Stability and Instability of Standing Waves for the Nonlinear Fractional Schrödinger Equation

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1. Equation

\[ iu_t - (-\Delta + k^2)^s u + \left( \frac{1}{|x|^\gamma} * |u|^2 \right) u = 0, \quad x \in \mathbb{R}^N, \quad (1) \]

\[ u = u(t, x) : [0, T) \times \mathbb{R}^N \to \mathbb{C}, \quad 0 < T \leq +\infty; \]

\[ (-\Delta + k^2)^s u = \mathcal{F}^{-1}[(|\xi|^2 + k^2)^s \mathcal{F}[u](\xi)], \quad s > 0; \]

\[ \left( \frac{1}{|x|^\gamma} * |u|^2 \right)(x) = \int \frac{|u(y)|^2}{|x-y|^\gamma} dy; \quad 0 < \gamma < 4s. \]

◊ Boson star, fractional quantum mechanics etc.
In particular, when $N = 3$, $s = \frac{1}{2}$ and $\gamma = 1$, Eq.(1) is the Boson star equation.


- Bao and Dong (2011): efficient and accurate numerical methods to compute the ground states and dynamics.
### Known Studies

For general $0 < s < 1$, Laskin(2000,2002): expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths.


Known Studies

Existence and stability of standing waves for the classic nonlinear Schrödinger equation (i.e. $s = 1$)


Existence and stability of standing waves for the fourth-order nonlinear Schrödinger equation (i.e. $s = 2$)

Problems and Arguments

- Problems

1. Existence of standing waves for (1).
2. Stability of these standing waves.

- Arguments

- Weinstein(1983): blow-up argument
Local Well-posedness

Denote $H^s := \{ v \in S'(\mathbb{R}^N) \mid \int (1 + |\xi|^2)^s |\widehat{v}(\xi)|^2 d\xi < +\infty \}$ and

$$E(v) := \frac{1}{2} \int \overline{v}(-\Delta)^s v dx - \frac{1}{4} \int \overline{v} \left( \frac{1}{|x|^\gamma} * |v|^2 \right) dx.$$

Impose

$$u(0, x) = u_0 \in H^s. \quad (2)$$

**Proposition 1** (Cho, Hwang, Hajaiej and Ozawa, 2012)

Let $N \geq 2$, $0 < s < 1$ and $0 < \gamma \leq 2s$. If $u_0 \in H^s$, then $\exists! u(t, x)$ of (1)-(2) on $I = [0, T)$ such that $u(t, x) \in C(I; H^s) \cap C^1(I; H^{-s})$. Moreover, for all $t \in I$, $u(t, x)$ satisfies the following conservation laws.

(i) Conservation of mass $\|u(t)\|_2 = \|u_0\|_2$.

(ii) Conservation of energy $E(u(t)) = E(u_0)$.
**Proposition 2** Let \( N \geq 2 \) and \( 0 < s < 1 \). If \( \{v_n\}_{n=1}^{+\infty} \) is a bounded sequence in \( H^s \), then there exists a subsequence of \( \{v_n\}_{n=1}^{+\infty} \) (still denoted \( \{v_n\}_{n=1}^{+\infty} \)), a family \( \{x^j_n\}_{j=1}^{+\infty} \) of sequences in \( \mathbb{R}^N \) and a family \( \{V^j\}_{j=1}^{+\infty} \) of \( H^s \) functions satisfying the following.

(i) For every \( k \neq j \), \( |x^k_n - x^j_n| \to +\infty \) as \( n \to +\infty \).

(ii) For every \( l \geq 1 \) and every \( x \in \mathbb{R}^N \), \( v_n(x) \) can be decomposed as

\[
v_n(x) = \sum_{j=1}^{l} V^j(x - x^j_n) + v^l_n(x)
\]

with \( \lim_{l \to +\infty} \lim_{n \to +\infty} \|v^l_n\|_p = 0 \) for every \( p \in (2, \frac{2N}{(N-2s)+}) \).

