Strategic Departure Decisions and Correlation in Dynamic Congestion Games

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Strategic Departure Problem

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  - Utilizing a common fixed capacity network.
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- Arrival time $a^* = 0$, departure time $d_i \in \mathbb{Z}_-$. 

Assume without loss $r_i(d_i) = -d_i$. 

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- Uniform Random Priority: Ties are broken uniformly.
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- Cost of player $i \in I = \{1, \ldots, n\}$ under pure profile $d$:
  \[
  R_i(d_i, d_{-i}) = r_i(d_i) + \mathbb{1}_{a_i > 0} \cdot f(a_i, C)
  \]
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- Examples:
  - $f(a_i, C) = a_i \cdot C$; pay $C$ for each period you are late.
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  - $f(a_i, C) = a_i \cdot C$; pay $C$ for each period you are late.
  - $f(a_i, C) = a_i + C$; pay $C$ once, pay for additional periods in transit.
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- We assume $f(a_i, C) = C$, $C$ large, and obtain results robust to model selection.
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- **IMPORTANT**: We consider fixed $n$ and assume $C$ is large w.r.t. $n$. 
A 3-player Example

Departures

\[-3 \quad -2 \quad -1 \quad 0\]

\[d\]

\[P_1\]

\[P_2, P_3\]

Each player is late with probability 1/3, thus \[R_1 = R_2 = R_3 = 2 + \frac{1}{3} \cdot C\]
A 3-player Example

Departures

-3  -2  -1  0

$P_1$  $P_2, P_3$

No player is ever late $\Rightarrow R_1 = 3, R_2 = R_3 = 2$. Each player is late with prob $\frac{1}{3}$ $\Rightarrow R_1 = R_2 = R_3 = 2 + \frac{1}{3} \cdot C$. 
A 3-player Example

Departures

\[ -3 \quad -2 \quad -1 \quad 0 \]

\[ d \quad P1 \quad P2, P3 \quad \text{Prob } \frac{1}{2} \]
A 3-player Example

Departures

-3  P1 exit
-2  P2 exit
-1  P3 exit
0  Prob \( \frac{1}{2} \)

Each player is late with prob \( \frac{1}{3} \) = \( R_1 = R_2 = R_3 = 2 + \frac{1}{3} \cdot C \)
A 3-player Example

Departures

\[ \begin{array}{c}
-3 & -2 & -1 & 0 \\
\end{array} \]

\[ \begin{array}{c}
P1 exit \\
\end{array} \]

\[ \begin{array}{c}
P2, P3 \\
\end{array} \]

\[ d \]

\[ \begin{array}{c}
P1 \\
\end{array} \]

\[ \text{No player is ever late} = R_1 = 3, R_2 = R_3 = 2. \]

\[ \text{Each player is late with prob} = \frac{1}{3} = R_1 = R_2 = R_3 = 2 + \frac{1}{3} \times C \]
A 3-player Example

Departures

\[ d \]

\[ P_1 \]
\[ P_2, P_3 \]
\[ \text{Prob } \frac{1}{2} \]

No player is ever late implies
\[ R_1 = 3, \quad R_2 = R_3 = 2. \]
A 3-player Example

Departures

\[ d \]

\[ P_1 \]

\[ P_2, P_3 \]

\[ P_2 \text{ exit} \]

\[ P_3 \text{ exit} \]

\[ P_1 \text{ exit} \]

\[ \text{Prob} \ \frac{1}{2} \]

\[ -3 \]

\[ -2 \]

\[ -1 \]

\[ 0 \]
A 3-player Example

Departures

\[ \begin{array}{|c|c|c|c|}
\hline
\text{Departures} & -3 & -2 & -1 & 0 \\
\hline
\text{d} & P_1 & P_2, P_3 & & \\
\hline
\text{P1 exit} & & & P_3 exit & \\
\hline
\text{P2 exit} & & & & \\
\hline
\end{array} \]

\[ \text{Prob} \frac{1}{2} \]

