Rest in Lounge or Wait in Queue

Sandeep Juneja, TIFR
Visiting Professor CAFRAL, Central Bank India

initial parts are joint work with Nahum Shimkin (Technion, Israel), Rahul Jain (USC),
latter with D. Manjunath (IITB).

Workshop on Congestion Games, NUS
December 16, 2015
Queue timing games considered
To rest in lounge or wait in queue

Concert queue, queue size not known
To rest in lounge or wait in queue

Concert queue, queue size known
To rest in lounge or wait in queue

M/M/1 queue, queue size known
Substantial transportation literature motivated by Vickrey’s seminal work on the morning commute problem:

- Drivers going to work via single bottleneck queue. They have a preferred time to reach office. Too early or too late not good.
- Vehicles modelled as fluid particles
- Focus primarily on coming up with toll regimes to manage congestion.

Earlier queueing literature focusses primarily on admissions control, routing, reneging, pricing etc. Largely summarized in monograph ‘To Queue or not to Queue’ by Hassin and Haviv (2003).
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Related Literature: Strategic Timing Decisions

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Concert or Cafeteria Queueing Problem

  - Too early may mean large queues and long wait.
  - Too late means that one gets poorer service, gets served late

- In this talk,
  1. we review the literature in fluid framework when the total population is fixed
  2. we also discuss the finite population case,
  3. as well as the dynamic setting where people in ‘lounge’ can observe queue lengths and decide when to join.
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Fluid model framework

- **Fluid model**
  - Fluid model where arrivals are points in a continuum \([0, 1]\), server serves at a fixed rate \(\mu\).

  - Server becomes active at time zero

  - Customers can come and queue up before or after time zero

  - Customer cost is additive and linear in waiting time and service completion time (equivalently, in time spent in lounge)
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Suppose that a customer $s$ chooses her arrival time from distribution $G_s(\cdot)$, $s \in [0, 1]$. The deterministic arrival profile is given by

$$F(t) = \int_0^1 G_s(t) ds$$

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Customer Cost Function

- Let $W_F(t)$ denote waiting time for arrival at time $t$ when all other customers have an arrival profile $F$.

- Arrival at time $t$ incurs cost

$$ C_F(t) = \alpha W_F(t) + \beta (t + W_F(t)) $$

- More generally, the expected cost incurred by a customer $s$ who selects her arrival by sampling from probability distribution $G_s$ is

$$ C_F(G) = \int_{-\infty}^{\infty} (\alpha W_F(t) + \beta (t + W_F(t))) \, dG_s(t). $$
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$W_F(t)$ is a function of $Q_F(t)$ that depends on $F$

- Queue length

\[ Q_F(t) = F(t), \]

for $t \leq 0$.

- Waiting time

\[ W_F(t) = Q_F(t)/\mu - t = F(t)/\mu - t \]

- When there is a jump at time $t$, replace $F(t)$ by $(F(t) + F(t^-))/2$
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- When there is a jump at time $t$, replace $F(t)$ by $(F(t) + F(t^-))/2$
For $t > 0$, let $t^* \in [0, t]$ denote the last time the queue was empty if at all. Then,

$$Q_F(t) = F(t) - F(t^*) - \mu(t - t^*).$$

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Evaluating $W_F(t), \ t > 0$

- Else, if queue was never empty

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Q_F(t) = F(t) - \mu t.
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and

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[Diagram showing queue dynamics over time]
Nash Equilibrium

A multi-strategy \( \{ G_s(\cdot), \ s \in [0, 1] \} \) with arrival profile \( F(t) = \int_0^1 G_s(t) ds \) for each \( t \) is a Nash equilibrium point if

- For any customer \( s \in [0, 1] \),
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  C_F(G_s) \leq C_F(\tilde{G}), \quad \text{for every CDF } \tilde{G}.
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  That is, no customer \( s \) can improve her cost by modifying her own arrival time distribution.

- This is equivalent to an existence of an arrival profile \( F \) such that \( C_F(t) \) is constant on support of \( F \) and at least as much elsewhere.
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- **This is equivalent to an existence of an arrival profile \( F \) such that \( C_F(t) \) is constant on support of \( F \) and at least as much elsewhere.**
Equilibrium profile implies equilibrium multi-strategy

Cost = \alpha W_t + \beta(t + W_t)

Support of F
Non-equilibrium profile implies non-equilibrium multi-strategy
Hunt for Equilibrium Profile
Useful Insights

- Let \( t^* = \inf\{t \geq 0 : F(t) < \mu t\} \).