Moreover, we have, as \( n \to +\infty \),

\[
\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|v^l_n\|_2^2 + o(1),
\]

\[
\|v_n\|_{H^s}^2 = \sum_{j=1}^{l} \|V^j\|_{H^s}^2 + \|v^l_n\|_{H^s}^2 + o(1).
\]
Let $k = 0$, $N \geq 2$ and $M > 0$. For any $0 < \gamma < 2s$, we define the following variational problem

$$d_M := \inf_{\{v \in H^s \mid \|v\|_2^2 = M\}} E(v) \quad (3)$$

where $E(v) := \frac{1}{2} \int |\xi|^{2s} |\hat{v}|^2 d\xi - \frac{1}{4} \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|^{\gamma}} dxdy$ is the energy functional. Define the set

$$S_M := \{v \in H^s \mid v \text{ is the minimizer of the variational problem (3)}\}. \quad (4)$$

From the Euler-Lagrange Theorem, we see that for any $v \in S_M$, there exists $\omega \in \mathbb{R}$ such that

$$(-\Delta)^s v + \omega v - \left(\frac{1}{|x|^{\gamma}} * |v|^2\right)v = 0, \quad v \in H^s.$$
Theorem 1 Let $k = 0$, $N \geq 2$ and $M > 0$. Assume $0 < s < 1$ and $0 < \gamma < 2s$. Then for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that for any $u_0 \in H^s$, if the initial data $u_0$ satisfies

$$\inf_{v \in S_M} \|u_0 - v\|_{H^s} < \delta,$$

then the corresponding solution $u(t, x)$ of the Cauchy problem (1)-(2) is such that

$$\inf_{v \in S_M} \|u(t, x) - v(x)\|_{H^s} < \varepsilon$$

for all $t > 0$, where $S_M$ is defined in (4).
Sketch of the Proof of Theorem 1

Key: The existence of minimizer of Variational Problem (3)

i.e. Let \( N \geq 2, 0 < \gamma < 2s \) and \( M > 0 \). Then,

\[
d_M := \inf_{\{v \in H^s \mid \|v\|_2^2 = M\}} E(v) = \min_{\{v \in H^s \mid \|v\|_2^2 = M\}} E(v).
\]

Firstly, the variational problem (3) is well defined. Using the Hölder inequality with \( 1 = \frac{\gamma}{2s} + \frac{2s-\gamma}{2s} \), we deduce that

\[
\int \frac{|v(x)|^2}{|x-y|^{\gamma}} dx \leq C\|x|^{-s}v(x+y)\|_2^\frac{\gamma}{s} \|v\|_2^\frac{2s-\gamma}{s} \leq C\|v\|_{\dot{H}^s}^\frac{\gamma}{s} \|v\|_2^\frac{2s-\gamma}{s}.
\]

Thus,

\[
\int \left( \frac{1}{|x|^{\gamma}} \ast |v(x)|^2 \right) |v(x)|^2 dx \leq \| \int \frac{|v(x)|^2}{|x-y|^{\gamma}} dx \|_\infty \|v(y)\|_2^2 \leq C\|v\|_{\dot{H}^s}^\frac{\gamma}{s} \|v\|_2^\frac{4s-\gamma}{s}.
\]

From Young inequality, \( \exists \ C(\varepsilon, \gamma, s, \sqrt{M}) > 0 \) such that for all \( 0 < \varepsilon < \frac{1}{2} \)

\[
E(v) \geq \frac{1}{2} \|v\|_{\dot{H}^s}^2 - C \|v\|_2^{\frac{4s-\gamma}{s}} \|v\|_{\dot{H}^s}^{\frac{\gamma}{s}} \geq \left( \frac{1}{2} - \varepsilon \right) \|v\|_{\dot{H}^s}^2 - C(\varepsilon, \gamma, s, \|v\|_2).
\]
Sketch of the Proof of Theorem 1

Secondly, there exists $C_0 > 0$,

$$d_M \leq -C_0 < 0.$$ 

Indeed, take $v_n = \rho_n^N R(\rho_n x)$, where $\rho_n > 0$ and $R$ is a function such that $\|v_n\|^2_2 = \|R\|^2_2 = M$. We deduce that

$$E(v_n) = \frac{1}{2} \int |\xi|^{2s} |\widehat{v_n}|^2 d\xi - \frac{1}{4} \int \int \frac{|v_n(x)|^2 |v_n(y)|^2}{|x-y|^\gamma} dxdy$$

$$= \frac{\rho_n^{2s}}{2} \int |\xi|^{2s} |\widehat{R}|^2 d\xi - \frac{\rho_n^\gamma}{4} \int \int \frac{|R(x)|^2 |R(y)|^2}{|x-y|^\gamma} dxdy.$$ 

Since $0 < \gamma < 2s$, we can choose $\rho_n > 0$ sufficiently small such that there exists $C_0 > 0$ such that $E(v_n) \leq -C_0 < 0$. 

