Dynamic Atomic Congestion Games
with Seasonal Flows

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Dynamic congestion games

- Most models of congestion games are static.
- The static game represents the *steady state* of a dynamic model with constant flow over time.
- Even if the flow of travellers is constant, *how* is the steady state reached?
- In real life traffic flows are rarely constant, although often (nearly) *periodic*. How does this affect the steady state?
Each edge had a *travel time* and a *capacity*. For example, $\tau_e = 2$ and $\gamma_e = 2$. 
Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$. 

$t = 0$

1, 2, 3

$V$ $W$
Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$. 

![Diagram](attachment:image.png)
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$t = 3$
Edge dynamics

Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$. 

$3$

$1$

$2$

$t = 0$
Edge dynamics

Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$.

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<thead>
<tr>
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<tbody>
<tr>
<td>$t = 0$</td>
<td>3</td>
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<tr>
<td>$t = 1$</td>
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</table>
Edge dynamics

Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$.

\[
\begin{array}{c|c|c}
3 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}
\]

$t = 0$  \hspace{1cm}  \begin{array}{c|c|c}
3 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}
\hspace{1cm} t = 1  \hspace{1cm} \begin{array}{c|c|c}
3 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}
\hspace{1cm} t = 2
Edge dynamics

Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$.

\[
\begin{align*}
\text{t = 0} & \quad \text{t = 1} & \quad \text{t = 2} & \quad \text{t = 3} \\
1 & \quad 2 & \quad 3 & \quad 1 & \quad 2 & \quad 3 & \quad \text{(empty)}
\end{align*}
\]
Related literature

Continuous time and flows

- Koch and Skutella (2011) provide a characterization of Nash flows over time via a sequence of thin flows with resetting.
- Cominetti, Correa and Larré (2011) prove existence and uniqueness of Nash flows over time.
- Macko, Larson and Steskal (2013) analyse Braess’s paradox for flows over time.

Discrete time and flows

Model

- A directed network $\mathcal{N} = (V, E, (\tau_e)_{e \in E}, (\gamma_e)_{e \in E})$ with a single source and sink, where
  - $\tau_e \in \mathbb{N}$ is the travel time,
  - $\gamma_e \in \mathbb{N}$ is the capacity.
- Time is discrete and players are atomic.
- Inflow is deterministic, but is allowed to be periodic.
At each stage $t$, a generation $G_t$ of $\delta_t$ players departs from the source. Players are ordered by priority $\prec$.

At time $t$, player $[it]$ observes the choices of players $[js] \prec [it]$ and chooses an edge $e = (s, v) \in E$.

Player $[it]$ arrives at time $t + \tau_e$ at the exit of $e$. 
At this exit a queue might have formed by
1. players who entered $e$ before $[it]$, 
2. players who entered $e$ at the same time as $[it]$, but have higher priority.

Recall at most $\gamma_e$ players can exit $e$ simultaneously.

When exiting edge $e = (s, v)$, player $[it]$ chooses an outgoing edge $e' = (v, v')$. This is repeated until player $[it]$ arrives at the destination.

This defines a game with perfect information $\Gamma(\mathcal{N}, K, \delta)$. 

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Latencies

- $c_{it}(\sigma) = \sum_{e \in r_{it}(\sigma)} \tau_{e}$ is the travel time of player $[it]$,
- $w_{it}(\sigma)$ is the waiting time of player $[it]$,
- $\ell_{it}(\sigma)$ is the total latency suffered by player $[it]$: 
  \[ \ell_{it}(\sigma) = c_{it}(\sigma) + w_{it}(\sigma). \]
- $\ell_{t}(\sigma) = \sum_{[it] \in G_t} \ell_{it}(\sigma)$ is the total cost of generation $G_t$. 

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Solution concepts

- **Equilibrium.** Each player minimizes her own total latency given the queues in the system.
  - Exists: multiple equilibria
  - Subgame perfect Markov equilibrium

- **Optimum.** A social planner controls all players and seeks to minimize the long-run total costs, averaged over a period.
Overview

1 Model

2 Parallel networks
   - Uniform departures
   - Periodic departures

3 Extensions
   - Chain-of-parallel networks
   - Braess’s networks
   - Series-parallel networks

4 Conclusion
In a parallel network each route is made of a single edge. The capacity of the network is $\gamma = \sum_e \gamma_e$.