- No player is ever late \( \implies R_1 = 3, R_2 = R_3 = 2. \)
A 3-player Example

\[ \begin{array}{cccccc}
-3 & -2 & -1 & 0 \\
\hline
D & P1 & P2, P3 & P2 exit & P3 exit & \text{Prob } \frac{1}{2} \\
\end{array} \]

\[ \begin{align*}
\text{No player is ever late} \implies R_1 &= 3, \quad R_2 = R_3 = 2. \\
\end{align*} \]
A 3-player Example

-3 -2 -1 0

Departures

\[ d \]

-3 -2 -1 0

\[ d \]

\[ P1 \]

\[ P2, P3 \]

\[ P2 \text{ exit} \]

\[ P3 \text{ exit} \]

\[ \text{Prob } \frac{1}{2} \]

\[ \text{Prob } \frac{1}{6} \]

\[ P1 \text{ exit} \]

\[ R_1 = 3, \ R_2 = R_3 = 2. \]
A 3-player Example

- Departures
  - $d$
  - $P_1, P_2, P_3$

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  - $P_1, P_2, P_3$

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$\begin{align*}
\text{Prob } \frac{1}{2} \\
\end{align*}$

$\begin{align*}
\text{Prob } \frac{1}{6} \\
\end{align*}$
A 3-player Example

\begin{align*}
\text{Departures} & \quad \quad -3 \quad \quad -2 \quad \quad -1 \quad \quad 0 \\
\quad d & \quad \quad P1 \quad \quad P2, P3 \quad \quad P2 \quad \quad \text{Prob } \frac{1}{2}
\end{align*}

\textbf{•} No player is ever late \quad \implies \quad R_1 = 3, \quad R_2 = R_3 = 2.

\begin{align*}
\text{Departures} & \quad \quad -3 \quad \quad -2 \quad \quad -1 \quad \quad 0 \\
\quad d & \quad \quad P1, P2, P3 \quad \quad P2 \quad \quad \text{Prob } \frac{1}{6}
\end{align*}
A 3-player Example

\[ d \]

Departures

\[ \begin{array}{c}
\text{Departures} \\
\text{Prob} \frac{1}{2}
\end{array} \]

\[ P1, P2, P3 \]

\[ P1 \text{ exit} \]

\[ P2 \text{ exit} \]

\[ P3 \text{ exit} \]

\[ R_1 = 3, \ R_2 = R_3 = 2. \]

\[\begin{array}{l}
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$\text{Departures}$

$\begin{align*}
  &-3 & -2 & -1 & 0 \\
  &P_1 & P_2, P_3 & \text{P2 exit} & \text{P3 exit} \\
  &d & & & \\
\end{align*}$

$\text{Prob} \frac{1}{2}$

$\text{Departures}$

$\begin{align*}
  &-3 & -2 & -1 & 0 \\
  &P_1, P_2, P_3 & \text{P3 exit} & \text{P2 exit} & \\
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\end{align*}$

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A 3-player Example

-3 −2 −1 0

Departures

<table>
<thead>
<tr>
<th>Departures -3</th>
<th>0</th>
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<td>d P1</td>
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⇒ No player is ever late

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⇒ Each player is late with prob \( \frac{1}{3} \)

\[ R_1 = R_2 = R_3 = 2 + \frac{1}{3} \cdot C \]

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Departures & \\
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\end{align*}$

$\begin{align*}
d & P1 \quad P2, P3 \quad P2 exit \\
\end{align*}$

$\begin{align*}
\text{Prob } & \frac{1}{2} \\
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- Interested in best/worst Nash Equilibrium payoffs.
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- $\sigma^{opt}$; one player departs at each time $t \in \{-n, -(n-1), \ldots, -1\}$. 

$\sigma^{opt}$ is not a Nash Equilibrium.

Deviation: $P_1$ can deviation to time $-2$.

