Online Learning with Feedback Graphs

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Joint work with:

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Theory of repeated games

James Hannan
(1922–2010)

David Blackwell
(1919–2010)

Learning to play a game (1956)

Play a game repeatedly against a possibly suboptimal opponent

Prediction with expert advice

N actions

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For $t = 1, 2, \ldots$

1. Loss $\ell_t(i) \in [0, 1]$ is assigned to every action $i = 1, \ldots, N$ (hidden from the player)

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3. Player gets feedback information: $\boldsymbol{\ell}_t = \big(\ell_t(1), \ldots, \ell_t(N)\big)$

Regret

The loss process $\langle \ell_t \rangle_{t \geqslant 1}$ is deterministic and unknown to the (randomized) player $I_1, I_2, \ldots$

Regret of player $I_1, I_2, \ldots$

$$R_T \stackrel{\text{def}}{=} \mathbb{E}\left[\sum_{t=1}^{T} \ell_t(I_t)\right] - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \ell_t(i) \stackrel{\text{want}}{=} o(T)$$

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Asymptotic lower bound for experts' game

$$\boxed{R_T = \big(1 - o(1)\big)\sqrt{\frac{T \ln N}{2}}}$$

Proof uses an i.i.d. stochastic loss process

Exponentially weighted forecaster

At time t pick action $I_t = i$ with probability proportional to

$$\exp\left(-\eta \sum_{s=1}^{t-1} \ell_s(i)\right)$$

the sum at the exponent is the total loss of action i up to now

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Regret bound

If $\eta = \sqrt{\frac{\ln N}{8T}}$ then $\boxed{R_T \leqslant \sqrt{\frac{T \ln N}{2}}}$

Matching asymptotic lower bound including constants

Dynamic choice $\eta_t = \sqrt{(\ln N)/(8t)}$ only loses small constants

The bandit problem: playing an unknown game

N actions

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Many applications

Ad placement, recommender systems, online auctions, . . .

Bandits as an instance of a general feedback model

- Besides observing the loss of the played action, the player also observes the loss some other actions
- For example, a recommender system can infer how the user would have reacted had similar products been recommended
- <u>However:</u> we do not insist on assuming that observability between actions implies similarity between losses

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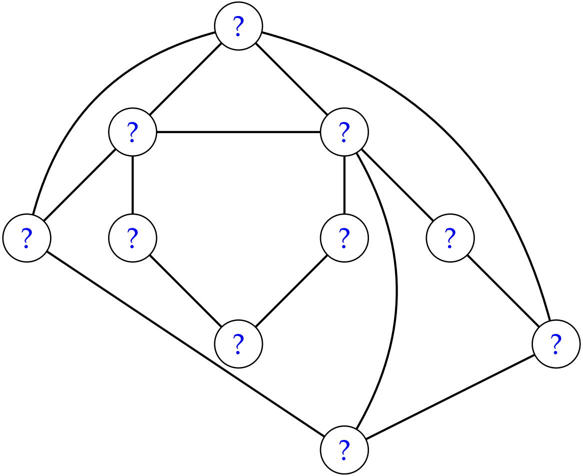
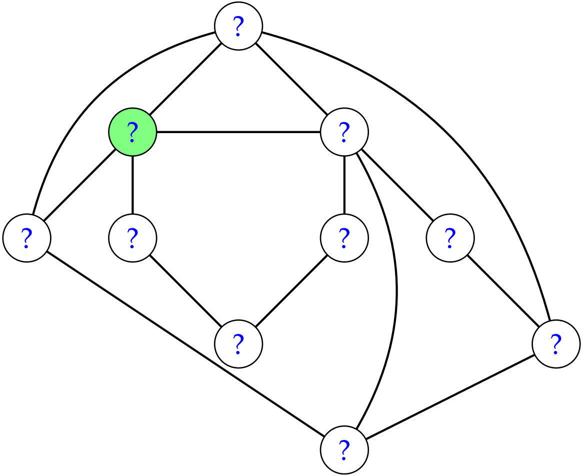
How does the observability structure influence regret?

