Learning in Network Games with Continuous Actions

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Stochastic Methods in Game Theory, IMS, Singapore

Outline

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Games on networks

Settings (simplified model for now):

- n players: $N = \{1, ..., n\}$;
- typically $S_i = [0, +\infty[$;
- $\mathbf{G}$ matrix such that $\mathbf{G}_{i,i} = 0$. $\mathbf{G}$ represents interactions: payoff of player i only depends on his own actions and a weighted sum of other agents efforts:

$$u_i(x_i, x_{-i}) = v_i(x_i, \sum_j \mathbf{G}_{i,j} x_j).$$

- Payoff of agent i strictly concave in his own action;
- "symmetric" externalities:

$$sgn\left(\frac{\partial u_i}{\partial x_j}(x)\right) = sgn\left(\frac{\partial u_j}{\partial x_i}(x)\right).$$

$\Rightarrow$ $\mathbf{G}$ defines a network/graph:

$\mathbf{G}_{i,j} \neq 0$ iif there is a link between i and j in the graph.

Two important Examples of Games on networks

Ballester, Calvo & Zenou, 2006

$$u_i(x_i, x_{-i}) = \underbrace{x_i - \frac{1}{2}x_i^2}_{idiosyncratic} + \underbrace{\delta x_i \sum_{j \in N_i} x_j}_{local\ interaction}, \delta > 0.$$

- $\mathbf{G}_{i,j} \in \{0, \delta\}$;
- linear quadratic idiosyncratic payoff;
- local interactions are strategic **complements**: $\frac{\partial BR_i}{\partial x_j}(x_{-i}) \geq 0$.

Nash equilibria: For $\lambda_{max}(\mathbf{G}) < 1/\delta$, unique interior Nash equilibrium $\mathbf{x}^*$:

$$\mathbf{x_i^*} = \sum_{k=0}^{+\infty} \delta^k b_i^k \quad (b_i^k : \#\text{of paths of length k starting from i})$$

Applications: education, crime, R & D...

Games on networks

Bramoulle and Kranton, 2007

cost $c > 0$, $b(\cdot)$ strictly increasing and concave, s.t. $b(0) = 0, b'(1) = c$,

$$u_i(x_i, x_{-i}) = b\left(x_i + \sum_{j \in N_i(g)} x_j\right) - c.x_i$$

- $\mathbf{G}_{i,j} \in \{0, 1\}$.
- local interactions are strategic **substitutes**: $\frac{\partial BR_i}{\partial x_j}(x_{-i}) \leq 0$.

Nash equilibria: Any strategy profile $\mathbf{x}^*$ such that

$$\forall i \in N, \quad \mathbf{x}_i^* = \max(0, 1 - \sum_{j \in N_i(g)} x_j^*)$$

Potentially infinite!

Applications: for instance provision of public goods.

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Maximal independent sets and Nash equilibria

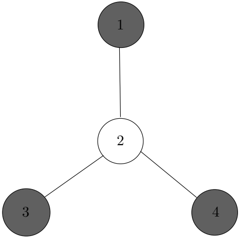

Figure: Maximal independent set of order 3

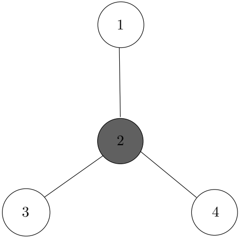

Figure: Maximal independent set of order 1

A strategy profile x is *specialized* if $x_i \in \{0, 1\}$. Let $A(x)$ be the set of active players in x.

1. x is a Nash equilibrium iif $A(x)$ is a maximal independent set;
2. x is ”stable” iif $A(x)$ is a maximal independent set of order at least 2.

