Multiscale methods and analysis for highly oscillatory differential equations

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Outline

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• Klein-Gordon equation in nonrelativistic limit regime
  • Classical numerical methods
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Oscillate:
- swing backward and forward like a pendulum;
- move or travel back and forth between two points;
- vary above and below a mean value.

Oscillations happen everywhere in our life.

macroscopic:

- vibrating spring system
- pendulum
Background of oscillatory problems

- microscopic: molecular vibrations, etc.
- others: signal
Background of oscillatory problems

Oscillations concerned here: solution of *second order differential equations*.

- A simple example: Newton’s equation of a harmonic oscillator
  \[ m\ddot{x}(t) = -kx(t), \quad t > 0. \] (1)

  - \( m \) is the mass, \( x(t) \) is the position, \( k > 0 \) is Young’s modulus.
  - \( x(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)x(0) + \sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)\dot{x}(0). \)
  - when \( k \) is large, \( x(t) \) is highly oscillatory.

Many other physical systems exhibit similar oscillatory phenomena:

- outer solar system; Hénon-Heiles model for stellar motion;
- molecular dynamics (MD).
- PDEs from quantum physics in some limit regimes.
- the right hand side of (1) is usually nonlinear in \( t \) and \( x \).
Background of oscillatory problems

Solution of the problem: highly oscillatory, non-periodic.

- Exact solution: impossible in general.
- Main way to study: effective approximations.
- Existing studies: $0 < \varepsilon \ll 1$,
  - averaging method (*Krylov etc. 1935*); stroboscopic averaging (*Sanders etc. 2007*):
    \[
    \dot{u}(t) = f \left( t, \frac{t}{\varepsilon}, u \right), \text{ with } f(t, \tau, u) \text{ periodic in } \tau.
    \]
  - quadrature for highly oscillatory integrals (*Iserles etc. 2005*):
    \[
    l = \int_0^1 f(t) e^{i \frac{1}{\varepsilon} g(t)} dt
    \]
    extensions to ODEs with oscillatory source term (*Condon etc. 2010*):
    \[
    \ddot{u} = f(u, \dot{u}, t) + g \left( \frac{t}{\varepsilon} \right).
    \]
Second order highly oscillatory differential equations

- exponential wave integrators (EWIs) \((Lubich etc. 1999)\); modulated Fourier expansion \((Hair etc. 2000)\):

\[
\begin{align*}
\ddot{u}(t) + \frac{1}{\varepsilon^2}Au(t) + f(u(t)) &= 0, \quad t > 0, \\
u(0) &= \varepsilon \Phi_0, \quad \dot{u}(0) = \Phi_1.
\end{align*}
\]

Model problem arising from MD.

The existing methods classified into two branches:

- analytical approach: averaging method; modulated Fourier expansion;
- numerical approach: EWIs; quadratures for oscillatory integrals;
- recent studies turn to combine both: \textit{multiscale methods}.

However all these methods are strongly problem dependent!
Still many interesting oscillation systems unsettled!

- We are interested: a class of oscillatory equations from quantum or plasma mechanics!
Second order highly oscillatory differential equations

We begin by considering a fundamental *model problem*: the *second order highly oscillatory differential equations* (HODEs):

\[
\begin{aligned}
\varepsilon^2 \ddot{y}(t) + Ay(t) + \frac{1}{\varepsilon^2} y(t) + f(y(t)) &= 0, \quad t > 0, \\
y(0) &= \Phi_0, \quad \dot{y}(0) = \frac{\Phi_1}{\varepsilon^2}.
\end{aligned}
\]

(2)

- \(y \in \mathbb{C}^d, \ 0 < \varepsilon \leq 1, \ A \in \mathbb{R}^{d \times d}\) is positive semi-definite.
- Motivated from the spatial discretized Klein-Gorden equation; \(\varepsilon\) inversely proportional to speed of light. \(f(e^{is}y) = e^{is}f(y)\).
- Different from the model problem arising from MD:

\[
\begin{aligned}
\dddot{u}(t) + \frac{1}{\varepsilon^2} Au(t) + f(u(t)) &= 0, \quad t > 0, \\
u(0) &= \varepsilon \Phi_0, \quad \dot{u}(0) = \Phi_1.
\end{aligned}
\]

- well-studied by Hair, Lubich etc.
- weaker nonlinearity and initial data compared to (2).
Second order highly oscillatory differential equations

- \( y(t) \) propagates waves with wavelength \( O(\varepsilon^2) \); high oscillations as \( 0 < \varepsilon \ll 1 \).