- In Nash Equilibrium, first time the server has spare capacity is when all customers are served. That is \( t^* = \frac{1}{\mu} \).

Similarly, in Nash equilibrium there can be no point masses in \( F \).
Useful Insights

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- Similarly, in Nash equilibrium there can be no point masses in $F$. 
Cost Function under Equilibrium

- If $F(\cdot)$ denotes an equilibrium profile Then, $W_F(t)$ equals

$$F(t)/\mu - t, \text{ for } t \leq 1/\mu.$$ 

- The cost function $C_F(t)$ equals

$$\beta(t + W_F(t)) + \alpha W_F(t) = (\alpha + \beta)F(t)/\mu - \alpha t$$
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Cost Function under Equilibrium

\[ C_F(t) = (\alpha + \beta) F(t)/\mu - \alpha t \]

- Last arrival at time \( \leq 1/\mu \). This must equal \( 1/\mu \) to improve cost.
- Hence, \( F(1/\mu) = 1 \), and \( C_F(t) = \beta/\mu \) on support of \( F \).
- Furthermore

\[ F(t) = \frac{(t - t_0)}{(t_1 - t_0)}, \]

where \( t_1 = 1/\mu \) and \( t_0 = -\beta/(\alpha \mu) \). \( F(t) = 0 \) for \( t < t_0 \) and \( F(t) = 1 \) for \( t > t_1 \).
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Queue Length Process

\[ F'(t) = \mu \left( \frac{\alpha}{\alpha + \beta} \right) \]

for \( t_0 < t < t_1 = 1/\mu \).
Socially Optimal Solution

- The smallest value of waiting time is zero.

- Total time to service is minimized if the server serves at the fastest possible rate. So the average service time is $1/(2\mu)$ and the overall cost is $\beta/(2\mu)$.

- Price of anarchy equals two.
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Finite Population Analysis
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- $N$ customers, each requires an exponentially distributed service.
- $\mathcal{F} = \{F_i : i \leq N\}$ denotes a collection of arrival time distributions of all customers.
- The expected cost

$$C_i(\mathcal{F}) = E_\mathcal{F}(\alpha W_i + \beta T_i).$$

- An arrival profile $\{F_i, i = 1, \ldots, N\}$ is a Nash equilibrium point if

$$C_i(F_i, \mathcal{F}^{-i}) \leq C_i(\tilde{F}_i, \mathcal{F}^{-i})$$

for every customer $i$ and every CDF $\tilde{F}_i$. 

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  for every customer $i$ and every CDF $\tilde{F}_i$. 
An arrival profile \( \{ F_i, \ i = 1, \ldots, N \} \) is a Nash equilibrium point if, and only if, for every \( i \) there exists a constant \( c_i \) so that

(i) \( C_i(t, F^{-i}) \geq c_i \) for all \( t \).

(ii) \( C_i(t, F^{-i}) = c_i \) on a set of \( F_i \)-measure 1.
Nash Equilibrium must be symmetric

- Suppose $c_i < c_j$. Then customer $j$ can improve cost by coming just before the earliest time in support of $F_i$.

- If the two df’s $F_i$ and $F_j$ differ, e.g. $F_i(t) = F_j(t)$ for $t \leq t^*$ and then $F_j(t) > F_i(t)$ over an interval just after $t^*$.

- This implies that $Q^{-i}(t) > Q^{-j}(t)$ for some $t$, implying that the costs incurred by $i$ and $j$ are different at $t$. 
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Setting the cost function derivative with respect to time to zero

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\frac{N-1}{\mu} F'(t) = \frac{\alpha}{\alpha + \beta} - P_0(t),
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for \( t \in [0, t_b) \). and

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Figure: Numerically computed equilibrium arrival densities $G'(t)$ for $\lambda = 2, \beta = 1$ and $\mu = 1$, and several values of $M = N - 1$. The plots present the normalized densities $MG'(Mt)$. 