Continuum of Nash equilibria

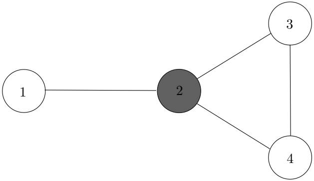

Figure: Isolated Nash equilibrium

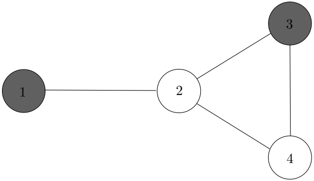

Figure: Continuum of NE

No maximal independent set of order 2 or more...What is stable?

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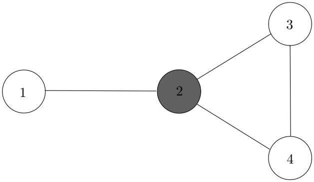

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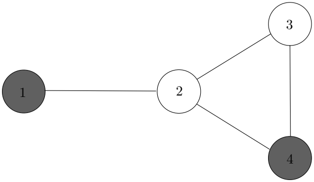

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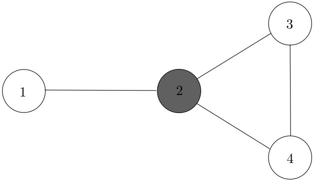

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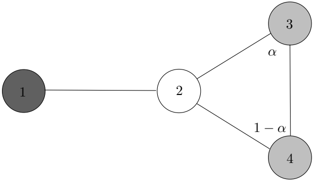

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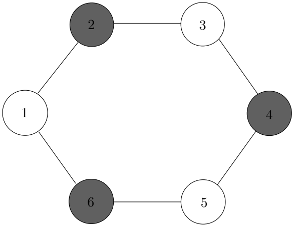

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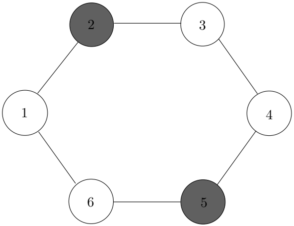

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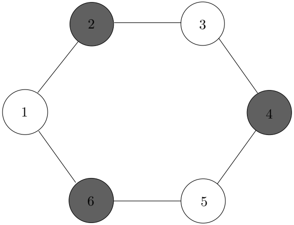

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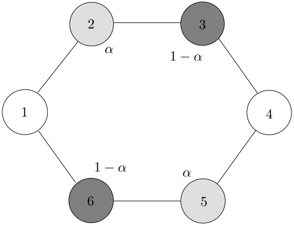

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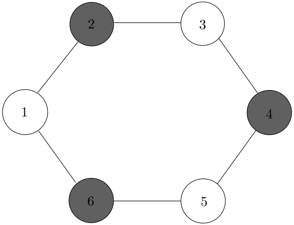

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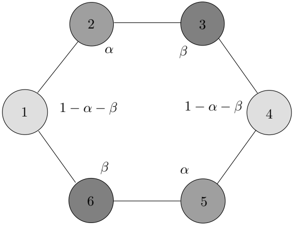

Figure: Continuum of NE

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Our Learning Process

Motivation: *Assume that agents infinitely play the same game (with underlying network structure as defined above). Are there simple global adaptive rules that lead to Nash equilibria? And which NE?*

We are interested in **global** learning procedures where:

- agents have a continuous action set
  ”Classical” reinforcement learning doesn’t work
- agents ignore their payoff function and don’t observe opponent’s actions
  Best response not possible
- agents ignore the structure of the network
  cannot infer the actions of the others from the observed payoff

agents only observe their own payoff and need to deal with it!

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Our Learning Process

Initially, at round 0, agents i plays some action x_0^i

For $n \geq 1$, round n is divided into two periods:

- First, agent i randomly tests a position e_n^i around x_{n-1}^i;
- he observe his own payoff;
- then he chooses x_n^i taking into account x_{n-1}^i and his previous payoffs.
- A new round starts

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Our Learning Process, more specifically

Let $(\epsilon_n^i)_n$ be iid and such that $\mathbb{P}(\epsilon_n^i = 1) = \mathbb{P}(\epsilon_n^i = -1) = 1/2$

For $n \geq 0$, round $n+1$ goes as follows:

1. ϵ_{n+1}^i is drawn and player i chooses $e_{n+1}^i := x_n^i(1 + \frac{1}{n+1}\epsilon_{n+1}^i)$;
2. he observes his realized payoff and computes $\Delta u_{i,n}$, the payoff difference between actions e_{n+1}^i and x_n^i;
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We study the convergence of the random process $(x_n)_n$.