$P_3$ is late, but $P_1$ and $P_2$ are not.
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\begin{array}{cccc}
-3 & -2 & -1 & 0 \\
\hline
\text{Departures} & & & \\
\text{$\sigma^{opt}$} & P_1 & P_2 & P_3
\end{array}
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\textit{Departures}

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\sigma^{opt} \\
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Departures

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$P1 \rightarrow P2 \rightarrow P3$

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- $P3$ is late, but $P1$ and $P2$ are not.
Nash Equilibrium Results

- We present our results for the class of games with $C > n$. 

Result 1: There are no pure Nash Equilibrium.

Result 2: For all $C > n$, the worst NE payoff is obtained by $\sigma_{\text{wst}}$ with $\text{supp}(\sigma_{\text{wst}}_i) = \{-n, -(n-1)\}$ for all $i \in I$.

$\sigma_{\text{wst}}$ is characterized by the symmetric strategy:

$$
\sigma_{\text{wst}}_i(n) = 1 - \frac{(nC)}{n-1} \quad \sigma_{\text{wst}}_i(n-1) = \frac{(nC)}{n-1}
$$

(Sketch of Proof):

Time $-n$ is a safe time so $R_i(\sigma) \leq n$ in any equilibrium $\sigma$.

$\sigma_{\text{wst}}$ is a NE that gives each player a payoff of exactly $n$. 

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- $\sigma^{wst}$ is characterized by the symmetric strategy:

  $$\sigma^{wst}_i(n) = 1 - \left(\frac{n}{C}\right)^{\frac{1}{n-1}} \quad \sigma^{wst}_i(n-1) = \left(\frac{n}{C}\right)^{\frac{1}{n-1}}$$
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The Price of Anarchy

- The social planner wants to minimize the sum of equilibrium payoffs.

- Define sum of costs as $SC(\sigma) := \sum_{i \in I} R_i(\sigma)$.

$$SC(\sigma^{opt}) = \frac{n(n + 1)}{2} \quad SC(\sigma^{wst}) = n^2$$
The Price of Anarchy

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- Corollary [The Price of Anarchy]:

$$PoA := \frac{SC(\sigma^{wst})}{SC(\sigma^{opt})} = 2 - \frac{2}{n + 1}$$
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- Corollary [The Price of Anarchy]:

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- Conclusion: The worst equilibrium costs are roughly twice the optimum.
Result 3: There exists $\bar{C} \in (n, n^2)$ such that for all $C > \bar{C}$ the best equilibrium payoffs are obtained by $\sigma^{bst}$ with

$$\text{supp} (\sigma^{bst}_i) = \{-n\} \text{ and } \text{supp} (\sigma^{bst}_j) = \{- (n-1), - (n-2)\} \text{ for all } j \neq i.$$
The Price of Stability

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- (Sketch of Proof):
The Price of Stability

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(Sketch of Proof):

- If no player departs at time $-n$ then at least one player is late for sure.
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▶ (Sketch of Proof):

▶ If no player departs at time $-n$ then at least one player is late for sure.

▶ As $C \rightarrow \infty$ the risk of being late becomes too large so there is a deviation to $-n$.
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\]

- (Sketch of Proof):

- If no player departs at time $-n$ then at least one player is late for sure.

- As $C \to \infty$ the risk of being late becomes too large so there is a deviation to $-n$.

- Hence for large $C$, there exists $i \in I$ such that $-n \in \text{supp} (\sigma_i^{bst})$. 
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(Sketch of Proof):

- If no player departs at time $-n$ then at least one player is late for sure.

- As $C \to \infty$ the risk of being late becomes too large so there is a deviation to $-n$.

- Hence for large $C$, there exists $i \in I$ such that $-n \in \text{supp} (\sigma_i^{bst})$.

