Imitation dynamics and dominated strategies

Ovvero: Does learning by imitation eliminate irrational behaviors?

Yannick Viossat (Université Paris-Dauphine)
Panayotis Mertikopoulos (CNRS, Université de Grenoble)
Bill Sandholm (Wisconsin University)

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Evolutionary game theory

Evolution of behaviour in population of weakly rational agents interacting

Strategies with currently good payoffs spread

Specification of this process: evolutionary dynamics

Central topic: link between outcome of dynamics and static concepts?

Today: elimination of pure strategies dominated by other pure strategies

Two big classes: *imitative* and *innovative* dynamics

*Imitative dynamics eliminate pure strategies dominated by other pure strategies; innovative dynamics need not.*

Claim: misleading picture. Studied imitative dynamics are special.

Dynamics based on imitation need not eliminate dominated strategies
Outline

- Imitative and innovative dynamics
- Innovative dynamics favour rare strategies
- Imitation dynamics favouring rare/frequent strategies
- Survival of dominated strategies under imitation dynamics
Framework 1: Single population dynamics

Interactions within a single, large population

- finite set of pure strategies \( I := \{1, \ldots, N \} \).
- \( x_i(t) \): frequency of strategy \( i \) at time \( t \)
- \( x(t) := (x_i(t))_{i \in I} \): state of the population
- evolves in \( S_N = \{ x \in \mathbb{R}_+^N, \sum_{i \in I} x_i = 1 \} \)
- Payoff for \( i \)-strategists: \( u_i(x(t)) \)
- Dynamics: \( \dot{x} = f(x, \text{payoffs}) \)
Agents from focal population interact against unspecified opponent.

At time $t$, opponent plays $y(t) \in S_{opp}$, with $S_{opp}$ compact.

Payoffs of $i$-strategist in focal population: $u_i(y)$

Dynamics: $\dot{x} = f(x, y, \text{payoffs})$

No assumption on evolution of $y(\cdot)$, except regular enough.

Single-population dynamics correspond to $S_{opp} = S_N$ and $y(t) = x(t)$. 
Domination and extinction

Strategy $i$ dominated by $j$ if: $\forall y \in S_{opp}, u_i(y) < u_j(y)$

Pure strategy $i$ goes extinct if $x_i(t) \to 0$ as $t \to +\infty$

Issue: do dominated strategies go extinct?
Replicator dynamics

\[ \dot{x}_i = x_i (u_i - \bar{u}) \quad \text{with} \quad \bar{u} = \sum_i x_i u_i \]

Introduced in biology (78), but later reinterpreted as imitation model.

Idea: assume \( i \)-strategists switch to strategy \( j \) at rate \( \rho_{ij}(x, \text{payoffs}) \)

\[ \leftrightarrow \quad \dot{x}_i = \sum_j x_j \rho_{ji} - x_i \sum_j \rho_{ij} \]

Several specifications of the \( \rho_{ij} \) based on imitation lead to REP:

\[ \rho_{ij} = x_j (K + u_j) \quad \text{(imitation of success)} \]

\[ \rho_{ij} = x_j [u_j - u_i]_+ \quad \text{(proportional pairwise imitation rule)} \]
“Revision protocol” specifies rate $\rho_{ij}$ at which $i$-strategists switch to $j$.

Imitative dynamics (Sandholm, 10): $\rho_{ij} = x_j r_{ij}$ with $u_i < u_j \iff r_{ij} > r_{ji}$

Models two step process:
Step 1: revising $i$-strategist meets $j$-strategist with probability $x_j$
Step 2: imitate him with “probability” $r_{ij}$ favouring successful strategies

Coincide with Nachbar’s (90) monotone dynamics: $\dot{x}_i = x_i (g_i - \bar{g})$
with $g_i = g_i(x, \text{payoffs})$, $\bar{g} = \sum_{i \in I} x_i g_i$, and $g_i < g_j \iff u_i < u_j \quad \forall i, j, x$
Theorem (Akin 1980, Nachbar, 1990)

Assume strategy i strictly dominated by strategy j. Then under any imitative dynamics, $x_i(t) \to 0$ as $t \to +\infty$.

Proof.

Simply use: $u_i < u_j \Rightarrow g_i < g_j \Rightarrow \frac{\dot{x}_i}{x_i} < \frac{\dot{x}_j}{x_j}$
In innovative dynamics, strategies initially not played may appear.

**Smith dynamics**: revising $i$-strategists pick a strategy $j$ at random, and adopt it with probability proportional to $[u_j - u_i]^+$. So $\rho_{ij} = \frac{1}{N}[u_j - u_i]^+$

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**Theorem (Hofbauer-Sandholm, 2011)**

Under Smith and all innovative dynamics satisfying 4 natural conditions (Positive correlation, Continuity, Innovation, Nash stationarity), pure strategies dominated by other pure strategies may survive!