We assume that $\delta_t = \gamma$ for all $t \in \mathbb{N}$. 
Example

Inflow = (3, 3, 3, ...). What happens in the equilibrium?
Equilibrium

\[ t = 1 \]

\[ \begin{array}{c}
3 \\
2 \\
1
\end{array} \]

Total costs $= 3 \times 5 = 15$. 
Equilibrium

Parallel networks

Uniform departures

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Equilibrium

<table>
<thead>
<tr>
<th>t = 1</th>
<th>t = 2</th>
<th>t = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>*</td>
<td>*</td>
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<tr>
<td></td>
<td>1</td>
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Total costs = 3 · 5 = 15.
Equilibrium

Parallel networks
Uniform departures

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Equilibrium

<table>
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<tr>
<th>Time ( t )</th>
<th>Network 1</th>
<th>Network 2</th>
<th>Network 3</th>
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<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Total costs: \( 3 \cdot 5 = 15 \).
Parallel networks

Uniform departures

Equilibrium

\[ t = 1 \]
\[ t = 2 \]
\[ t = 3 \]
\[ t = 4 \]
\[ t = 5 \]

\[ t = 6 \]

Total costs = $3 \cdot 5 = 15$. 

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Equilibrium

Parallel networks | Uniform departures

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Equilibrium

$\text{Parallel networks \ Uniform departures}$

... Total costs $= 3 \cdot 5 = 15$. 

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Optimum

Can we do better in the long-run than 15 per generation?
Can we do better in the long-run than 15 per generation?
Can we do better in the long-run than 15 per generation?
Optimum

Can we do better in the long-run than 15 per generation?

... Total costs = 1 + 3 + 5 = 9.
Steady state

**Proposition**

Let $\mathcal{N}$ be a parallel network. Then

$$WEq(\mathcal{N}, \gamma) = \gamma \cdot \max_{e \in E} \tau_e,$$

$$Opt(\mathcal{N}, \gamma) = \sum_{e \in E} \gamma_e \cdot \tau_e.$$
Parallel networks

Uniform departures

Steady state

Proposition

Let $\mathcal{N}$ be a parallel network. Then

$$WEq(\mathcal{N}, \gamma) = \gamma \cdot \max_{e \in E} \tau_e,$$

$$Opt(\mathcal{N}, \gamma) = \sum_{e \in E} \gamma_e \cdot \tau_e.$$

Equilibrium flows eventually coincide with optimal flows, but equilibrium costs are higher.
Let $\mathcal{N}$ be a parallel network. Then

$$PoA(\mathcal{N}, \gamma) = \frac{WEq(\mathcal{N}, \gamma)}{Opt(\mathcal{N}, \gamma)} \leq \frac{\max_e \tau_e}{\min_e \tau_e}.$$ 

The price of anarchy is unbounded over the class of parallel networks.

**Example** Bad network: $\tau_1 = 1$, $\gamma_1 = N$, $\tau_2 = N$, $\gamma_2 = 1$.

$$PoA(\mathcal{N}, \gamma) = \frac{(N + 1) \cdot N}{2N}.$$
**Periodic departures**

- Inflow is a $K$-periodic vector:

  \[
  \delta = (\delta_1, \ldots, \delta_K) \in \mathbb{N}^K
  \]

  such that $\sum_{k=1}^{K} \delta_k = K \cdot \gamma$. We denote $\mathbb{N}_K(\gamma)$ the set of such sequences.

- When $\delta$ is not-uniform, queues have to be created when there is a surge of players.
Example

Equilibrium for inflow (4,2,3).
**Example**

Equilibrium for inflow \((4,2,3)\).
Example

Equilibrium for inflow (4,2,3).