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Iterative Process

Lemma

The iterative process can be written as

$$x_{n+1} = x_n + \frac{1}{n+1}\left(G(x_n) + U_{n+1} + \xi_{n+1}\right)$$

where

- for any i, $G_i(x) = x_i \cdot \frac{\partial u_i}{\partial x_i}(x_i, x_{-i})$
- we have $U^i_{n+1} = \epsilon^i_{n+1} \sum_{j \neq i} \epsilon^j_{n+1} x^j_n \frac{\partial u_i}{\partial x_j}(x_n)$;
- U_{n+1} is a bounded martingale difference ($\mathbb{E}\left(U_{n+1} \mid \mathcal{F}_n\right) = 0$)
- $\xi_n = \mathcal{O}(1/n)$

Mean dynamics

Stochastic process:

$$x_{n+1} = x_n + \frac{1}{n+1}\Bigg(\underbrace{G(x_n)}_{drift} + \underbrace{U_{n+1}}_{=0 \text{ in average}} + \underbrace{\xi_{n+1}}_{\text{small}}\Bigg)$$

Close from the Cauchy-Euler scheme:

$$x_{n+1} = x_n + \frac{1}{n+1} G(x_n)$$

which approximates the solutions of the ODE:

$$\dot{x} = G(x)$$

Set of stationary points of the ODE:

$$Z(G) = NE \cup \underbrace{\left\{x : \; G(x) = 0 \text{ and } \exists i \text{ s.t. } x_i = 0, \; \frac{\partial u_i}{\partial x_i}(x) > 0\right\}}_{\text{"fake zeroes": one agent at least would like to deviate}}$$

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The limit set

(All the results on $\mathcal{L}(x_n)$ have to be understood on the event $\{\sup_n \|x_n\| < +\infty\}$.)

Definition (Limit set of $(x_n)_n$)

Let ω be a realization of the random process.
$\mathcal{L}((x_n)_n(\omega)) := \{x \in S;\ \exists \text{ a sequence } (n_k)_k;\ x_{n_k}(\omega) \to x\}$

Limit sets are **random** objects

Good scenarios:

- a.s. there exists $x \in NE$ s.t. $\mathcal{L}(x_n) = x$;
  (i.e. $(x_n)_n$ converges)
- a.s. $\mathcal{L}(x_n) \subset NE$;
  (i.e. $\lim_n d(x_n, NE) = 0$)

Ideally one of these two cases, but it does not need to be the case. What can we say in full generality??

General properties of the limit set

Theorem (Benaïm, 1996)

The limit set of $(x_n)_n$ is almost surely internally chain transitive (ICT), i.e. it is compact, invariant and it cannot contain a proper attractor.

Consequence: ICT sets are connected (cannot be a finite union of equilibria for instance)

Some examples of ICT sets:

- any equilibrium;
- any periodic orbit;
- more generally any ω-limit set.
- not every ICT sets are omega limit sets: continuum of equilibria

Attractors for the deterministic dynamics

Definition (Attractor)

A compact invariant set A is an attractor for $\dot{x} = G(x)$ if there exists an open neighborhood U of A that is uniformly attracted by A:

$$\lim_{t \to +\infty} \sup_{x \in U} d(\phi_t(x), A) = 0.$$

examples: linearly stable stationary points, linearly stable periodic orbits, more complicated stuff...

Theorem

If an attractor A is attainable by the random process then

$$\mathbb{P}(\mathcal{L}(x_n) \subset A) > 0.$$

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$$\mathbb{P}\left(\mathcal{L}(x_n) \subset A\right) > 0.$$

What's the deal with the "fake zeroes"?