- But then, at least $n-1$ players must mix over time $-(n-1)$. 
The Price of Stability

▶ Corollary [Price of Stability]: There exists $\bar{C} \in (n, n^2]$ such that for all $C > \bar{C}$

$$PoS := \frac{SC(\sigma^{bst})}{SC(\sigma^{opt})} = \frac{n + (n - 1)^2}{\frac{n(n+1)}{2}} = 2 + \frac{2}{n(n+1)} - \frac{4}{n+1}$$
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▶ **Corollary [Price of Stability]:** There exists $\bar{C} \in (n, n^2]$ such that for all $C > \bar{C}$

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▶ **Conclusion:** The best Nash equilibrium cost is also roughly twice the social optimum.
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- **Conclusion:** The best Nash equilibrium cost is also roughly twice the social optimum.

- **Question:** Is there any way to coordinate the players actions to obtain an outcome closer to the social optimum?
Correlated Equilibrium Example

- The planner draws an outcome $s \sim Q \in \Delta(S)$ and tells each player to play $s_i$. 
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- If playing $s_i$ is optimal for each $i \in I$ and $s_i$ in the support of $Q$

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  - Thomas J. Rivera, Marco Scarsini, Tristan Tomala
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- Then \( Q \) is a correlated equilibrium.
Example: 4 players, \( C=20 \).

\[
\begin{array}{cccc}
-4 & -3 & -2 & -1 \\
\end{array}
\]

\textit{Departures}

\( s' := P1 \quad P2 \quad P3 \quad P4 \)

\( s'' := P1 \quad P2 \quad P3, P4 \)

\( s''' := P1 \quad P2, P3, P4 \)

\( \star \quad \star \quad \star \quad \star \)

\( Q(\star) = \frac{59}{100} \)

\( Q(\star) = \frac{21}{100} \)

\( Q(\star) = \frac{20}{100} \)

\( \rightarrow \text{Claim: No deviation by } P1 \text{ to time } -3. \)

\( R_1(Q(\star)) = 4 \leq 3 + 20 \cdot \frac{1}{100} \cdot C = 4 = R_1(-3, Q(\star)) \)

\( Q(\star) \) is a CE that yields the best SC:

\( SC(\sigma_{bst}) = 13 \)

\( SC(\sigma_{opt}) = 10 \)

\( SC(Q(\star)) = 10 \)
Example: 4 players, C=20.

\[
\begin{array}{c}
\text{Departures} \\
\hline
-4 & -3 & -2 & -1 \\
Q^*(s') = \frac{59}{100} & s' := P1 & P2 & P3 & P4 \\
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<table>
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Claim: No deviation by $P1$ to time $-3$. 

$R_1(Q^*) = 4 \leq 3 + \frac{20}{100} \cdot C = 4 = R_1(−3, Q^*)$

$Q^*$ is a CE that yields the best SC:

$SC(σ_{bst}) = 13$ \quad $SC(σ_{opt}) = 10$ \quad $SC(Q^*) = 10$. 

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\begin{array}{cccccc}
-4 & -3 & -2 & -1 & \\
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\textit{Departures}\hspace{1cm} |

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Q^*(s') = \frac{59}{100} \quad s' := \quad P1 \rightarrow P2 \quad P3 \quad P4
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\[
R_1(Q^*) = 4
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\[
R_1(Q^*) = 4 \leq 3 + \frac{20}{100} \cdot \frac{C}{4}
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▶ \(Q^*)\) is a CE that yields the best SC:

\[
SC(\sigma^{bst}) = 13 \quad SC(\sigma^{opt}) = 10 \quad SC(Q^*) = 10.81
\]
Characterizing Best Correlated Equilibrium

- $S = \mathbb{Z}^n_-$, we look for CE $Q \in \Delta(S)$ that minimize

$$SC(Q) := \sum_{s \in S} Q(s)SC(s)$$

- Only interested in $Q \in \Delta(S^Y)$: set of outcomes where no player is late.

- Enforcing strategies: $s \in S$ enforces time $k$ for player $i$ if when $i$ is told to depart at time $k$, she is late with positive probability when departing at time $k - 1$ instead, when others play $s_{-i}$.

- $Z^{i,k}$ set of strategies that enforce $k$ for player $i$.