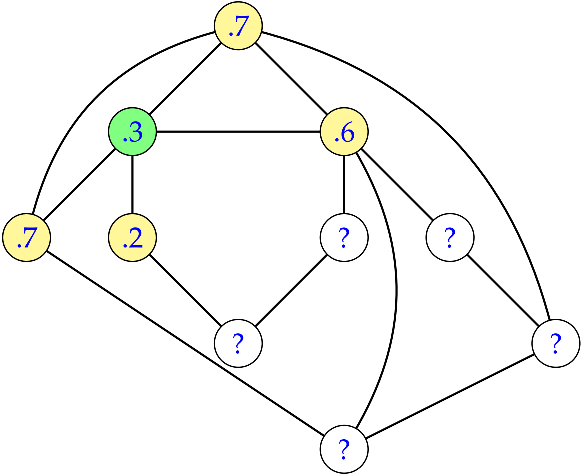
Feedback graph



Feedback graph

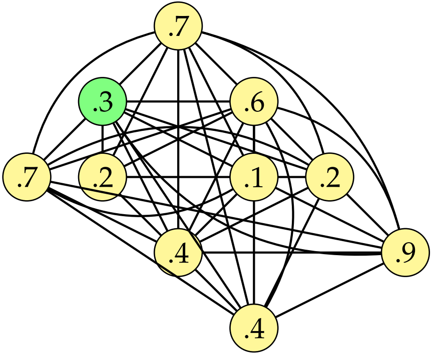


Feedback graph

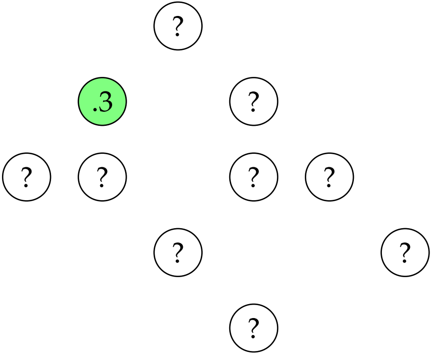


Recovering expert and bandit settings

Experts: clique



Bandits: empty graph



Exponentially weighted forecaster — Reprise

Player's strategy

- $\mathbb{P}_t(I_t = i) \quad \propto \quad \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) \qquad i = 1, \dots, N$
- $\widehat{\ell}_t(i) = \begin{cases} \dfrac{\ell_t(i)}{\mathbb{P}_t\big(\ell_t(i) \text{ observed}\big)} & \text{if } \ell_t(i) \text{ is observed} \\ 0 & \text{otherwise} \end{cases}$

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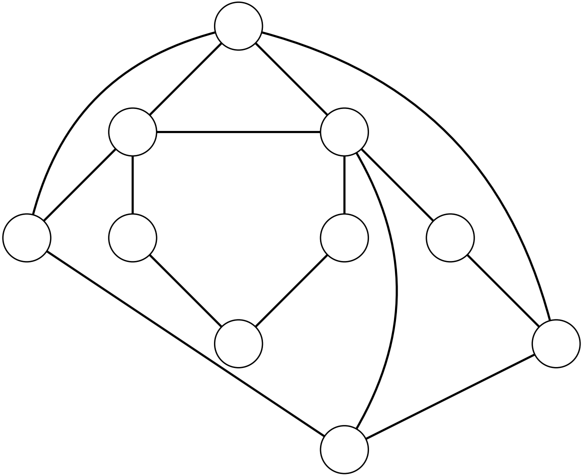
Importance sampling estimator

$\mathbb{E}_t\left[\widehat{\ell}_t(i)\right] = \ell_t(i)$ unbiasedness

$\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right] = \dfrac{\ell_t(i)^2}{\mathbb{P}_t\big(\ell_t(i) \text{ observed}\big)}$ variance control

Independence number $\alpha(G)$

The size of the largest independent set



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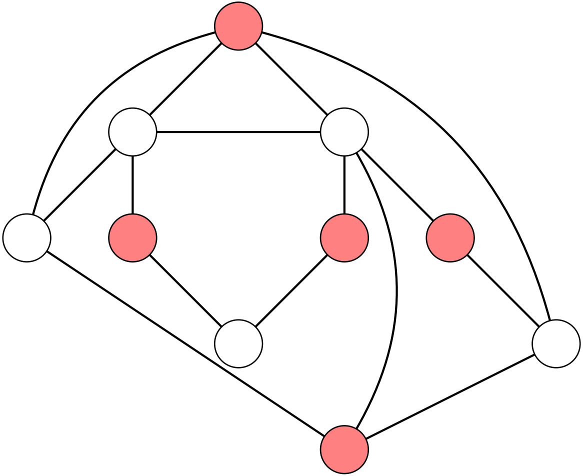

Regret bounds

Analysis (undirected graphs)

$$R_T \leqslant \frac{\ln N}{\eta} + \frac{\eta}{2}\,\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{\mathbb{P}_t(i \text{ is played})}{\mathbb{P}_t(\ell_t(i) \text{ is observed})}\right]$$

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Lemma

For any undirected graph $G = (V, E)$ and for any probability assignment $p_1, \ldots, p_N$ over its vertices

$$\sum_{i=1}^{N}\frac{p_i}{\underbrace{p_i + \sum_{j \in N_G(i)} p_j}_{\mathbb{P}_t(\text{loss of } i \text{ observed})}} \leqslant \alpha(G)$$

Regret bounds

Analysis (undirected graphs)

$$R_T \leqslant \frac{\ln N}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \alpha(G) = \sqrt{T\alpha(G) \ln N} \qquad \text{by choosing } \eta$$

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$$R_T \leqslant \frac{\ln N}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \alpha(G) = \sqrt{T\alpha(G) \ln N} \qquad \text{by choosing } \eta$$

