Imitation dynamics and dominated strategies

Ovvero : **Does learning by imitation eliminate irrational behaviors?**

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Evolutionary game theory

Evolution of behaviour in population of weakly rational agents interacting

Strategies with currently good payoffs spread

Specification of this process: evolutionary dynamics

Central topic: link between outcome of dynamics and static concepts?

Today: elimination of pure strategies dominated by other pure strategies

Literature's big picture

Incredibly good survey: Viossat (2015, Bulletin Economic Theory) !

Two big classes: *imitative* and *innovative* dynamics

Imitative dynamics eliminate pure strategies dominated by other pure strategies; innovative dynamics need not.

Claim: misleading picture. Studied imitative dynamics are special.

Dynamics based on imitation need not eliminate dominated strategies

Outline

- Imitative and innovative dynamics
- Innovative dynamics favour rare strategies
- Imitation dynamics favouring rare/frequent strategies
- Survival of dominated strategies under imitation dynamics

Framework 1: Single population dynamics

Interactions within a single, large population

- finite set of pure strategies $I := \{1, ..., N\}$.
- $x_i(t)$: frequency of strategy i at time t
- $\mathbf{x}(t) := (x_i(t))_{i \in I}$: state of the population
- evolves in $S_N = \left\{\mathbf{x} \in \mathbb{R}_+^N, \sum_{i \in I} x_i = 1\right\}$
- Payoff for i-strategists : $u_i(\mathbf{x}(t))$
- Dynamics: $\dot{\mathbf{x}} = f(\mathbf{x}, \textit{payoffs})$

Framework 2 : Games against the environment

Agents from focal population interact against unspecified opponent

At time t, opponent plays $\mathbf{y}(t) \in S_{opp}$, with S_{opp} compact.

Payoffs of i-strategist in focal population: $u_i(\mathbf{y})$

Dynamics: $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{y}, \textit{payoffs})$

No assumption on evolution of $\mathbf{y}(\cdot)$, except regular enough.

Single-population dynamics correspond to $S_{opp} = S_N$ and $\mathbf{y}(t) = \mathbf{x}(t)$.

Domination and extinction

Strategy i dominated by j if: $\forall \mathbf{y} \in S_{opp}, u_i(\mathbf{y}) < u_j(\mathbf{y})$

Pure strategy i goes extinct if $x_i(t) \to 0$ as $t \to +\infty$

Issue: do dominated strategies go extinct?

Replicator dynamics

$$\dot{x}_i = x_i\,(u_i - \overline{u}) \qquad \text{with } \overline{u} = \sum_i x_i u_i$$

Introduced in biology (78), but later reinterpreted as imitation model.

Idea: assume i-strategists switch to strategy j at rate $\rho_{ij}(\mathbf{x}, \textit{payoffs})$

$$\hookrightarrow \quad \dot{x}_i = \sum_j x_j \rho_{ji} - x_i \sum_j \rho_{ij}$$

Several specifications of the ρ_{ij} based on imitation lead to REP:

$\rho_{ij} = x_j(K + u_j)$ (imitation of success)

$\rho_{ij} = x_j[u_j - u_i]_+$ (proportional pairwise imitation rule)

Imitative dynamics

"Revision protocol" specifies rate ρ_{ij} at which i-strategists switch to j.

Imitative dynamics (Sandholm, 10): $\rho_{ij} = x_j r_{ij}$ with $u_i < u_j \Leftrightarrow r_{ij} > r_{ji}$

Models two step process:

Step 1: revising i-strategist meets j-strategist with probability x_j

Step 2: imitate him with "probability" r_{ij} favouring successful strategies

Coincide with Nachbar's (90) monotone dynamics: $\dot{x}_i = x_i \left(g_i - \overline{g}\right)$

with $g_i = g_i(\mathbf{x}, payoffs)$, $\overline{g} = \sum_{i \in I} x_i g_i$, and $g_i < g_j \Leftrightarrow u_i < u_j \quad \forall i, j, \mathbf{x}$

Imitative dynamics and dominated strategies

Theorem (Akin 1980, Nachbar, 1990)

Assume strategy i strictly dominated by strategy j. Then under any imitative dynamics, $x_i(t) \to 0$ as $t \to +\infty$.