The initial parts are joint work with Nahum Shimkin (Technion, Israel), Rahul Jain (USC), and later with D. Manjunath (IITB).
Convergence to the fluid limit

**Theorem**

$F(Nt)$ converges to a uniform distribution on $[-\frac{1}{\mu \alpha}, \frac{1}{\mu}]$ at rate $o\left(\frac{\log N}{\sqrt{N}}\right)$, in the sense that

$$\frac{N-1}{\mu} F'(t) = \frac{\alpha}{\alpha + \beta} - P_0(t) 1\{t \geq 0\}, \quad t \in (t_a, t_b)$$

is satisfied with

$$\frac{t_a}{N-1} = -\frac{1}{\mu \alpha} - o\left(\frac{\log N}{\sqrt{N}}\right) < -\frac{1}{\mu \alpha}$$ \hspace{1cm} (1)

$$\frac{t_b}{N-1} = \frac{1}{\mu} + o\left(\frac{\log N}{\sqrt{N}}\right)$$ \hspace{1cm} (2)

and

$$P_0(t) \leq \frac{1}{N} \text{ for } \frac{t}{N} \leq \frac{1}{\mu} - o\left(\frac{\log N}{\sqrt{N}}\right).$$ \hspace{1cm} (3)
Proof idea: Inductively show that $P_0(t)$ is small

1. Suppose $P_0(t)$ small
2. Large queue at beginning and end
3. So $P_0(t)$ small
To rest in lounge or wait in queue

Concert queue, queue size known
Finite Population, can observe the queue at a ‘Concert’

- Finite population of size $N$.

- Customers start by all being in the lounge. They can observe the lounge as well as the queue size.

- Queue service starts at time zero. Exponential service times.

- Lounge unit time cost $\beta < \alpha$ unit wait cost.

- We assume that if many people in lounge wish to join the queue, only one succeeds at random.
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Equilibrium policy has the threshold \( \{ m(n) \} \) form.

- If \( n \) people in lounge, then for queue \( m \leq m(n) \), people in lounge compete to join the queue.

- For \( m > m(n) \), people in the lounge choose to wait.
- Equilibrium policy has the threshold \( \{m(n)\} \) form.

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Illustrative threshold equilibrium policy

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m(n)$</th>
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<tr>
<td>2</td>
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The related equations are simple

- For $m \leq m(n)$,

$$C(m, n) = \frac{1}{n} \alpha m / \mu + \frac{n - 1}{n} C(m + 1, n - 1) \leq C(m + 1, n - 1)$$

- For $m > m(n)$,

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For $m > m(n)$,

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These are easily solved

- $C(m, 1) = m\beta/\mu$

\[ m(n) = \max\{m : \alpha m/\mu \leq C(m + 1, n - 1)\} \]

- For $m \leq m(n)$,

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  C(m, n) = \frac{1}{n} \alpha m/\mu + \frac{n - 1}{n} C(m + 1, n - 1).
  \]

- For \( m > m(n) \),
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  C(m, n) = \beta/\mu + C(m - 1, n).
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These are easily solved

- \( C(m, 1) = m\beta/\mu \)

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\[ \frac{m(n)}{n} \xrightarrow{n} \frac{\beta}{\alpha - \beta} \]

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- **Strategy** \{m(n)\} is a weakly dominant strategy when the server is on. Thus, unique Nash equilibrium.

- Price of anarchy is bounded from above by 2.
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To rest in lounge or wait in queue

M/M/1 queue, queue size known
$M/M/1$ queue with lounge facility

- Need to get service arises as a Poisson process.

- Arrival joins lounge. Can see others in the lounge.

- For

  \[
  \frac{\alpha}{\beta} \leq \frac{\mu}{\mu - \lambda}
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  dominant strategy to join the queue.

- Lounge remains empty. Everyone joins the queue.

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$$\frac{\alpha}{\beta} > \frac{\mu}{\mu - \lambda}$$

- Equilibrium policy has the threshold $\{m(n)\}$ form.

- If $n$ people in lounge, then for queue $m \leq m(n)$, people in lounge compete to join the queue.

- Equations similar to before. Price of anarchy is bounded from above by

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Limiting Queue Buffer to Control PoA

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  by limiting queue size, customers forced to remain in lounge.

- Can numerically see that lounge may have customers in this case!

- Can bound PoA by
  \[ \frac{\alpha}{\beta}(1 - \rho^{k+1}) + \rho^{k+1}, \]

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Conclusion

- Discussed strategic arrivals in the concert queue setting when the queue is not observed by customers; customers are non-atomic as well as when they are atomic.

- Strategic arrivals in concert queue setting when customers in lounge can observe the queue.

- Considered lounge in $M/M/1$ setting where we do steady state analysis of strategic queue joining behaviour.
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