The limit set cannot be contained in the set of fake zeroes:

Theorem

$$\mathbb{P}\left(\mathcal{L}(x_n) \subset Z(G) \setminus NE\right) = 0.$$

Sketch of the proof (simplified):

- Pick a compact set $K \subset Z(G) \setminus NE$ such that, on K, $x_i = 0$ and $\frac{\partial u_i}{\partial x_i}(x) > 0$;
- on the event $\{\mathcal{L}(x_n) \subset K\}$, the random sequence $1/x_n^i$ is a positive supermartingale and therefore converges almost surely: we cannot have $x_n^i \to_n 0$;
- thus the event $\{\mathcal{L}(x_n) \subset K\}$ occurs with null probability.

What about linearly unstable equilibria

Under right assumptions), $(x_n)_n$ cannot converge to an unstable equilibrium:

Theorem

Let $\hat{x}$ be an interior linearly unstable equilibrium. In the following cases

- *"nondegenerate" games with strategic complements (supermodular);*
- *non bipartite interaction graphs;*

we have

$$\mathbb{P}\left(\lim_n x_n = \hat{x}\right) = 0.$$

What could go wrong?

- the noise needs to be "exciting" in an unstable direction;
- However in general our noise may vanish in some directions;
- for strategic complements noise is always exciting in the unstable direction;
- for non-bipartite graphs, the noise cannot cancel out in any direction;

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Potential Games

Definition (Potential Games)

A game $\mathcal{G}$ is said to be a potential game if there is a function $P : X \to \mathbb{R}$ such that

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i})$$

$\mathcal{G}$ is said to be a generalized ordinal potential game if

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) > 0 \Longrightarrow P(x_i, x_{-i}) - P(x_i', x_{-i}) > 0.$$

Monderer and Shapley, 96

Lemma (Lyapunov function)

Assume $\mathcal{G}$ is a generalized ordinal potential game. Then P is a Lyapunov function for G with respect to $Z(G)$:

- *If $x \in Z(G)$ then $t \mapsto P(\phi_t(x))$ is constant;*
- *If $x \notin Z(G)$ then $t \mapsto P(\phi_t(x))$ is strictly increasing.*

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Limit set with Lyapunov function

We assume that G admits a Lyapunov function P, with respect to $Z(G)$.

Theorem

The limit set of $(x_n)_n$ is contained in $Z(G)$.

Theorem (Attractors)

Let Λ be an isolated component of $Z(G)$. Then Λ is an attractor iif

- *P is constant on Λ: $v = P(\Lambda)$;*
- *there exists an open neighborhood U of Λ such that $v > P(x)\ \forall x \in U \setminus \Lambda$.*

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Back to Examples

Public good game:

$$u_i(x_i, x_{-i}) = b(x_i + \sum_{j \in N_i} x_j) - c.x_i$$

with

- b strictly increasing and concave;
- $b(0) = 0$ and $b'(1) = c$.

What does $Z(G)$ look like?

$$Z(G) = \Lambda_1 \cup \ldots \cup \Lambda_K,$$

where Λ_k is an isolated component that can contain fake zeroes.

Potential

This game admits a Lyapunov function:

$$
\begin{aligned}
P(x) &= \sum_i x_i - \frac{1}{2} x_i^2 - \frac{1}{2} \sum_{j \in N_i} x_i x_j \\
&= \langle x, \mathbf{1} \rangle - \frac{1}{2} \|x\|^2 - \frac{1}{2} \langle x, \mathbf{G} x \rangle .
\end{aligned}
$$

Theorem

- *P is constant on Λ_k, $k = 1, \ldots, K$ (Sard),*
- *if $x^* \in Z(G)$ then $P(x^*) = \frac{1}{2} \sum_i x_i^*$,*
- *if x^* is a specialized Nash equilibrium then $P(x^*) = \frac{1}{2} |A(x^*)|$*