Proof.

Simply use: $u_i < u_j \Rightarrow g_i < g_j \Rightarrow \frac{\dot{x}_i}{x_i} < \frac{\dot{x}_j}{x_j}$ $\square$

Innovative dynamics

In innovative dynamics, strategies initially not played may appear.

Smith dynamics: revising i-strategists pick a strategy j at random, and adopt it with probability proportional to $[u_j - u_i]_+$. So $\rho_{ij} = \frac{1}{N}[u_j - u_i]_+$

Theorem (Hofbauer-Sandholm, 2011)

Under Smith and all innovative dynamics satisfying 4 natural conditions (Positive correlation, Continuity, Innovation, Nash stationarity), pure strategies dominated by other pure strategies may survive!

Hofbauer and Sandholm use two 4 strategy games:

- Rock-Paper-Scissors + feeble Twin
- Hypnodisk game + feeble Twin

A simpler example for games against the environment

Consider 3×2 game:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} A \\ B \\ T \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\alpha & 1-\alpha \end{pmatrix} \end{array}$$

Consider dynamics satisfying the following conditions:

- Continuity: $\dot{\mathbf{x}}$ depends continuously on $\mathbf{x}, \mathbf{y}$, *payoffs*.
- Innovation: if i is an unplayed best-reply to $\mathbf{y}$, but not $\mathbf{x}$, then $\dot{x}_i > 0$
- Positive correlation: if $\mathbf{x}$ not a best-reply to $\mathbf{y}$, then $\sum_i \dot{x}_i u_i > 0$

Theorem

$\forall \alpha$ *small enough*, $\exists \mathbf{y} : \mathbb{R} \to S_{opp}$ *such that strategy* T *survives*

Innovative dynamics favour rare strategies

Innovation: if i unused best-reply, then $\dot{x}_i > 0$. Hence $\dot{x}_i / x_i = +\infty$.

So by Continuity: if i almost best-reply and $x_i << 1$, then $\dot{x}_i / x_i$ huge.

Thus, if $x_i << 1$, we may have: $u_i < u_j$ but $\dfrac{\dot{x}_i}{x_i} > \dfrac{\dot{x}_j}{x_j}$

$\hookrightarrow$ favours rare strategies.

Imitative dynamics neutral: $u_i < u_j \Rightarrow \frac{\dot{x}_i}{x_i} < \frac{\dot{x}_j}{x_j}$ whatever $x_i, x_j > 0$.

But imitation dynamics might favour rare/frequent strategies ; then same survival results should hold.

Imitation protocol favouring rare/frequent strategies

Step 1:

1a) a revising agent meets 3 randomly drawn agents, e.g, (j, k, k)

1b) make a list of strategies played by these agents; here: {j, k}

1c) pick one at random: here j with probability 1/2

Step 2: decides whether to imitate him according to standard r_{ij}.

Leads to: $\rho_{ij} = p_j(x)r_{ij}$ where $p_j(x)$ proba of picking j in step 1.

Step 1 favours rare strategies: $x_i < x_j \Rightarrow p_i/x_i > p_j/x_j$.

If instead, when meeting (j, k, k), agent 1 focuses on the "majoritarian choice" k, favours frequent strategies: $x_i < x_j \Rightarrow p_i/x_i < p_j/x_j$.

Result 1 -Distorted Imitation of success

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare strategies.
- in step 2, $r_{ij} = K + u_j$, or $r_{ij} = f(u_j)$, with f positive increasing.

Theorem

There are two-strategy games with a strictly dominated strategy that survives in proportion almost $1/2$ for most initial conditions.

With advantage to frequent strategies, survival in proportion almost 1!