![Graphs showing oscillatory behavior](image)

**Figure**: Solutions of (2) with \( d = 2, f_1(y_1, y_2) = y_1^2 y_2, f_2(y_1, y_2) = y_2^2 y_1, A = \text{diag}(2, 2), \Phi_1 = (1, 0.5)^T \) and \( \Phi_2 = (1, 2)^T \) for different \( \varepsilon \).

- **Numerical difficulties**: small step size to resolve oscillations.
- **Stronger and wider oscillation**: Existing studies give no clues to find good numerical integrators to solve our problem for \( 0 < \varepsilon \ll 1 \).
For simplicity of notation, we shall present the numerical methods by the degenerated scalar equation:

\[
\begin{align*}
\epsilon^2 \ddot{y} + \left( \alpha + \frac{1}{\epsilon^2} \right) y + f(y) &= 0, \quad t > 0, \\
y(0) &= \phi_1, \quad \dot{y}(0) = \frac{\phi_2}{\epsilon^2},
\end{align*}
\]

where \( \alpha \geq 0, \ 0 < \epsilon \leq 1, \ y(t) \in \mathbb{C} \).

Goal:
- Find good numerical integrators that work uniformly well for all \( 0 < \epsilon \leq 1 \) for the model problem.
- Later extend the methods to wave or dispersive PDEs from quantum and plasma physics.
Properties of the model problem

- In most applications, nonlinearity is the pure power law:
  \[ f(y) = g(|y|^2)y, \quad \text{with} \quad g(\rho) = \lambda \rho^p, \quad p \in \mathbb{N}, \quad \lambda \in \mathbb{R}. \]

- Energy conservation:
  \[
  E(t) \ := \ \varepsilon^2 |\dot{y}(t)|^2 + \left( \alpha + \frac{1}{\varepsilon^2} \right) |y(t)|^2 + F(|y(t)|^2)
  \equiv \ \frac{1}{\varepsilon^2} |\phi_2|^2 + \left( \alpha + \frac{1}{\varepsilon^2} \right) |\phi_1|^2 + F(|\phi_1|^2) := E(0),
  \]
  with \( F(\rho) = \int_0^\rho g(\rho')d\rho'. \)

- \( y(t) = O(1), \ \dot{y}(t) = O\left(\frac{1}{\varepsilon^2}\right), \ldots, \) for fixed \( t \geq 0. \)
Finite difference (FD) integrators

\[ \delta_t^2 y^n := \frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2}, \quad \text{with} \quad \tau = \Delta t > 0. \]

- Conservative Crank-Nicolson finite difference (CNFD) integrator:

\[ \varepsilon^2 \delta_t^2 y^n + \left( \alpha + \frac{1}{\varepsilon^2} \right) \frac{y^{n+1} + y^{n-1}}{2} + \hat{F} \left( y^{n+1}, y^{n-1} \right) = 0, \quad n \geq 0, \]

with \( \hat{F} \left( y^{n+1}, y^{n-1} \right) := \frac{F \left( |y^{n+1}|^2 \right) - F \left( |y^{n-1}|^2 \right)}{|y^{n+1}|^2 - |y^{n-1}|^2} \cdot \frac{y^{n+1} + y^{n-1}}{2} \).

- implicit; nonlinear solver; time consuming.
- unconditionally stable when \( \lambda \geq -\varepsilon^{-2} \) (oscillatory nature).
- conserves a discrete energy:

\[ E^n := \varepsilon^2 \left| \delta_t^+ y^n \right|^2 + \left( \alpha + \frac{1}{\varepsilon^2} \right) \frac{|y^{n+1}|^2 + |y^n|^2}{2} + \frac{F \left( |y^{n+1}|^2 \right) + F \left( |y^n|^2 \right)}{2} \equiv E^0 \quad n \geq 0. \]
Semi-implicit finite difference (SIFD) integrator:

\[ \varepsilon^2 \delta_t^2 y^n + \left( \alpha + \frac{1}{\varepsilon^2} \right) \frac{y^{n+1} + y^{n-1}}{2} + f(y^n) = 0, \quad n \geq 0. \]

- solve a linear equation.
- unconditionally stable when \(|\lambda| \leq \varepsilon^{-2}\).

Explicit finite difference (EXFD) or Störmer-Verlet integrator:

\[ \varepsilon^2 \delta_t^2 y^n + \left( \alpha + \frac{1}{\varepsilon^2} \right) y^n + f(y^n) = 0, \quad n \geq 0. \]

- fully explicit.
- stable under condition: \(\tau \lesssim \varepsilon^2\).
Error bounds of FD integrators for $t \in [0, T]$:

$$e^n := y(t_n) - y^n, \quad 0 \leq n \leq \frac{T}{\tau}.$$ 

For CNFD, SIFD and EXFD with $0 < \tau \leq \tau_0$ where $\tau_0 > 0$ is a constant independent of $\varepsilon$, with $0 < \tau \lesssim \varepsilon^3$ (W. Bao and X. Dong, Numer. Math. 2011):

$$|e^n| \lesssim \frac{\tau^2}{\varepsilon^6}, \quad 0 \leq n \leq \frac{T}{\tau}.$$ 

To get good approximations, one need to use:

$$\tau \lesssim \varepsilon^3 \quad (\varepsilon\text{-scalability}).$$

Too much numerical burden when $0 < \varepsilon \ll 1$. 

