Zero-sum Revision Games

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Players have to prepare their actions in a pre-play phase preceding the payoff-relevant play in a one shot game.

During the pre-play phase:
- Prepared actions are commonly observed.
- Prepared actions can be change only at the bell of a Poisson clock.

Only the last prepared action profile matters for the payoff.

Some examples
- Preopening in the stock market (Nasdaq, Euronext, Toronto SE, daily from 7a.m. to 9 a.m.)
- Interaction through internet servers (e-bay auctions).
- Preparatory meetings to negotiate the terms of a treaty.
- Armies deploying their troops on the ground
- ...
Component game: Zero-sum game

- 2 players.
- \( X_i \): player \( i \)'s finite set of actions.
- \( U : X_1 \times X_2 \rightarrow \mathbb{R} \): player 1’s payoff matrix (generic).

\[
BR^U_1(x) := \arg \max_{y_1} U(y_1, x_2) \ ; \ BR^U_2(x) := \arg \min_{y_2} U(x_1, y_2)
\]

- Stackelberg payoff where 1 plays first:

\[
S_1 = \max_{x_1 \in X_1} \min_{x_2 \in X_2} U(x_1, x_2)
\]

- Stackelberg payoff where 2 plays first

\[
S_2 = \min_{x_2 \in X_2} \max_{x_1 \in X_1} U(x_1, x_2)
\]

- Value of the game: \( V \)

\[
S_1 \leq V \leq S_2
\]

with equality if pure Nash.
At $t = 0$, starting prepared action $x(0) \in X$ exogenous.

Between time 0 and $T$, Poisson arrivals of revision times independent for each player (Asynchronous moves).

Each player can change his prepared actions only at his revision times.

At $T$ players get their only payoff and this results from players playing, in the component game, their last prepared actions.
Finite time horizon $[0, T]$.

Game $\Gamma_{[\tau, T]}(x)$ with $\tau < T$ and $x \in X$.

Time $\eta$ is drawn from an exponential distribution with parameter $\lambda$.

If $\eta + \tau > T$, then the game is over and players’ payoff is $\{U(x), -U(x)\}$

if $\eta + \tau < T$, then
- With Pr $q \in (0, 1)$ player 1 chooses an action $y_1 \in X_1$ and the game $\Gamma_{[\tau+\eta, T]}(y_1, x_2)$ starts.
- With Pr $1 - q$ player 2 chooses an action $y_2 \in X_2$ and the game $\Gamma_{[\tau+\eta, T]}(x_1, y_2)$ starts.

Initial game: $\Gamma_{[0, T]}(x(0))$

Calcagno, Kamada, Lovo and Sugaya (2014): In $2 \times 2$ conflicting interest games (generic Battle of the sexes),
- the revision game equilibrium is unique;
- the slow players has an advantage over the fast players;
- revision game equilibrium payoff = component game Nash equilibrium payoff;
- All action occurs at the beginning of the revision game.

Cheap talk games: Farrell (1987), Rabin (1994), Aumann and Hart (2003), ...

Switching cost games: Lipman and Wang (2000) and Caruana and Einav (2008), ...
Research question

What are we after?

- Under what conditions does a player prefer to play the revision game rather than the straight zero-sum game?
- Characterization of equilibrium payoff.
- Characterization of equilibrium behavior.
1 Preliminaries
2 General results
3 $2 \times 2$ equilibrium characterization
Preliminaries
Notation, histories and strategies

- Set of states: player who can revise and the resulting new profile of action
  \[ K = \{1, 2\} \times X \]
- History of past revision time and chosen actions
  \[ h_n = \{x, \tau_1, k_1, \ldots, \tau_n, k_n\} \in X \times ([0, T] \times K)^n \]
- Strategy: mapping histories and revision times into a (mixed) action
  \[ \sigma_i : \bigcup_{n \geq 0} (H_n \times [0, T]) \to \Delta X_i \]
- A Markov strategy is a measurable mapping
  \[ \sigma_i : X \times [0, T] \to \Delta X_i \]
- Expected payoff given \( \sigma \):
  \[ u_\sigma(T, x) := \mathbb{E}_\sigma[U(x(T)) | x(0) = x] \]
  where \( x(T) \) is the last prepared action profile at time \( T \).
Preliminaries
Existence

Theorem

(Lovo and Tomala (2015)) The revision game has a Markov perfect equilibrium. With

- $t$ is the remaining time.
- $u(x, t)$: equilibrium payoff of the game of length $t$ with starting action profile $x$.
- $u(t) := \{u(t, x)\}_{x \in X}$
- $\sigma_i(t, x) \in BR_i^{u(t)}(x_{-i})$
Remark that $u(t, x)$ is Lipschitz.