Stability for Nonlinear FLS

1. Equation
2. Known Studies
3. Problems
4. Preliminaries
5. Orbital Stability
6. Instability
Thirdly, we prove the existence of minimizers of (3). Take the minimizing sequence \( \{v_n\}_{n=1}^{+\infty} \subset H^s \) of (3) satisfying that as \( n \to +\infty \),

\[
E(v_n) \to d_M \quad \text{and} \quad \|v_n\|_2^2 \to M.
\]  

(6)

Then, for \( n \) large enough, \( E(v_n) \) satisfies \( E(v_n) < d_M + 1 \). From (5), for any \( 0 < \varepsilon < \frac{1}{2} \) and \( n \geq 1 \) large enough, we deduce that

\[
\left( \frac{1}{2} - \varepsilon \right) \|v_n\|_{H^s}^2 \leq d_M + 1 + C(\varepsilon, \gamma, s, M).
\]

Hence, \( \{v_n\}_{n=1}^{+\infty} \) is bounded in \( H^s \). Apply Proposition 2, we see that

\[
v_n(x) = \sum_{j=1}^{l} V^j(x - x^j_n) + v^l_n,
\]  

(7)

\[
\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j_n\|_2^2 + \|v^l_n\|_2^2 + o(1), \quad \|v_n\|_{H^s}^2 = \sum_{j=1}^{l} \|V^j_n\|_{H^s}^2 + \|v^l_n\|_{H^s}^2 + o(1),
\]

\[
\int \int \frac{|v_n(x)|^2|v_n(y)|^2}{|x - y|^{\gamma}} \, dxdy = \sum_{j=1}^{l} \int \int \frac{|V^j_n(x)|^2|V^j_n(y)|^2}{|x - y|^{\gamma}} \, dxdy + \int \int \frac{|v^l_n(x)|^2}{|x - y|^{\gamma}} \, dxdy.
\]
Thus, we have

\[ E(v_n) = \sum_{j=1}^{l} E(V^j(x - x^j_n)) + E(v^l_n) + o(1). \] (8)

Take the scaling transformation \( V^j_{\rho_j} = \rho_j V^j(x - x^j_n) \) with \( \rho_j = \frac{\sqrt{M}}{\|V^j(x - x^j_n)\|_2} \geq 1 \). We have \( \|V^j_{\rho_j}\|_2^2 = M \). We deduce that as \( n \to +\infty \) and \( l \to +\infty \)

\[ d_M \geq E(v_n) \]
\[ = \sum_{j=1}^{l} \left( \frac{E(V^j_{\rho_j})}{\rho_j^2} + \frac{\rho_j^2 - 1}{4} \int \int \frac{|V^j(x - x^j_n)|^2 |V^j(y - x^j_n)|^2}{|x-y|^{\gamma}} \, dx \, dy \right) + E(v^l_n) + o(1) \]
\[ \geq \sum_{j=1}^{l} \frac{d_M}{\rho_j^2} + \inf_{j \geq 1} \frac{\rho_j^2 - 1}{4} \left( \sum_{j=1}^{l} \int \int \frac{|V^j(x - x^j_n)|^2 |V^j(y - x^j_n)|^2}{|x-y|^{\gamma}} \, dx \, dy \right) + \frac{\|v^l_n\|_2^2}{M^2} d_M \]
\[ \geq d_M + C_0 \left( \frac{M}{\|V^j_0\|_2^2} - 1 \right) + o(1), \]

where \( C_0 > 0 \). Therefore, there exists only one term \( V^{j_0} \neq 0 \) in the decomposition (7) such that \( \|V^{j_0}\|_2^2 = M \). \( V^{j_0} \) is the minimizer of (3).
Theorem 2 Let $k = 0$ and $N \geq 2$. Assume $0 < \frac{\gamma}{2} = s < 1$. Then, the ground state solitary waves $e^{i\omega t}Q(x)$ of the fractional nonlinear Schrödinger equation (1) are unstable in the following sense: For arbitrary $\varepsilon > 0$, there exist the radial initial data $\{u_{0,n}\}_{n=1}^{+\infty} \subset H^{s_0}$ with $s_0 = \max\{2s, \frac{2s+1}{2}\}$ satisfying $|x|u_{0,n} \in L^2$, $x \cdot \nabla u_{0,n} \in L^2$, 

$$\|u_{0,n} - Q\|_{H^s} < \varepsilon$$

and the corresponding solution $\{u_n(t,x)\}_{n=1}^{+\infty}$ blows up in the finite time, where $Q$ is the ground state solution of 

$$(-\Delta)^s Q + Q - (\frac{1}{|x|^{2s}} * |Q|^2)Q = 0.$$
Sketch of the Proof of Theorem 2

**Proposition 3** Let $N \geq 2$ and $0 < s < 1$. Then,

\[
\int |v|^2 \, dx \leq \frac{2}{N} \left( \int \overline{vx}(-\nabla)^{1-s}xv \, dx \right)^{\frac{1}{2}} \left( \int \overline{v}(-\nabla)^s v \, dx \right)^{\frac{1}{2}} , \quad \forall \ v \in H^s. \]

