Hotelling games on networks

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Hypothesis on buyers

1. Infinite number of buyers, distributed on the network.

2. They want to buy one share of a particular good whose price is fixed: they shop to the closest location.

Hypothesis on sellers

1. A fixed number of sellers cover the demand on this network.

2. They simultaneously choose their locations.

3. They want to sell as much as possible.
$G = (X, E), \quad \lambda : E \to \mathbb{R}_+^*$
Uniform density
Non uniform density

The model
Congestion games
Existence results
Efficiency results

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Not a potential game:

- There is no pure equilibrium for 3 players in the unit interval.

Finite number $k$ of possible locations:

- At equilibrium with a large number of players, every location is occupied.
- The network is divided into $k$ parts of lengths $L_1, \ldots, L_k$.
- Such an equilibrium is an equilibrium in the congestion game with parallel edges with cost $\frac{L_i}{n}$ when $n$ users choose the edge $i$. 
Results with uniform density

- Existence of pure Nash equilibrium for any graph when the number of player is large enough.

- Efficiency of these equilibria in terms of distance consumers have to travel: asymptotic convergence.
The unit interval

1. For $n = 2$, there exists a pure Nash equilibrium.
2. For $n = 3$, there is no pure Nash equilibrium.
3. For $n \geq 4$, there exists a pure Nash equilibrium.

\[
\begin{align*}
\forall i \in [1, n-5], & \quad 0 \leq \eta_i \leq 2\xi, \\
\forall i \in [1, n-6], & \quad \frac{\eta_i + \eta_{i+1}}{2} \geq \xi \\
\sum_{i=1}^{n-5} \eta_i + 6\xi = 1
\end{align*}
\]
The star $S_k(r)$

1. For $n \leq k$, there exists a pure Nash equilibrium.
2. For $n \in [k, 3k - 1[$, there is no pure Nash equilibrium.
3. For $n \geq 3k - 1$, there exists a pure Nash equilibrium.

Equilibrium with $4k + r$ players

$(2r\xi/k \leq y \leq 2(r + 1)\xi/k)$
Asymptotic existence of pure Nash equilibrium

On any finite graph Hotelling games always have pure Nash equilibrium, provided the number of players is larger than

\[ N := 3 \text{card}(E) + \sum_{e \in E} \left\lceil \frac{5\lambda(e)}{\lambda^*} \right\rceil. \]

\[ \lambda^* = \min_{E} \lambda \text{ (the length of the shortest edge)}. \]
1/ The graph $G = (X, E)$ and $n$ are fixed. We want to construct a pure Nash equilibrium with $n$ players on $G$. We fix a general dilatation parameter $\xi > 0$.

2/ On each edge, we put a number of players $n(e)$ that only depends on the length $\lambda(e)$ of the edge and on $\xi$.

Where $\alpha$ is such that the number of players on $e$ is $n(e)$. 
3/ We prove that if $\xi$ is small enough this profile of location is an equilibrium, with a number of player equal to

$$\sum_e n(e) = 3 \text{card}(E) + \sum_{e \in E} \left\lceil \frac{\lambda(e)}{2\xi} \right\rceil$$

4/ Can we find $\xi$ such that $f(\xi) = n$?

5/ No but we can find $n'$ such that there exists $\xi$ such that $f(\xi) = n'$, $n' \geq n$, and $n' - n \leq \text{card}(E)$.

6/ We select the equilibrium with $n'$ players. We can remove up to one unnecessary player on each edge to have an equilibrium with $n$ player.
Results with uniform density

- Existence of pure Nash equilibrium for any graph when the number of player is large enough.

- Efficiency of these equilibria in terms of distance consumers have to travel: asymptotic convergence.
Travelling distances of consumers, in equilibrium and in social optimum.

Equilibrium social cost: ?
Optimum social cost: ?
Social costs in equilibrium and in social optimum.