Let

$$u^+(t, x) := \max_{y_1 \in X_1} u(t, y_1, x_2) \quad ; \quad u^-(t, x) := \min_{y_2 \in X_2} u(t, x_1, y_2);$$

$$\lambda_1 := \lambda q \quad ; \quad \lambda_2 := \lambda (1 - q)$$

Then

$$u(t, x) = U(x) e^{-\lambda t} + \int_{s=0}^{t} e^{-\lambda (t-s)} \left( \lambda_1 u^+(s, x) + \lambda_2 u^-(s, x) \right) ds,$$

$$\frac{\partial u(t, x)}{\partial t} = \lambda_1 (u^+(t, x) - u(t, x)) + \lambda_2 (u^-(t, x) - u(t, x)),$$

$$u(0, x) = U(x).$$
Proposition

1. The revision game has a Markov perfect equilibrium in pure strategy.
2. The equilibrium payoff $u(t, x)$ is Lipschitz in $t$, $U$ and is continuous in $(q, \lambda) \in (0, 1) \times (0, \infty)$.
3. The equilibrium payoff $u(t, x)$ is non-decreasing in $q$. 
General results
Pure strategies: sketch of the proof

Take a MPE and suppose that for some time $t$, $\sigma_i(t, x)$ is not pure. For this $t$ replace $\sigma_i(t, x)$ by a pure action in $\sigma'_i(t, x) \in BR^u_i(x)$. Observe that $u^+$ and $u^-$ do not change with $\sigma$ or $\sigma'$. Hence

$$u(t, x) = u(x)e^{-\lambda t} + \int_{s=0}^{t} e^{-\lambda(t-s)} \left( \lambda_1 u^+(s, x) + \lambda_2 u^-(s, x) \right) ds$$

is the same under $\sigma$ and $\sigma'$. Zero sum structure is crucial.
General results
Continuity of $u(t, x)$: sketch of the proof

- 1-Lipschitz in $t$:

  $$|u(t, x) - u(t + \varepsilon, x)| \leq \|u(t)\|(1 - e^{-\lambda\varepsilon})$$

- 1-Lipschitz in $U$: Take $U' \neq U$, then

  $$|u(t, x) - u'(t, x)| \leq \max_{y \in X} |U(y) - U'(y)|$$

- Continuous in $\lambda$, take $\lambda' \neq \lambda$, then

  $$u(t, x)|_{\lambda} = u\left(\frac{\lambda'}{\lambda} t', x\right)|_{\lambda'}$$

- Continuous in $q$: Payoff continuously depends on the distribution of revision time that is continuous in $q$. 

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General results
Monotonicity of $u(t, x)$ in $q$: sketch of the proof

Let $q' < q$.

$$\frac{\partial u(t, x)}{\partial t} = \lambda (qu^+(t, x) + (1 - q)u^-(t, x) - u(t, x))$$

If for some $t$, $u(t, x)|_{q'} = u(t, x)|_{q}$, then

$$\left. \frac{\partial u(t, x)}{\partial t} \right|_{q'} \leq \left. \frac{\partial u(t, x)}{\partial t} \right|_{q}$$

implying $u(\tau, x)|_{q} \leq u(\tau, x)|_{q'}$, for some $\varepsilon > 0$, and all $\tau \in (t, t + \varepsilon)$.

but

$$u(0, x)|_{q'} = u(0, x)|_{q} = U(x)$$

so it can never be that

$$u(\tau, x)|_{q'} > u(\tau, x)|_{q}$$
Consider a revision game where the starting action profile is $x$ and let

$$\underline{R}(x) := \liminf_{t \to \infty} u(t, x) \quad \text{and} \quad \overline{R}(x) := \limsup_{t \to \infty} u(t, x)$$

If $\underline{R}(x) = \overline{R}(x) = R(x)$ then we say that the revision game value is $R(x)$.
Proposition

1. Irrelevance of revision when $V$ is achieved with pure strategies:

$$ S_1 \leq R(x) \leq \overline{R}(x) \leq S_2 $$

2. Ergodicity:

$$ R(x) = \overline{R}(x) = R, \forall x \in X $$

for some constant $C$

3. $R$ is $1$-Lipschitz in $U$, and continuous in $(q, \lambda) \in (0, 1) \times (0, \infty)$.