**Proof.**

For all $v \in H^s$, we have

\[
\int |v|^2 \, dx = \frac{2}{N} \int \mathcal{F}[\nabla \cdot (xv)] \mathcal{F}^{-1}[\overline{v}] \, d\xi \\
= \frac{2}{N} \int \overline{\mathcal{F}[v]} \xi \cdot \nabla \xi \mathcal{F}[v] \, d\xi \\
\leq \frac{2}{N} \int |\xi|^s |\overline{\mathcal{F}[v]}| |\xi|^{1-s} |\nabla \xi \mathcal{F}[v]| \, d\xi \\
\leq \frac{2}{N} \left( \int \overline{v}(-\nabla)^s v \, dx \right)^{\frac{1}{2}} \left( \int \overline{vx}(-\nabla)^{1-s}xv \, dx \right)^{\frac{1}{2}}.
\]
**Proposition 4** (see [Cho, Hwang, Kwon and Lee, 2012]) Let $k = 0$, $N \geq 2$, $0 < s < 1$ and $\gamma = 2s$. Assume that $u_0 \in H^{s_0}$ with $s_0 = \max\{2s, \frac{\gamma + 1}{2}\}$ is radial symmetric, and $|x|u_0 \in L^2$ and $x \cdot \nabla u_0 \in L^2$. If $u(t, x)$ is the solution of the Cauchy problem (1)-(2), then for all $t \in I$ (the maximal time interval), $\int \bar{u}x(-\Delta)^{1-s}udx$ is nonnegative and

$$\int \bar{u}x(-\Delta)^{1-s}xudx \leq 2sE(u_0)t^2 + Ct + C.$$
Sketch of the Proof of Theorem 2

Let \( \{c_n\}_{n=1}^{+\infty} \subset \mathbb{C}\setminus\{0\} \) be such that \( |c_n| > 1 \) and \( \lim_{n\to+\infty} |c_n| = 1 \), and \( \{\rho_n\}_{n=1}^{+\infty} \subset \mathbb{R}^+ \) be such that \( \lim_{n\to+\infty} \rho_n = 1 \). We take the initial data

\[
    u_{0,n} = c_n^{\frac{N}{2}} \rho_n^2 Q(\rho_n x),
\]

where \( Q \) is the ground state solution of (10). We see that for all \( n \geq 1 \),

\[
    \|u_{0,n}\|_2 > \|Q\|_2 \quad \text{and} \quad \lim_{n\to+\infty} \|u_{0,n}\|_2 = \lim_{n\to+\infty} |c_n| \|Q\|_2 = \|Q\|_2
\]

and

\[
    \lim_{n\to+\infty} \|u_{0,n}\|_{\dot{H}^s} = \lim_{n\to+\infty} |c_n| \rho_n^s \|Q\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}.
\]

Hence, from the Brézis-Lieb Lemma, \( \forall \varepsilon > 0 \), \( \|u_{0,n} - Q\|_{H^s} < \varepsilon \) as \( n \) is sufficiently large.

On the other hand, we see that

\[
    E(u_{0,n}) = \frac{(|c_n|^2 - |c_n|^4)\rho_n^{2s}}{2} \|Q\|_{\dot{H}^s}^2 \leq -C_0 < 0.
\]
Then,
\[ \int \overline{u_n} x (-\Delta)^{1-s} x u_n dx \leq -2sC_0 t^2 + Ct + C, \]
which implies that there exists \(0 < T < +\infty\) such that
\[ \lim_{t \to T} \int \overline{u_n} x (-\Delta)^{1-s} x u_n dx = 0. \]

Finally, using the conservation of mass and applying the inequality in Proposition 3 to \(u_n\), we see that for all time \(t\)
\[ \|u_{0,n}\|_2^2 = \|u_n(t)\|_2^2 \leq \frac{2}{N} \left( \int \overline{u_n} x (-\Delta)^{1-s} x u_n dx \right)^{\frac{1}{2}} \left( \int \overline{u_n} (-\Delta)^s u_n dx \right)^{\frac{1}{2}} \leq \frac{2}{N} \left( \int \overline{u_n} x (-\Delta)^{1-s} x u_n dx \right)^{\frac{1}{2}} \|u_n(t)\|_{\dot{H}^s}. \]

Therefore, there exists \(0 < T < +\infty\) such that
\[ \lim_{t \to T} \|u_n(t)\|_{\dot{H}^s} = +\infty. \]
Thanks!