- $S^{i,k} = \{s \in S : s_i = k\}$. 
Lemma: $Q \in \Delta(S^Y)$ is a correlated equilibrium of SD game with penalty $C$ if and only if for all $i \in I$

$$\sum_{s \in Z^{i,k}} Q(s) \geq \frac{k}{C} \left[ \sum_{s \in S^{i,k}} Q(s) \right]$$

for $k = 2, \ldots, n$

Proof:
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- $s \in Z^{i,k}$ means exactly $k - 1$ other players depart at time $-(k - 1)$.
- Hence, if the outcomes is $s$ and player $i$ departs instead at $-(k - 1)$ he is late with probability $\frac{1}{k}$.
- So player $i$, being told to depart at $-k$ does not want to deviate to $-(k - 1)$ only if

\[k \leq k - 1 + \mathbb{P}(s \in Z^{i,k} | s_i = -k) \cdot \frac{C}{k}\]
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and

$$\mathbb{P}(s \in Z^{i,k} | s_i = -k) = \frac{\sum_{s \in Z^{i,k}} Q(s)}{\sum_{s \in S^{i,k}} Q(s)}$$
From Strategies to Outcomes

\[ s = (4, 3, 3, 3) \rightarrow y^s = (1, 3, 0, 0) \]

- Working with strategies is difficult so we switch to distributions \( Q^o \in \Delta(Y) \). Y outcomes where no one is late.
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  \[ S(y) = \{(4, 3, 3, 3), (3, 4, 3, 3), (3, 3, 4, 3), (3, 3, 3, 4)\} \]
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- We show it is without loss to restrict attention to distributions over outcomes with this implementation.
A Best Correlated Equilibrium

Let \( y^k = (1, \ldots, 1, k - 1, 0, \ldots, 0) \). \( y^2 \) is the socially optimal outcome.
A Best Correlated Equilibrium

- Let $y^k = (1, ..., 1, k - 1, 0, ..., 0)$. $y^2$ is the socially optimal outcome.

**Theorem:** There exists $\bar{C}$ such that for all $C > \bar{C}$, the best correlated equilibrium payoff is generated by $Q^* \in \Delta(S^Y)$:

$$Q^*(s) = \frac{1}{|S(y^s)|} \hat{Q}^o(y^s)$$

and $\hat{Q}^o(y) \in \Delta(Y)$ satisfies

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$$\hat{Q}^o(y^2) = 1 - \sum_{j=3}^{n} \hat{Q}^o(y^j)$$
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**Corollary:** As $C \to \infty$, $Q^*(\sigma^{opt}) \to 1$. 
A Mechanism For Implementing The Social Optimum With Arbitrary Probability

Consider the following *toll pricing mechanism* $M_\tau$: Any player exiting the road after time 0 pays a large toll of $\tau$. 

**Corollary:** For every $\epsilon > 0$ there exists $\tau > 0$ such that $Q^\star$ is implementable with the mechanism $M_\tau$ and $Q^\star(\sigma_{\text{opt}}) = 1 - \epsilon$.

**Proof:** $M_\tau$ effectively increases $C \rightarrow C + \tau$.
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$$\text{CPoS} := \frac{SC(Q^*)}{SC(\sigma^{opt})} = 1 + \delta(C)$$

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Small $C$ and Model Robustness

- Example: 3 players, $0 \leq C \leq 3$: Unique Nash Equilibrium $\sigma^{NE}$
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$$C \leq 1 \quad \implies \quad \text{supp}(\sigma^{NE}) = \{0\}$$
Small $C$ and Model Robustness

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1 < C \leq 2 & \implies \text{supp}(\sigma^{NE}) = \{1, 0\} \\
C = 3 & \implies \text{supp}(\sigma^{NE}) = \{2\} \\
\end{align*}
\]

As $C$ varies the equilibrium support varies. Exacerbated if $f(a_i, C) \neq C$.

Corollary: There exists $\bar{C} \in \mathbb{R}$ such that for all $C > \bar{C}$ our results regarding the PoA, PoS, and CPoS are robust to changes in $C$ and to the specification of $f(a_i, C)$.
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Thank you!

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