Nash equilibria, MI2

isolated Nash equilibrium: $\Lambda_1 = \{(0,0,1,1,0,1,1,0)\}$

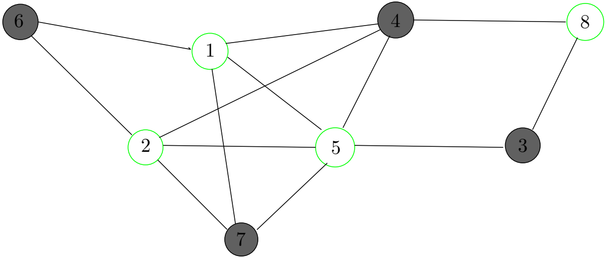

Figure: maximal independent set of order 2

$$P = 2$$

Nash equilibria, continuum 1

Component of Nash: $\{(1,1,1-\alpha,0,0,0,0,\alpha):\ \alpha\in[0,1]\}$

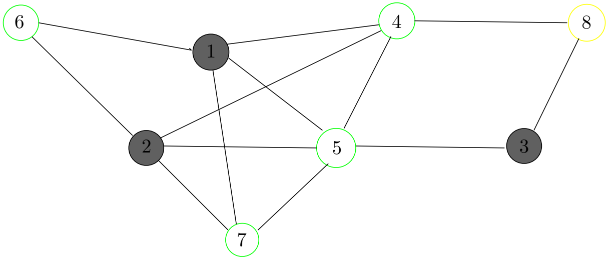

Figure: two maximal independent sets of order 1 in the same component

$P = 3/2$

Nash equilibria, continuum 1

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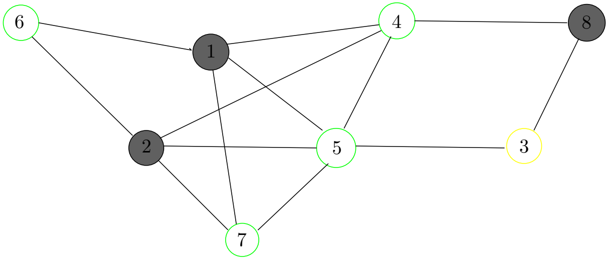

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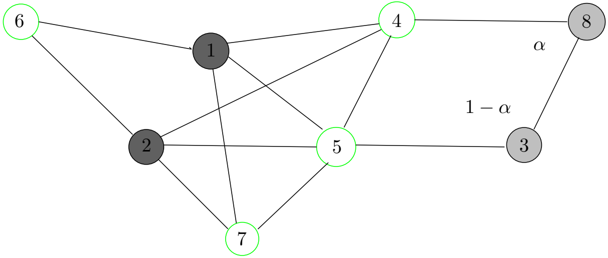
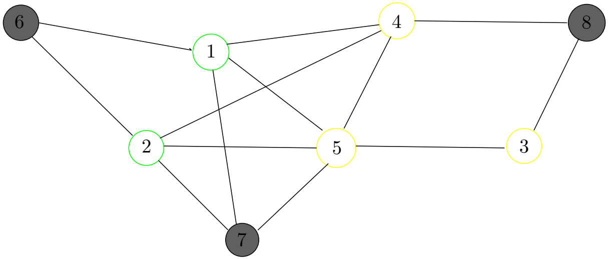


Figure: two maximal independent sets of order 1 in the same component

$P = 3/2$

Nash equilibria, continuum 2

Component of Nash: $\{(0,0,0,0,\alpha,1,1-\alpha,1):\ \alpha\in[0,1]\}$



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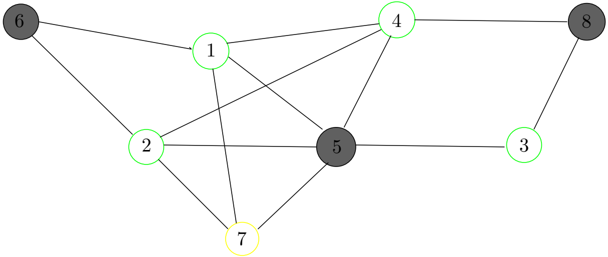

