Q$ is the traffic intensity defined (as a measure on $\Omega$) through

$$\langle i_Q, \phi \rangle := \int_C dQ(\gamma) \int_0^1 \phi(\gamma(t))|\gamma'(t)|dt.$$  

Also this equilibrium problem is a potential game, and solutions can be found by solving

$$\min \left\{ \int_{\Omega} G(i_Q(x))dx : Q \text{ admissible} \right\}.$$  


An example

Figure: Traffic intensity $i_Q$ at equilibrium in a city with a river and a bridge, with two sources $S_1$ and $S_2$, and two targets $T_1$ and $T_2$. Traffic concentrates close to origins, destinations, and concave corners of the domain.

Wardrop vs MFG – difference and similarities

- Wardrop equilibrium is a *statical* problem, while MFG are *dynamical* (more refined modeling).
- Additive costs versus multiplicative ones (i.e. *conformal Riemannian distances*): different mathematical techniques.
- Prescribing the final density $\rho_1$ or a final cost $\Psi$ is just a matter of taste (but the former is impossible in the stochastic case).
- In Wardrop we usually prescribe $(e_0, e_1) \# Q = \pi \in \mathcal{P}(\Omega \times \Omega)$ (*who-goes-where* problem: agents are not indistinguishable), while in MFG we usually give $\rho_0$ and $\Psi$.
- In the indistinguishable case for the Wardrop problem (i.e. prescribing $(e_0) \# Q = \rho_0$, $(e_1) \# Q = \rho_1$), then there is a divergence-constrained formulation

$$\min \left\{ \int G(v(x)) dx : \nabla \cdot v = \rho_0 - \rho_1 \right\}$$

(MFG, instead, contains a space-time divergence constraints).
- In both cases, we should give a meaning to the integral of $h (= g(\rho_1)$ of $g(i_Q)$) on curves, which requires regularity, or at least summability.
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- In the indistinguishable case for the Wardrop problem (i.e. prescribing $(e_0) \# Q = \rho_0$, $(e_1) \# Q = \rho_1$), then there is a divergence-constrained formulation

\[
\min \left\{ \int G(\nu(x))dx : \nabla \cdot \nu = \rho_0 - \rho_1 \right\}
\]

(MFG, instead, contains a space-time divergence constraints).
- In both cases, we should give a meaning to the integral of $h (= g(\rho_t)$ or $g(i_Q))$ on curves, which requires regularity, or at least summability.
Obtaining classical solutions to the MFG system is a hard question. One possible strategy, suggested by P-L Lions, is to reduce everything to a (non-linear and degenerate) elliptic equation in $\varphi$. For instance, if $g(\rho) = \rho$, we can replace $\rho$ with $-\partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2$ and obtain

$$
\partial_{tt} \varphi + \frac{1}{2} \Delta_4 \varphi - 2 \partial_t \nabla \varphi \cdot \nabla \varphi - \partial_t \Delta \varphi = 0.
$$

This PDE is degenerate elliptic and corresponds to the minimization of $\iint (\partial_t \varphi - \frac{1}{2}|\nabla \varphi|^2)^2$ (with suitable boundary conditions; actually, this is just the dual problem).

It is easier when $g(\rho) = \log \rho$, which reduces degeneracy

$$
\Delta_{t,x} \varphi + \nabla \varphi \cdot D^2 \varphi \cdot \nabla \varphi - 2 \partial_t \nabla \varphi \cdot \nabla \varphi = 0
$$

(it is actually non-degenerate as soon as $|\nabla \varphi|$ is bounded). This corresponds to $\min \iint e^{(\partial_t \varphi - \frac{1}{2}|\nabla \varphi|^2)}$.

Yet, let us see a different technique, based on duality (originating again from Brenier’s works on incompressible Euler).
Using duality

Take arbitrary \((\rho, v)\) and \((\phi, p)\) admissible in the primal and dual problem. Compute

\[
\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p) = \int_{\Omega} (\Psi - \phi T)\rho_T + \int_0^T \int_{\Omega} (G(\rho) + G^*(p_+ - p\rho)) + \frac{1}{2} \int_0^T \int_{\Omega} \rho |v + \nabla \phi|^2.
\]

Notice

\[
(G(\rho) + G^*(p_+ - p\rho)) \geq \frac{1}{2} |\rho - g^{-1}(p_+)|^2 \quad \text{where} \quad \lambda = \inf g'.
\]

Suppose \(\lambda > 0\).