4. $\lim_{q \to 0} = S_1$ and $\lim_{q \to 1} = S_2$
Let $X_1 = X_2 = \{\alpha, \beta\}$.

\[
\begin{array}{cc}
\alpha & \beta \\
\hline
\alpha & U(\alpha, \alpha) & U(\alpha, \beta) \\
\beta & U(\beta, \alpha) & U(\beta, \beta)
\end{array}
\]

Then the component game where $U$ is generic, implying $x \neq x' \Rightarrow U(x) \neq U(x')$. 
2 × 2 games
Possible scenarios for $u(t)$

- **Scenario DD**: Each player has a dominant action. For example:

  \[
  \begin{array}{cc}
  0 & 2 \\
  -1 & 1 \\
  \end{array}
  \]

- **Scenario DN**: One player has a dominant action whereas the other player does not. For example:

  \[
  \begin{array}{cc}
  0 & 2 \\
  -1 & -5 \\
  \end{array}
  \]

- **Scenario NN**: No pure Nash Eq. For example:

  \[
  \begin{array}{cc}
  0 & 2 \\
  3 & -5 \\
  \end{array}
  \]
Proposition

Suppose $U$ is in scenario DD, and let $\hat{x}_i$ be player $i$'s dominant action in the component game. Then for all $t$,

1. $u(t)$ is in scenario DD
2. $\sigma_i(t, x) = \hat{x}_i, \forall x \in X$
3. $R = V$
Intuition:
- By continuity with respect to $t$ each player prepares his dominant action when $t$ is close to 0.
- If for all $\tau > t$ the other player uses a fixed action no matter what you do, then you strictly prefer preparing your dominant action at $t$.

Algebraic:
Solve the ODE and verify that $BR_i^{u(t)}(x)$ does not depend on $t$. 
Proposition

Suppose $U$ is in scenario DN, and $\hat{x}_1$ is player 1's dominant action in the component game. Then there is $t^*$ finite such that

1. For $t < t^*$:
   - $u(t)$ is in scenario DN
   - $\sigma_1(t, x) = \hat{x}_1, \forall x$
   - $\sigma_2(t, x) = BR^U_2(x_1)$

2. For $t \geq t^*$:
   - $u(t)$ is in scenario DD
   - $\sigma_1(t, x) = \hat{x}_1, \forall x$
   - $\sigma_2(t, x) = BR^U_2(\hat{x}_1), \forall x$

3. $R = V$
$2 \times 2$ games
Scenario DN

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<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-1</td>
<td>-5</td>
</tr>
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1. $t > t^*$: At the beginning of the revision phase players prepare the action forming the component game pure Nash equilibrium.
2. $t < t^*$: Once reached these actions they do not move.

\[ u(t) \text{ is in scenario DD} \]

\[ u(t) \text{ is in scenario DN} \]

\[ t^* \quad t = 0 \]
Proposition

If $U$ is in NN scenario, then there are $0 < t^{**} < t^*$, $i^*$ and $x_{i^*}$ such that:

1. For $t < t^{**}$:
   - $u(t)$ is in scenario NN
   - $\sigma_i(t, x) = BR_i^U(x_{-i})$

2. For $t^{**} < t < t^*$:
   - $u(t)$ is in scenario DN

3. For $t \geq t^*$:
   - $u(t)$ is in scenario DD

4. Generically

\[ R = u(t^{**}, x^*) \neq V \]
surplice and wrestle

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<thead>
<tr>
<th></th>
<th>0</th>
<th>0.2</th>
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<tbody>
<tr>
<td>0.3</td>
<td>−0.5</td>
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1. $t > t^*$: At the beginning of the revision phase players prepare a “surplace” action, that they keep until $t^*$
2. $t < t^*$: starting to $t^*$ players actions cycle