\( t = 1 \)

\( t = 2 \)
**Example**

Equilibrium for inflow (4,2,3).

\[
\begin{align*}
&\text{t = 1} & &\text{t = 2} & &\text{t = 3} \\
&3 & 1 & * & 2 & 3 & * \\
&2 & & 1 & * & & 2 \\
&1 & * & & 1 & * & *
\end{align*}
\]
Example

Equilibrium for inflow (4,2,3).
Example

Equilibrium for inflow $(4,2,3)$. 

\begin{align*}
t = 1: & \quad \begin{array}{c}
3 \\
2 \\
1
\end{array} \\
\quad \begin{array}{c}
1
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
*
\end{array} \\
\quad \begin{array}{c}
*
\end{array}
\end{align*}

\begin{align*}
t = 2: & \quad \begin{array}{c}
2 \\
3 \\
1 \\
* \\
*
\end{array} \\
\quad \begin{array}{c}
3
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
*
\end{array}
\end{align*}

\begin{align*}
t = 3: & \quad \begin{array}{c}
3 \\
1 \\
2 \\
* \\
*
\end{array} \\
\quad \begin{array}{c}
2
\end{array} \\
\quad \begin{array}{c}
1
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array}
\end{align*}

\begin{align*}
t = 4: & \quad \begin{array}{c}
1 \\
2 \\
3 \\
* \\
*
\end{array} \\
\quad \begin{array}{c}
3
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
*
\end{array}
\end{align*}

\begin{align*}
t = 5: & \quad \begin{array}{c}
3 \\
2 \\
* \\
* \\
*
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
* \\
*
\end{array} \\
\quad \begin{array}{c}
*
\end{array}
\end{align*}
Example

Equilibrium for inflow (4,2,3).

$t = 1$

$t = 2$

$t = 3$

$t = 4$

$t = 5$

$t = 6$
**Example**

**Equilibrium** for inflow \((4,2,3)\).

- \(t = 1\): 3, 1, 2
- \(t = 2\): 2, 3, *
- \(t = 3\): 3, 1, 2
- \(t = 4\): 1, 2, 3
- \(t = 5\): 3, 2, *
- \(t = 6\): 2, 3, 1
- \(t = 7\): 1, 2, 3

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Example

**Equilibrium** for inflow (4,2,3).

- **t = 1**: 3, 1, *
- **t = 2**: 2, 3, *
- **t = 3**: 3, 1, 2
- **t = 4**: 3, 1, *
- **t = 5**: 3, *
- **t = 6**: 2, 3, *
- **t = 7**: 1, 2, *
- **t = 8**: 1, 2, 3

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Example

Equilibrium for inflow (4,2,3).

<table>
<thead>
<tr>
<th>Time</th>
<th>Network 1</th>
<th>Network 2</th>
<th>Network 3</th>
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<tr>
<td>t = 1</td>
<td>3 * 1</td>
<td>2 * 3</td>
<td>1 * 2</td>
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<tr>
<td>t = 2</td>
<td>2 * 3</td>
<td>1 * 2</td>
<td>3 * 1</td>
</tr>
<tr>
<td>t = 3</td>
<td>3 * 2</td>
<td>1 * 2</td>
<td>1 * 3</td>
</tr>
<tr>
<td>t = 4</td>
<td>1 * 2</td>
<td>3 * 1</td>
<td>2 * 3</td>
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<tr>
<td>t = 5</td>
<td>3 * 2</td>
<td>1 * 3</td>
<td>* 2</td>
</tr>
<tr>
<td>t = 6</td>
<td>2 * 3</td>
<td>1 * 2</td>
<td>* 1</td>
</tr>
<tr>
<td>t = 7</td>
<td>1 * 2</td>
<td>3 * 1</td>
<td>* 2</td>
</tr>
<tr>
<td>t = 8</td>
<td>* 2</td>
<td>* 1</td>
<td>* 3</td>
</tr>
<tr>
<td>t = 9</td>
<td>* 2</td>
<td>* 1</td>
<td>* 3</td>
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</table>
Example

**Optimum** for inflow (4,2,3).
Example

Optimum for inflow (4,2,3).
**Example**

**Optimum** for inflow (4,2,3).

\[
\begin{array}{c}
1 & 2 & 3 \\
1 & 2 & 3 \\
t = 1 & t = 2 \\
\end{array}
\]

Both in the equilibrium as in the optimum, the fourth player behaves as if he was postponed by one stage.
Example

**Optimum** for inflow (4,2,3).

Both in the equilibrium as in the optimum, the fourth player behaves as if he was postponed by one stage.
Optimum for inflow (4,2,3).