Proof (advantage to rare strategies)

Dynamics:

$$\dot{x}_i = \sum_j x_j p_i f(u_i) - x_i \sum_j p_j f(u_j)$$

Let $p_i = x_i(1+\varepsilon_i)$. For two strategies i and j with same payoff u:

$$\frac{\dot{x}_i}{x_i} - \frac{\dot{x}_j}{x_j} = (\varepsilon_i - \varepsilon_j) f(u)$$

With only these strategies: since $x_i < x_j \Rightarrow \varepsilon_i > \varepsilon_j$, $x_i \to 1/2$

Now perturb: assume payoff of x_i is $u - \alpha$ so i dominated

We get: $\forall \eta > 0$, $\exists \overline{\alpha} > 0$, $\forall \alpha < \overline{\alpha}$, $x_i > \eta \Rightarrow \liminf x_i > \frac{1}{2} - \eta$.

With advantage to frequent strategies: $x_i > 1/2 + \eta \Rightarrow \liminf x_i > 1 - \eta$.

Other dynamics?

Distorted imitation of success does not satisfy Positive Correlation, has non Nash interior rest-points, and not all Nash equiibria are rest-points!

Can we have similar results for dynamics with more usual properties?

$\hookrightarrow$ Yes, but requires more elaborate examples.

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Distorted imitation of success does not satisfy Positive Correlation, has non Nash interior rest-points, and not all Nash equiibria are rest-points!

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Result 2 - Games against the environment

Consider again 3×2 game:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} A \\ B \\ T \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\alpha & 1-\alpha \end{pmatrix} \end{array}$$

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare strategies.
- in step 2, $r_{ij} = [u_j - u_i]_+$, or same sign.

Theorem

$\forall \eta > 0, \exists \overline{\alpha} > 0, \forall \alpha < \overline{\alpha}, \exists \mathbf{y}(\cdot), \forall \mathbf{x}(0) \in int(S_3), \liminf x_T > \frac{1}{2} - \eta$

Same results if $r_{ij} = [u_j - \overline{u}]$, or same sign.

Favour frequent strategies: $x_T(0) > x_B(0) + \eta \Rightarrow \liminf x_T > 1 - \eta$

Result 3 - Single population dynamics

Consider dynamics derived from imitation revision protocol such that:

- step 1 favours rare (resp. frequent) strategies.
- in step 2, $r_{ij} = [u_j - u_i]_+$, or same sign.

Theorem

There are 4 strategy games such that for large sets of initial conditions, a pure strategy dominated by another pure strategy survives in proportion roughly 1/6 (resp. 1/3).

Same results if $r_{ij} = [u_j - \overline{u}]$, or same sign.

Proportion may be increased to $1/2 - \eta$ (resp. $1 - \eta$).

Sketch of proof

We mimick Hofbauer and Sandholm (2011).

They consider dynamics satisfying Positive Correlation (PC):

$$\dot{x} \neq 0 \Rightarrow \dot{x} \cdot u(x) > 0.$$

Geometrically: acute angle between $\dot{x}$ and payoff vector $u(x)$.

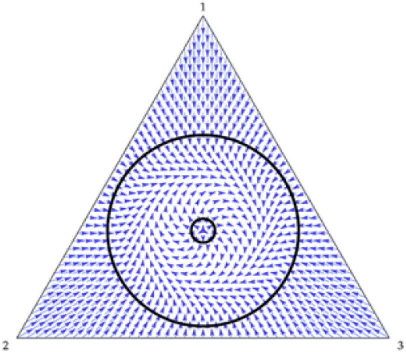
Also: acute angle between $\dot{x}$ and projection of payoff vector on simplex

Lemma

Under theorem's assumptions, our imitation dynamics satisfy (PC)

Hypnodisk game (Hofbauer and Sandholm)

3-strategy game with projected payoff vector field:



Projected payoff vector field for the hypnodisk game

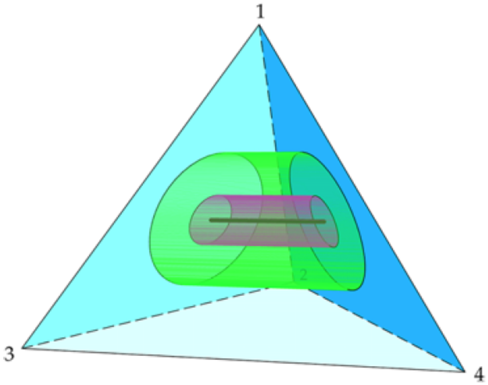
Due to (PC), all interior solutions enter annulus, except Nash equilibrium.