$u(t)$ is in scenario DD $u(t)$ is in scenario DN $u(t)$ is in scenario NN

$t^*$ $t^*$ $t = 0$
If $U$ is in NN scenario, then without loss of generality we have

$$\begin{array}{c|cc}
\alpha & \alpha & \beta \\
\hline
\alpha & 0 & b \\
\beta & c & b + c - 1 \\
\end{array}$$

with

$$0 < b, c, < 1$$
2 × 2 games
Scenario NN: Sur-place action

Let \( \sigma := |\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2|^{\frac{1}{2}} \) and let \( \hat{t}(A, B, \lambda_1, \lambda_2) \) be the smallest positive \( t \) such that

\[
e^{-\frac{\lambda_1+\lambda_2}{2}t} + A \cos \left( \frac{\sigma t}{2} \right) + \frac{(\lambda_2 - \lambda_1)A + 2\lambda_2 B}{\sigma} \sin \left( \frac{\sigma t}{2} \right) = 0 \quad (1)
\]

Set \( \hat{t}(A, B, \lambda_1, \lambda_2) \) to infinity. Let

\[
\begin{align*}
t_{\alpha,\alpha} &= \hat{t}(2c - 1, 2b - 1, \lambda_1, \lambda_1) \\
t_{\alpha,\beta} &= \hat{t}(2b - 1, 1 - 2c, \lambda_2, \lambda_1) \\
t_{\beta,\alpha} &= \hat{t}(1 - 2b, 2c - 1, \lambda_2, \lambda_1) \\
t_{\beta,\beta} &= \hat{t}(1 - 2c, 1 - 2b, \lambda_1, \lambda_1)
\end{align*}
\]

Then then

\[
\hat{x} = \arg \min_{y \in \{(\alpha,\alpha), (\alpha,\beta), (\beta,\alpha), (\beta,\beta)\}} t_y
\]

\[
t_\ast = t_{\hat{x}}
\]
$2 \times 2$ games

Scenario NN: sur-place actions for $q = 1/2$ and $0 < b, c < 1$

\[
\begin{array}{c|cc}
\alpha & \beta \\
\hline
\alpha & 0 & b \\
\beta & c & b + c - 1 \\
\end{array}
\]
2 × 2 games
Scenario NN: \( R \) and \( V \) for \( q = 1/2 \) and \( 0 < b, c < 1 \)

Theorem

If \( 0 < b, c < 1 \) and \( q = 1/2 \), then

- The value of the game is \( V = bc \)
- The revision game value is:

\[
R = \frac{1}{4}(2c + 2b - 1) + \frac{1}{2}(c + b - 1)(b - c) \sin(2\mu) \\
+ \frac{1}{4}(2b - 1)(2c - 1) \cos(2\mu),
\]

where \( \mu \) is the smallest \( t \) in \( \mathbb{R}_+ \) satisfying:

\[
e^{-t} = \max\{(1 - 2c) \cos(t) + (1 - 2b) \sin(t), (1 - 2b) \cos(t) - (1 - 2c) \sin(t), \\
-(1 - 2b) \cos(t) + (1 - 2c) \sin(t), -(1 - 2c) \cos(t) - (1 - 2b) \sin(t)\}.
\]
2 × 2 games
Scenario NN: \( R \) and \( V \) for \( q = \frac{1}{2} \) and \( 0 < b, c < 1 \)

\[
\begin{array}{c|cc}
\alpha & \alpha & \beta \\
\hline
\alpha & 0 & b \\
\beta & c & b + c - 1 \\
\end{array}
\]

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$2 \times 2$ games
Scenario NN: Sur-place action, $R$ and $V$

$q = 1/2$
A zero-sum revision game always has a pure strategy equilibrium.

When the component game Nash equilibrium is in pure, then players should be indifferent between paling the game with our without a (long) revision phase.

When the component game Nash equilibrium is not pure, then

- A player gain from being faster than the other player.
- Generically the revision game value is different from the one-shot game value.
- For $2 \times 2$ games, the unique equilibrium consists in players waiting on a sur-place action profile until the the deadline is close and then wrestle.