We know \(\min \mathcal{A} + \min \mathcal{B} = 0\). Take \((\rho, v), (\phi, p)\) optimal. We get

\[
\rho = g^{-1}(p_+), \quad \Psi = \phi_T \quad \text{on} \quad \{\rho_T > 0\}, \quad v = -\nabla \phi \quad \text{on} \quad \{\rho > 0\},
\]

i.e. (again) a solution of the MFG system, in a suitable sense.
Suppose for simplicity $\Omega = \mathbb{T}^d$ to be the flat torus. We go on from

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p) \geq c \int_0^T \int_\Omega |\rho - g^{-1}(p_+)|^2.$$

Again, take $(\rho, v), (\phi, p)$ optimal. Take $(\rho^\delta, v^\delta)$ translation of $(\rho, v)$ (i.e. $\rho^\delta(t, x) = \rho(t, x + \delta)$, up to some cut-off functions to correct at $t = 0$ and $t = T$).

From the fact that $\delta \mapsto \mathcal{A}(\rho^\delta, v^\delta)$ is smooth and minimal for $\delta = 0$, we can prove $\mathcal{A}(\rho^\delta, v^\delta) \leq \mathcal{A}(\rho, v) + C|\delta|^2$. We get

$$\int_0^T \int_\Omega |\rho^\delta - \rho|^2 = \int_0^T \int_\Omega |\rho^\delta - g^{-1}(p_+)|^2 \leq \mathcal{A}(\rho^\delta, v^\delta) + \mathcal{B}(\phi, p) \leq C|\delta|^2,$$

which means $\rho \in L^2_{loc}((0, T); H^1(\Omega))$. We can also adapt to time translation and obtain $\rho \in H^1_{loc(t,x)}$. We can also get $\iint \rho |D^2\phi|^2 < \infty$. 
How to define a mean field game if we want to replace the penalization $+ g(\rho)$ with the (capacity) constraint $\rho \leq 1$?

**Naive idea:** when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \leq 1$. But if $\rho$ already satisfies $\rho \leq 1$, one extra agent will not violate the constraint (it’s a *non-atomic game*). Hence the constraint becomes empty.

**Instead,** let’s look at the variational problem

$$
\min \left\{ \int_0^T \int_\Omega \frac{1}{2} \rho_t |v_t|^2 + \int_\Omega \Psi \rho_T : \rho \leq 1 \right\}.
$$

It means $G(\rho) = 0$ for $\rho \in [0, 1]$ and $+\infty$ otherwise. There is a dual

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\sup \left\{ \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega p_+ : \phi_T \leq \Psi, -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\}.
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This problem is also obtained as the limit $m \to \infty$ of $g(\rho) = \rho^m$. Indeed the functional $\frac{1}{m+1} \int \rho^{m+1}$ Γ-converges to the constraint $\rho \leq 1$.

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\sup \left\{ \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega p_+ : \phi_T \leq \Psi, -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\}.
\]

This problem is also obtained as the limit \( m \to \infty \) of \( g(\rho) = \rho^m \). Indeed the functional \( \frac{1}{m+1} \int \rho^{m+1} \) \( \Gamma \)-converges to the constraint \( \rho \leq 1 \).

The system we get is

\[
\begin{aligned}
\begin{cases}
-\partial_t \varphi + \frac{|\nabla \varphi|^2}{2} = p, \\
\partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0,
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
p \geq 0, \rho \leq 1, p(1 - \rho) = 0, \\
\varphi(T, x) = \Psi(x), \quad \rho(0, x) = \rho_0(x).
\end{aligned}
\]

Each agent solves 
\[
\min \int_0^T \left( \frac{|x'(t)|^2}{2} + p(t, x(t)) \right) dt + \Psi(x(T)).
\]

Here \( p \) is a \textbf{pressure} arising from the incompressibility constraint \( \rho \leq 1 \) but finally acts as a \textbf{price}. In order to give a meaning to the above problem we need a bit of regularity. The same kind of duality argument, as in the works by Brenier and Ambrosio-Figalli, allow to get

\[
p \in L^2_{loc}((0, T); BV(\Omega)).
\]

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, preprint

Continuous Wardrop equilibria with capacity constraints

Open Problem Given \( \pi \in \mathcal{P}(\Omega \times \Omega) \) (or - which is easier - given \( \rho_0, \rho_1 \in \mathcal{P}(\Omega) \)), find \( Q \in \mathcal{P}(C) \) and \( p \) smooth enough such that

- \( p \geq 0, \ i_Q \leq 1, \ p(1 - i_Q) = 0 \)
- \( Q \)-a.e. curve \( \gamma \) is geodesic for the distance \( d_k \) with \( k = i_Q + p \)
- \((e_0, e_1) \# Q = \pi \) (or \((e_0) \# Q = \rho_0, (e_1) \# Q = \rho_1))

The corresponding variational problem are

\[
\min \left\{ \int |i_Q| dx : i_Q \leq 1 \right\}
\]

and

\[
\min \left\{ \int |v(x)| dx : \nabla \cdot v = \rho_0 - \rho_1, |v| \leq 1 \right\}.
\]

The difficult issue is the regularity of the pressure/price \( p \), which is a priori a measure (since \( p \geq 0 \)) but must be integrated on each trajectory.
The End

Thank you for your attention