Equilibrium social cost: $\frac{1}{8}$
Optimum social cost: $\frac{1}{16}$
For $x \in S^n$, the social cost $\sigma(x)$ is given by:

$$\sigma(x) := \int_S \min_{i \in \{1, \ldots, n\}} d(x_i, y) dy$$

The price of anarchy is given by:

$$\text{IPoA}(n) := \frac{\max_{x \in \mathcal{E}_n(\mathcal{H})} \sigma(x)}{\min_{x \in S^n} \sigma(x)}$$

The price of stability is given by:

$$\text{IPoA}(n) := \frac{\min_{x \in \mathcal{E}_n(\mathcal{H})} \sigma(x)}{\min_{x \in S^n} \sigma(x)}$$

where $\mathcal{E}_n(\mathcal{H})$ is the set of equilibrium with $n$ players.
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Figure: Social optimum $\bar{x}$ with $n$ players.

Figure: Best equilibrium $\tilde{x}$ with $n$ players.

Figure: Worst equilibrium $\hat{x}$ with $n$ players ($n$ odd).

Figure: Worst equilibrium $\hat{x}$ with $n$ players ($n$ even).
On the unit interval, we have:

\[
\text{IPoA}(n) = \begin{cases} 
2 & \text{if } n \text{ is even}, \\
2 \left( \frac{n}{n+1} \right) & \text{if } n > 3 \text{ is odd}.
\end{cases}
\]

For \( n \geq 4 \)

\[
\text{IPoS}(n) = \frac{n}{n-2}
\]
Theorem

Suppose that the game $\mathcal{H}(n, S)$ has an equilibrium. Then

(a) $\text{IPoA}(n) \to 2$ as $n \to \infty$

(b) $\text{IPoS}(n) \to 1$ as $n \to \infty$
Stochastic dominance / Majorization

For a vector $z = (z_1, \ldots, z_n)$, we denote $z[1] \geq \cdots \geq z[n]$ its decreasing rearrangement.

**Definition**

Let $x, y \in [0, 1]^n$ be such

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

if, for all $k \in \{1, \ldots, n\}$

$$\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i].$$

then we say that $x$ is majorized by $y$ ($x \prec y$).
**Definition**

A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said **Schur-convex** if $x \prec y$ implies $\phi(x) \leq \phi(y)$.

**Proposition**

If $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function,

$$
\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \psi(x_i),
$$

then $\phi$ is Schur-convex.
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Counter-example
The result

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No general equilibrium with 4 players

There exists a pure Nash equilibrium on the unit interval with 4 players and with density $f$ if and only if $f$ satisfies $Q_{\frac{1}{2}} = \frac{Q_{\frac{1}{4}} + Q_{\frac{3}{4}}}{2}$.
Asymptotic existence of \( \epsilon \)-equilibrium.

Suppose that:

1. \( f \) is \( K \)-Lipschitz
2. There exist \( m \) and \( M \) such that for all \( x \), \( 0 < m \leq f(x) \leq M \)

Then:

\[
\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \geq N(\epsilon),
\]

there exists an \( \epsilon \)– pure equilibrium in the game with \( n \) players and density distribution \( f \).
Sketch of the proof:

1/ Fix an $\epsilon > 0$.

2/ Approximate $f$ by a step function $g$ with precision $\epsilon_2$.

3/ Construct an exact equilibrium on the game with density distribution $g$. It exists if the number of player is larger than of bound $N(\epsilon_1)$.

4/ Prove that if $\epsilon_1$ is small enough, the equilibrium is an $\epsilon$-equilibrium in the original game, with density distribution $f$.

During this constructive proof, we found that

$$N(\epsilon) := 4\text{card}(E) + \frac{2L(M + \epsilon)}{(m - \epsilon)} \left( \frac{K}{\epsilon} + \frac{2}{\min \lambda_e} \right) + \frac{3LK}{2\epsilon}$$
Thank you