Both in the equilibrium as in the optimum, the fourth player behaves as if he was postponed by one stage.
Measure of periodicity

**Definition**

For any two elements $\delta, \delta' \in \mathbb{N}_K(\gamma)$, we say that $\delta'$ is obtained from $\delta$ by an **elementary operation** if there exist an $i$ with $\delta_i > \gamma$ such that

$$\delta'_i = \delta_i - 1, \quad \delta'_{i+1} = \delta_i + 1.$$ 

Let $D(\delta)$ be the minimal number of elementary operations one has to perform to transform $\delta$ into $\gamma_K$. 
Measure of periodicity

Figure: 1 operation needed to transform (3, 1, 2) into (2, 2, 2).
Measure of periodicity

Figure: 1 operation needed to transform \((3, 1, 2)\) into \((2, 2, 2)\).

Figure: 2 operations needed to transform \((3, 2, 1)\) into \((2, 2, 2)\).
Steady state

Theorem

Let $\mathcal{N}$ be a parallel network and $\delta \in \mathbb{N}_K(\gamma)$. Then

$$WEq(\mathcal{N}, K, \delta) = K \cdot \gamma \cdot \max_{e \in E} \tau_e + D(\delta),$$

$$Opt(\mathcal{N}, K, \delta) = K \cdot \sum_{e \in E} \gamma_e \cdot \tau_e + D(\delta).$$
Steady state

Theorem

Let $\mathcal{N}$ be a parallel network and $\delta \in N_K(\gamma)$. Then

$$\text{WEq}(\mathcal{N}, K, \delta) = K \cdot \gamma \cdot \max_{e \in E} \tau_e + D(\delta),$$

$$\text{Opt}(\mathcal{N}, K, \delta) = K \cdot \sum_{e \in E} \gamma_e \cdot \tau_e + D(\delta).$$

Equilibrium flows eventually coincide with optimal flows.
Overview

1. Model

2. Parallel networks
   - Uniform departures
   - Periodic departures

3. Extensions
   - Chain-of-parallel networks
   - Braess’s networks
   - Series-parallel networks

4. Conclusion
Parallel network below capacity

$\tau_1 = 1$

$\tau_2 = 2$

$\tau_3 = 3$

$\tau_4 = 3$
Parallel network below capacity

\[ \tau_1 = 1 \]
\[ \tau_2 = 2 \]
\[ \tau_3 = 3 \]
\[ \tau_4 = 3 \]

Equilibrium.

- If \( \delta = 3 \), then \( WEq(\mathcal{N}, 1, \delta) = 9 \).
Parallel network below capacity

Equilibrium.

- If $\delta = 3$, then $WEq(\mathcal{N}, 1, \delta) = 9$.
- If $\delta = (6, 0)$, then $WEq(\mathcal{N}, 2, \delta) = 16 < 18$. 
Steady state below capacity

Proposition
Let $\mathcal{N}$ be a parallel network with capacity $\gamma$ and let $\delta \in \mathbb{N}_K(\gamma')$, where $\gamma' \leq \gamma$. Then

$$WEq(\mathcal{N}, K, \delta) \leq K \cdot \gamma' \cdot \max_{e \in E} \tau_e + D(\delta).$$
Chain-of-parallel network

\[ \tau_1 = 1 \]
\[ \tau_2 = 2 \]
\[ \tau_3 = 3 \]
\[ \tau_4 = 3 \]
\[ \tau_5 = 1 \]
\[ \gamma_5 = 3 \]
**Chain-of-parallel network**

\[ \tau_1 = 1 \]
\[ \tau_2 = 2 \]
\[ \tau_3 = 3 \]
\[ \tau_4 = 3 \]
\[ \tau_5 = 1 \]
\[ \gamma_5 = 3 \]

**Equilibrium.** If \( \delta = (6, 0) \), then

\[ t = 3 \]
Equilibrium. If $\delta = (6, 0)$, then
Equilibrium. If $\delta = (6, 0)$, then
**Chain-of-parallel network**

Equilibrium. If $\delta = (6, 0)$, then

$$
\begin{array}{cccc}
6 & 3 & 4 & 5 \\
3 & 1 & 2 & * \\
1 & * & * & * \\
* & * & * & * \\
* & * & * & * \\
t = 3
\end{array}
$$
Chain-of-parallel network