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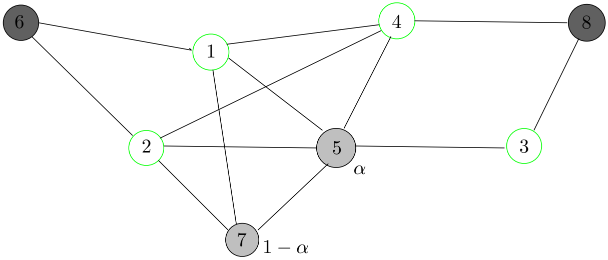

Figure: two maximal independent sets of order 1 in the same component

$P = 3/2$

What is the difference?

In terms of Nash equilibria, both components look similar

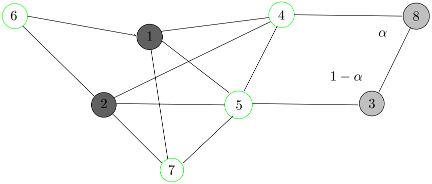

Figure: Continuum 1

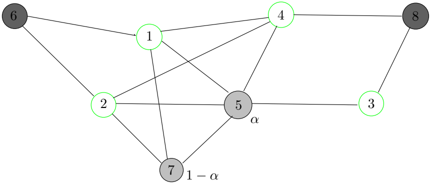

Figure: Continuum 2

But Component 2 also contains fake zeroes!

What is the difference?

In terms of Nash equilibria, both components look similar

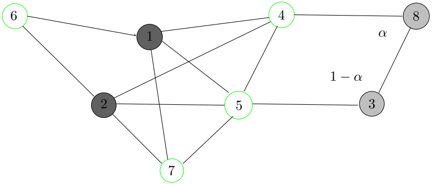

Figure: Continuum 1

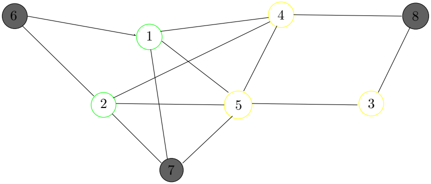

Figure: Continuum 2

But Component 2 also contains fake zeroes!

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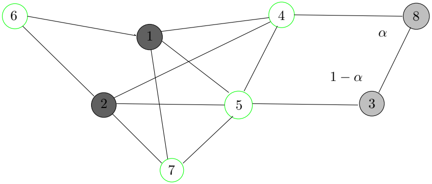


Figure: Continuum 1

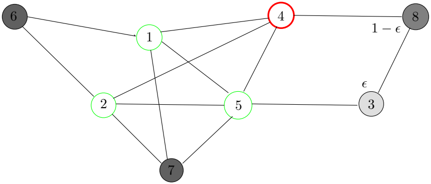


Figure: Continuum 2

But Component 2 also contains fake zeroes!

Identifying attractors

Lemma

Let x^* be an interior NE and $u \in \mathbb{R}^N$ s.t. $x + u \in \mathbb{R}_+^N$. Then

$$P(x+u) = P(x) - \frac{1}{2}\left(\|u\|^2 + \langle u, \mathbf{G}u\rangle\right)$$

Lemma

Let Λ be a component of $Z(G)$. If $\Lambda \cap Spec = \emptyset$ then Λ is not an attractor.

Theorem

Assume that $\Lambda \cap Spec \neq \emptyset$. Then there is equivalence between:

1. *Λ is an attractor;*
2. *$\Lambda \subset NE$*

   (no "fake zeroes" in the component)
3. *$\forall x \in \Lambda \cap Spec$, x is a local maximum of P*

   (only need to check the specialized)

How to use this?