Hofbauer and Sandholm use two 4 strategy games:
- Rock-Paper-Scissors + feeble Twin
- Hypnodisk game + feeble Twin
A simpler example for games against the environment

Consider a $3 \times 2$ game:

\[
\begin{pmatrix}
L & R \\
A & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
B & \begin{pmatrix} -\alpha & 1 - \alpha \end{pmatrix}
\end{pmatrix}
\]

Consider dynamics satisfying the following conditions:

- **Continuity**: $\dot{x}$ depends continuously on $x, y, \text{payoffs}$.
- **Innovation**: if $i$ is an unplayed best-reply to $y$, but not $x$, then $\dot{x}_i > 0$.
- **Positive correlation**: if $x$ not a best-reply to $y$, then $\sum_i \dot{x}_i u_i > 0$.

**Theorem**

$\forall \alpha$ small enough, $\exists y : \mathbb{R} \to S_{opp}$ such that strategy $T$ survives.
Innovative dynamics favour rare strategies

**Innovation:** if \( i \) unused best-reply, then \( \dot{x}_i > 0 \). Hence \( \dot{x}_i / x_i = +\infty \).

So by **Continuity:** if \( i \) almost best-reply and \( x_i << 1 \), then \( \dot{x}_i / x_i \) huge.

Thus, if \( x_i << 1 \), we may have: \( u_i < u_j \) but \( \frac{\dot{x}_i}{x_i} > \frac{\dot{x}_j}{x_j} \)

\( \leftrightarrow \) favours rare strategies.

**Imitative dynamics neutral:** \( u_i < u_j \Rightarrow \frac{\dot{x}_i}{x_i} < \frac{\dot{x}_j}{x_j} \) whatever \( x_i, x_j > 0 \).

But imitation dynamics might favour rare/frequent strategies ; then same survival results should hold.
Imitation protocol favouring rare/frequent strategies

Step 1:
1a) a revising agent meets 3 randomly drawn agents, e.g, (j, k, k)
1b) make a list of strategies played by these agents; here: {j, k}
1c) pick one at random: here j with probability 1/2

Step 2: decides whether to imitate him according to standard $r_{ij}$.

Leads to: $\rho_{ij} = p_{j}(x)r_{ij}$ where $p_{j}(x)$ proba of picking j in step 1.

Step 1 favours rare strategies: $x_{i} < x_{j} \Rightarrow p_{i}/x_{i} > p_{j}/x_{j}$.

If instead, when meeting (j, k, k), agent 1 focuses on the “majoritarian choice” k, favours frequent strategies: $x_{i} < x_{j} \Rightarrow p_{i}/x_{i} < p_{j}/x_{j}$.
Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare strategies.
- in step 2, \( r_{ij} = K + u_j \), or \( r_{ij} = f(u_j) \), with \( f \) positive increasing.

**Theorem**

*There are two-strategy games with a strictly dominated strategy that survives in proportion almost 1/2 for most initial conditions.*

With advantage to frequent strategies, survival in proportion almost 1!
Proof (advantage to rare strategies)

Dynamics:

\[ \dot{x}_i = \sum_j x_j p_i f(u_i) - x_i \sum_j p_j f(u_j) \]

Let \( p_i = x_i (1 + \varepsilon_i) \). For two strategies \( i \) and \( j \) with same payoff \( u \):

\[ \frac{\dot{x}_i}{x_i} - \frac{\dot{x}_j}{x_j} = (\varepsilon_i - \varepsilon_j) f(u) \]

With only these strategies: since \( x_i < x_j \Rightarrow \varepsilon_i > \varepsilon_j, \ x_i \rightarrow 1/2 \)

Now perturb: assume payoff of \( x_i \) is \( u - \alpha \) so \( i \) dominated

We get: \( \forall \eta > 0, \ \exists \bar{\alpha} > 0, \ \forall \alpha < \bar{\alpha}, \ x_i > \eta \Rightarrow \lim \inf x_i > \frac{1}{2} - \eta \).

With advantage to frequent strategies: \( x_i > 1/2 + \eta \Rightarrow \lim \inf x_i > 1 - \eta. \)
Other dynamics?

Distorted imitation of success does not satisfy Positive Correlation, has non Nash interior rest-points, and not all Nash equilibria are rest-points!

Can we have similar results for dynamics with more usual properties?

↪ Yes, but requires more elaborate examples.
Other dynamics?

Distorted imitation of success does not satisfy Positive Correlation, has non Nash interior rest-points, and not all Nash equilibria are rest-points!

Can we have similar results for dynamics with more usual properties?

→ Yes, but requires more elaborate examples.
Consider again $3 \times 2$ game:

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T & \begin{pmatrix}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
$$

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare strategies.
- in step 2, $r_{ij} = [u_j - u_i]_+$, or same sign.