Special cases

Experts (clique): $\alpha(G) = 1$ $\quad R_T \leqslant \sqrt{T \ln N}$

Bandits (empty graph): $\alpha(G) = N$ $\quad R_T \leqslant \sqrt{TN \ln N}$

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Special cases

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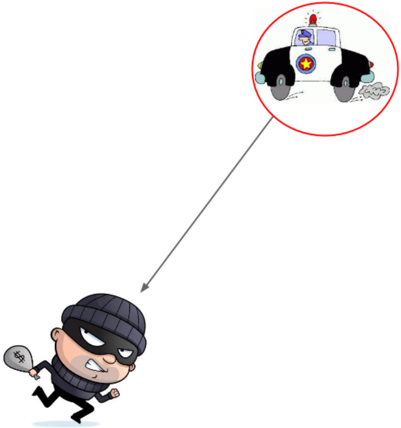
Minimax rate

The general bound is tight: $R_T = \widetilde{\Theta}\left(\sqrt{T\alpha(G) \ln N}\right)$

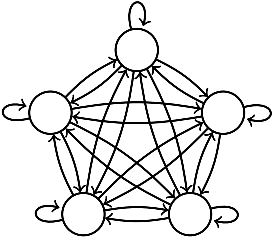
More general feedback models

Directed

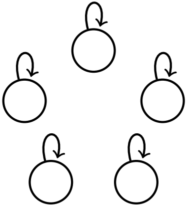
Interventions



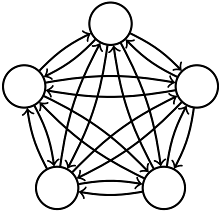
Old and new examples



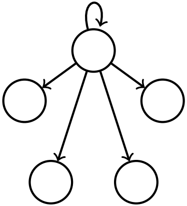
Experts



Bandits



Cops & Robbers



Revealing Action

Exponentially weighted forecaster with exploration

Player's strategy

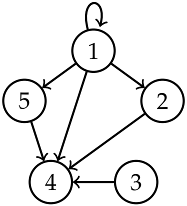
- $$\mathbb{P}_t(I_t = i) = \frac{1-\gamma}{Z_t} \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) + \gamma\, U_G \qquad i = 1, \dots, N$$
- $$\widehat{\ell}_t(i) = \begin{cases} \dfrac{\ell_t(i)}{\mathbb{P}_t\big(\ell_t(i) \text{ observed}\big)} & \text{if } \ell_t(i) \text{ is observed} \\ 0 & \text{otherwise} \end{cases}$$

U_G is uniform distribution supported on a subset of V

A characterization of feedback graphs

A vertex of G is:

- observable if it has at least one incoming edge (possibly a self-loop)
- strongly observable if it has either a self-loop or incoming edges from all other vertices
- weakly observable if it is observable but not strongly observable



- 3 is not observable
- 2 and 5 are weakly observable
- 1 and 4 are strongly observable

Characterization of minimax rates

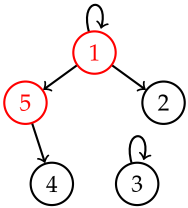
G is strongly observable $\quad R_T = \widetilde{\Theta}\left(\sqrt{\alpha(G)T}\right)$

U_G is uniform on V

G is weakly observable $\quad R_T = \widetilde{\Theta}\left(T^{2/3}\delta(G)\right)$ for $T = \widetilde{\Omega}(N^3)$

U_G is uniform on a weakly dominating set

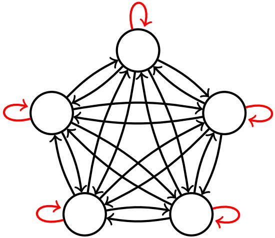
G is not observable $\quad R_T = \Theta(T)$



Weakly dominating set

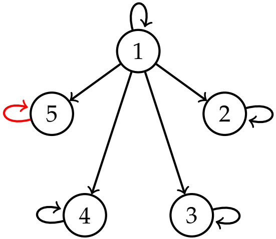
$\delta(G)$ is the size of the smallest set that dominates all weakly observable nodes of G

Some curious cases



Experts vs. Cops & Robbers

Presence of red loops does not affect minimax regret $R_T = \Theta\left(\sqrt{T \ln N}\right)$



Sharp transitions

With red loop: strongly observable with

$\alpha(G) = N - 1 \qquad R_T = \widetilde{\Theta}\left(\sqrt{NT}\right)$

Without red loop: weakly observable with

$\delta(G) = 1 \qquad R_T = \widetilde{\Theta}\left(T^{2/3}\right) \qquad$ for $T = \widetilde{\Omega}(N^3)$

Final remarks

- Theory extends to time-varying feedback graphs
- In the strongly observable case, algorithm can predict without knowing the graph
- Entire framework is a special case of partial monitoring, but our rates exhibit sharp problem-dependent constants

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Graph over actions: more interpretations

- Relatedness (rather than observability) structure on loss assignment
- Delay model for loss observations