Consequence: we only need to check the specialized Nash in the component. Given $x^* \in Spec$. Define, for $i \in A(x^*)$,

$$C_i := \{i\} \quad \cup \quad \underbrace{\{j \in N_i : \; N_j \cap A(x^*) = \{i\}\}}_{\text{i's neighbors that have no other active neighbor}}$$

Theorem

Let $x^* \in Spec$. Then x^* local maximum iif C_i form a complete graph, for any $i \in A$.

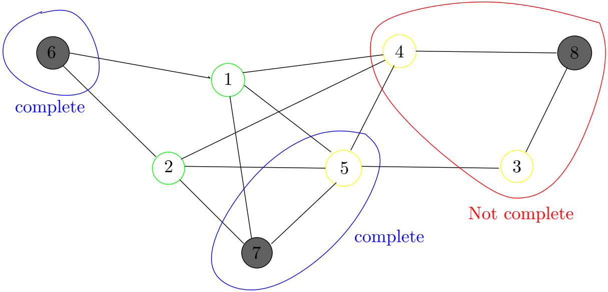
.

Corollary

If $A(x^*)$ is a maximal independent set of order 2 then it is linearly stable (attractor).

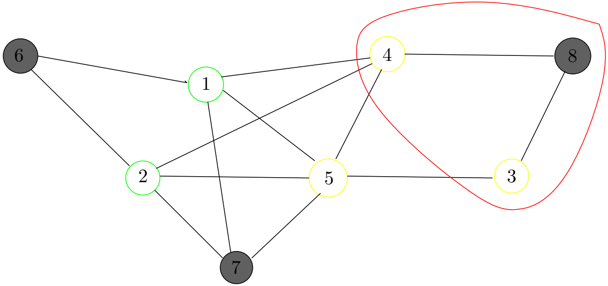
Back to previous example

Figure: Specialized equilibrium that is not a local maximum



Back to previous example

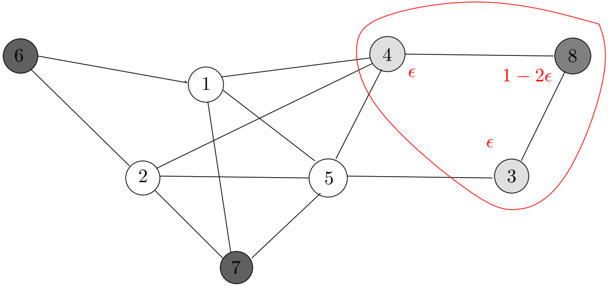
Figure: Specialized equilibrium that is not a local maximum



$P = 3/2$

Back to previous example

Figure: Specialized equilibrium that is not a local maximum



$$P = 3/2 + \epsilon^2$$

Outline of contents

General results on games with strategic complements

$$u_i(x_i, x_{-i}) = v_i(x_i, \sum_j \mathbf{G}_{i,j} x_j)$$

with

- strategic complements $\frac{\partial^2 u_i}{\partial x_i \partial x_j}(x) \geq 0$;
  (Hence $\dot{x} = G(x)$ is a *cooperative system*: $DG(x)$ has non-negative off-diagonal entries)
- v_i strictly concave in the first variable;
- "symmetric" externalities:

$$sgn\left(\frac{\partial u_i}{\partial x_j}(x)\right) = sgn\left(\frac{\partial u_j}{\partial x_i}(x)\right).$$

Theorem

Assume that the interaction graph is non bipartite. Then we have

$$\mathbb{P}\left(\exists x^* \in SNE : \lim_n x_n = x^*\right) = 1$$

on the event $\{\mathcal{L}(x_n) \subset Int(S)\}$.

SNE is the set of Nash equilibria that are not linearly unstable (no eigenvalue with positive real part).

very strong result: $(x_n)_n$ converges!

Elements of proof:

- no continuum of equilibria
- Symmetric externalities and the existence of an odd cycle imply that the noise doesn't cancel in any direction.
- use Benaïm and F. (2012): *if G is a cooperative irreducible dynamics. If the noise goes in every directions then almost surely $(x_n)_n$ converges to a stable zero of G.*
- stable zeroes of G are among Nash