**Theorem**

$$
\forall \eta > 0, \exists \alpha > 0, \forall \alpha < \alpha, \exists y(\cdot), \forall x(0) \in \text{int}(S_3), \lim \inf x_T > \frac{1}{2} - \eta
$$

Same results if $r_{ij} = [u_j - \bar{u}]_+$, or same sign.

Favour frequent strategies: $x_T(0) > x_B(0) + \eta \Rightarrow \lim \inf x_T > 1 - \eta$
Result 3 - Single population dynamics

Consider dynamics derived from imitation revision protocol such that:
- step 1 favours rare (resp. frequent) strategies.
- in step 2, \( r_{ij} = [u_j - u_i]_+ \), or same sign.

**Theorem**

*There are 4 strategy games such that for large sets of initial conditions, a pure strategy dominated by another pure strategy survives in proportion roughly 1/6 (resp. 1/3).*

Same results if \( r_{ij} = [u_j - \bar{u}] \), or same sign.

Proportion may be increased to \( 1/2 - \eta \) (resp. \( 1 - \eta \)).
We mimick Hofbauer and Sandholm (2011).

They consider dynamics satisfying Positive Correlation (PC):

\[ \dot{x} \neq 0 \Rightarrow \dot{x} \cdot u(x) > 0. \]

Geometrically: acute angle between \( \dot{x} \) and payoff vector \( u(x) \).

Also: acute angle between \( \dot{x} \) and projection of payoff vector on simplex.

**Lemma**

*Under theorem’s assumptions, our imitation dynamics satisfy (PC)*
Hypnodisk game (Hofbauer and Sandholm)

3-strategy game with projected payoff vector field:

Due to (PC), all interior solutions enter annulus, except Nash equilibrium.
Hypnodisk game with a twin (Hofbauer and Sandholm)

Add as strategy 4 a twin of strategy 3.

- Segment of equilibrium: $x_1 = x_2 = x_3 + x_4 = 1/3$.

- Attracting annulus becomes attracting "intercylinder zone"
Effect of advantage to rare strategies

Advantage to rare strategies: \( \frac{x_3}{x_4} \rightarrow 1 \).

Attractor \( A \) in intersection of intercylinder zone and plane \( x_3 = x_4 \).

Basin of attraction \( B(A) = \text{int}(S_4) \setminus \text{Nash equilibria} \)
Subtract $\varepsilon$ to strategy 4 $\rightarrow$ makes it dominated

By standard results on continuation of attractors, for $\varepsilon$ small enough, most solutions still converge to an attractor $A_\varepsilon$ in the neighborhood of $A$.

$\hookrightarrow$ under most solutions, strategy 4 survives, and $\liminf x_4 \geq 1/6 - r$, with $r$ radius of outer cylinder

Rk: $1/6$ may be changed to anything $< 1/2$ by modifying base game.
Conjecture for Josef

Done: survival of dominated strategies under imitation dynamics for

- games against the environment: symmetric bimatrix games, many dynamics;
- single-population dynamics: specific dynamics, or many dynamics but with hypnodisk game.

Conjecture

Similar results for many single-population dynamics in Hofbauer and Sandholm’s Rock-Paper-Scissors-feeble Twin game

Problem: prove instability of segment of Nash equilibria.
Elimination of dominated strategies "requires":

$$\forall x, u_i = u_j \Rightarrow \dot{x}_i/x_i = \dot{x}_j/x_j.$$ 

Fragile property, destroyed by appropriate small perturbation.

May be a pinch of innovation, or a twist in imitation process.
When evolutionary game theorists imported replicator dynamics in economics, they justified it through an imitation model.

They probably did not think to imitation models ex-nihilo.

Doing so leads to different kinds of imitation dynamics, which need not eliminate pure strategies dominated by other pure strategies.

Dichotomy innovative/imitative should be supplemented by "treat strategies differently as function of their frequencies or not".
Literature (non exhaustive)

- Akin (80): replicator dynamics; Nachbar (90): monotone dynamics
- Samuelson & Zhang (92): aggregate monotone dynamics
- Hofbauer & Weibull (96): convex monotone dynamics
- Dekel & Scotchmer (92), Cabrales & Sobel (92), Björnerstedt et al (96): discrete-time dynamics

More recently:
- Cressman & Hofbauer (05); Cressman et al (06); Heifetz et al (07a, 07b), Jouini et al. (13): continuum of pure strategies
- Fudenberg & Harris (92), Cabrales (00), Imhof (05), Hofbauer & Imhof (09); Mertikopoulos & Moustakas (10), Mertikopoulos & Viossat (15): stochastic dynamics
- Berger & Hofbauer (06), Hofbauer & Sandholm (11): innovative dynamics