Hypnodisk game with a twin (Hofbauer and Sandholm)

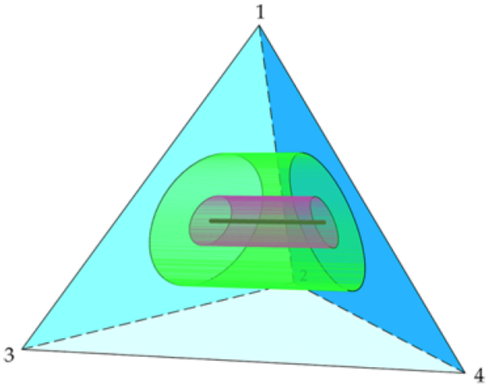
Add as strategy 4 a twin of strategy 3.

- Segment of equilibrium: $x_1 = x_2 = x_3 + x_4 = 1/3$.

- Attracting annulus becomes attracting "intercylinder zone"



Effect of advantage to rare strategies



Advantage to rare strategies: $x_3/x_4 \to 1$.

Attractor A in intersection of intercylinder zone and plane $x_3 = x_4$.

Basin of attraction $B(A) = int(S_4) \backslash Nash\ equilibria$

Continuation of attractors (Hofbauer and Sandholm)

Substract ε to strategy 4 $\rightarrow$ makes it dominated

By standard results on continuation of attractors, for ε small enough, most solutions still converge to an attractor A_ε in the neighborhood of A.

$\hookrightarrow$ under most solutions, strategy 4 survives, and $\liminf x_4 \geq 1/6 - r$, with r radius of outer cylinder

Rk: $1/6$ may be changed to anything $< 1/2$ by modifying base game.

Conjecture for Josef

Done: survival of dominated strategies under imitation dynamics for

◇ games against the environment: symmetric bimatrix games, many dynamics;

◇ single-population dynamics: specific dynamics, or many dynamics but with hypnodisk game.

Conjecture

Similar results for many single-population dynamics in Hofbauer and Sandholm's Rock-Paper-Scissors-feeble Twin game

Problem: prove instability of segment of Nash equilibria.

Discussion - I

Elimination of dominated strategies "requires":

$$\forall x, u_i = u_j \Rightarrow \dot{x}_i / x_i = \dot{x}_j / x_j.$$

Fragile property, destroyed by appropriate small perturbation.

May be a pinch of innovation, or a twist in imitation process.

Discussion - II

When evolutionary game theorists imported replicator dynamics in economics, they justified it through an imitation model.

They probably did not think to imitation models ex-nihilo.

Doing so leads to different kinds of imitation dynamics, which need not eliminate pure strategies dominated by other pure strategies.

Dichotomy innovative/imitative should be supplemented by "treat strategies differently as function of their frequencies or not".

Literature (non exhaustive)

- Akin (80): replicator dynamics; Nachbar (90): monotone dynamics
- Samuelson & Zhang (92): aggregate monotone dynamics
- Hofbauer & Weibull (96): convex monotone dynamics
- Dekel & Scotchmer (92), Cabrales & Sobel (92), Björnerstedt et al (96): discrete-time dynamics

More recently:

- Cressman & Hofbauer (05); Cressman et al (06); Heifetz et al (07a, 07b), Jouini et al. (13): continuum of pure strategies
- Fudenberg & Harris (92), Cabrales (00), Imhof (05), Hofbauer & Imhof (09); Mertikopoulos & Moustakas (10), Mertikopoulos & Viossat (15): stochastic dynamics
- Berger & Hofbauer (06), Hofbauer & Sandholm (11): innovative dynamics