Equilibrium. If $\delta = (6, 0)$, then

$t = 3$

$t = 4$
Chain-of-parallel network

Equilibrium. If $\delta = (6, 0)$, then

\[
\begin{align*}
\tau_1 &= 1 \\
\tau_2 &= 2 \\
\tau_3 &= 3 \\
\tau_4 &= 3 \\
\tau_5 &= 1 \\
\gamma_5 &= 3
\end{align*}
\]
**Equilibrium.** If $\delta = (6, 0)$, then

\[
\begin{array}{cccc}
6 & 3 & 4 & 5 \\
3 & * & * & * \\
1 & 2 & * & * \\
\end{array}
\]  \quad \text{and} \quad
\begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{array}
\]  \quad \text{for } t = 3, \quad t = 4, \quad t = 5
**Chain-of-parallel network**

Equilibrium. If $\delta = (6, 0)$, then

- For $t = 3$, the network has the following profile:
  - $\tau_1 = 1$
  - $\tau_2 = 2$
  - $\tau_3 = 3$
  - $\tau_4 = 3$
  - $\tau_5 = 1$
  - $\gamma_5 = 3$

- For $t = 4$, the network has the following profile:

- For $t = 5$, the network has the following profile:
Chain-of-parallel network

Equilibrium. If $\delta = (6, 0)$, then

- $\text{WEq}(\mathcal{N}, 2, \delta) = 22$ (earliest-arrival property).
- $\text{WEq}^*(\mathcal{N}, 2, \delta) = 25$ (no overtaking).
- $\text{WEq}^{**}(\mathcal{N}, 2, \delta) = 27$ (allow overtaking).
Let $F^*$ be the (static) min-cost flow. Define $M^r_p(\sigma) = \sum_{t=pK+1}^{(p+1)K} N^r_p(\sigma)$.

**Theorem**

Let $\delta \in \mathbb{N}_K(\gamma)$. Then there exists an optimal strategy profile $\sigma$ such that $M^r_p(\sigma) = K \cdot F^*_r$ for each route $r$ and each period $p$, and

$$\text{Opt}(\mathcal{N}, K, \delta) = \text{Opt}(\mathcal{N}, K, \gamma) + D(\delta).$$
Braess’s network

\begin{align*}
\tau_1 &= 0 \\
\tau_2 &= 1 \\
\tau_3 &= 0 \\
\tau_4 &= 1 \\
\tau_5 &= 0
\end{align*}
Worst equilibrium.

- Player [12] chooses $e_1 e_3 e_5$ and [22] chooses $e_2 e_5$.
- Player [13] chooses $e_1 e_3 e_5$ and [23] chooses $e_1 e_4$.
- For $t \geq 5$, player [1t] chooses $e_1 e_4$ and [2t] chooses $e_2 e_5$.

Total costs $= 3 + 3 = 6$. 
Braess’s network

Best equilibrium.

- For $t \geq 2$, player [1$t$] chooses $e_1e_4$ and [2$t$] chooses $e_2e_5$.

Total costs=1+1=2.
Braess’s network

Proposition
For every even integer $n$, there exists a network $\mathcal{N}$ in which $|V| = n$ such that

$$
\text{PoA}(\mathcal{N}, \gamma) = \frac{\text{WEq}(\mathcal{N}, \gamma)}{\text{BEq}(\mathcal{N}, \gamma)} = \text{BR}(\mathcal{N}, \gamma) = n - 1.
$$
Series-parallel network

\[ \tau_1 = 1 \]

\[ \tau_2 = 0 \quad \gamma_2 = 2 \]

\[ \tau_3 = 0 \]

\[ \tau_4 = 1 \]
Equilibrium.

- For $t \geq 2$, [1t] chooses $e_1$, [2t] chooses $e_2e_3$, [3t] chooses $e_2e_4$.

Total costs = $1 + 1 + 2 = 4$. 

Series-parallel network
Series-parallel network

- Suppose $e_3$ contains a queue, then total costs decrease to 3.
- Another view on Braess’s paradox: initial queues can improve total costs.
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Summary

Two main contributions:
- We propose a measure of periodicity that characterizes the additional delay due to periodicity.
- We illustrate a new form of Braess’s paradox: the presence of initial queues in a network may decrease the long-run costs in equilibrium.
Open problems

- General networks
- Multiple sources and destinations
- Connection with continuous time and flows
- Stochastic inflow
Apologies for congesting your brain.