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Exponential wave integrators (EWIs): based on the variation of constant formula of the ODE,

\[ y(t_{n+1}) + y(t_{n-1}) = 2 \cos(\omega \tau) y(t_n) - \int_0^\tau \frac{\sin(\omega(\tau - \theta))}{\varepsilon^2 \omega} [f^n(\theta) + f^n(-\theta)] d\theta. \]

where \( f^n(\theta) = f(y(t_n + \theta)). \)

  \[ y^{n+1} = -y^{n-1} + 2 \cos(\omega \tau) y^n - 2 \frac{1 - \cos(\omega \tau)}{\varepsilon^2 \omega^2} f(y^n), \]
  where \( \omega = \frac{\sqrt{1+\varepsilon^2 \alpha}}{\varepsilon^2}. \)

- Deuflhard’s type (EWI-D) (P. Deuflhard, ZAMP., 1979):
  \[ y^{n+1} = -y^{n-1} + 2 \cos(\omega \tau) y^n - 2 \frac{\tau \sin(\omega \tau)}{2\varepsilon^2 \omega} f(y^n). \]

Both are explicit; stable under condition: \( \tau \lesssim \varepsilon^2. \)
Error bounds of EWIs.

For $0 < \tau \lesssim \varepsilon^2$, EWI-G and EWI-D (W. Bao and X. Dong, Numer. Math. 2011):

$$|e^n| \lesssim \frac{\tau^2}{\varepsilon^4}, \quad 0 \leq n \leq \frac{T}{\tau}.$$

The $\varepsilon$-scalability is improved to $\tau \lesssim \varepsilon^2$ compared to FD. In practice, still not satisfactory when $0 < \varepsilon \ll 1$.

In all, FD and EWI good enough for $\varepsilon = O(1)$; unsatisfactory for $0 < \varepsilon \ll 1$.

We want to improve $\varepsilon$-scalability further, and look for numerical integrator works uniformly well for $0 < \varepsilon \leq 1$. 
Ansatz (multiscale expansion): from $t_n$ to $t_{n+1}$, let $y(t) = y(t_n + s)$:

$$y(t_n + s) = e^{is/\varepsilon^2} z_+^n(s) + e^{-is/\varepsilon^2} \bar{z}_-^n(s) + r^n(s), \quad 0 \leq s \leq \tau.$$ 

Plugging the ansatz into the HODE:

$$\begin{align*}
[2i \dot{z}_+^n(s) + \varepsilon^2 \ddot{z}_+^n(s) + \alpha z_+^n(s)] e^{is/\varepsilon^2} & \\
+ \left[ -2i \ddot{z}_-^n(s) + \varepsilon^2 \bar{z}_-^n(s) + \alpha \bar{z}_-^n(s) \right] e^{-is/\varepsilon^2} & \\
+ \varepsilon^2 \ddot{r}^n(s) + \left( \alpha + \frac{1}{\varepsilon^2} \right) r^n(s) + f(y(t_n + s)) &= 0, \quad 0 < s \leq \tau.
\end{align*}$$

Decompose the nonlinearity (Masmoudi, 2002):

$$f_{\pm}(z_+, z_-) = \frac{1}{2\pi} \int_0^{2\pi} f(z_\pm + e^{i\theta} \bar{z}_+) d\theta,$$

$$f_r(z_+, z_-, r; s) = f\left( e^{is/\varepsilon^2} z_+ + e^{-is/\varepsilon^2} \bar{z}_- + r \right) - f_+ (z_+, z_-) e^{is/\varepsilon^2} - \bar{f}_- (z_+, z_-) e^{-is/\varepsilon^2}. $$
HODEs-Multiscale decomposition

- Decomposition by frequency (MDF): for $0 < s \leq \tau$,

\[
\begin{cases}
2i\dot{z}^n_\pm(s) + \varepsilon^2\ddot{z}^n_\pm(s) + \alpha z^n_\pm(s) + f_\pm(z^n_+(s), z^n_-(s)) = 0, \\
\varepsilon^2\ddot{r}^n(s) + \left(\alpha + \frac{1}{\varepsilon^2}\right) r^n(s) + f_r(z^n_+(s), z^n_-(s), r^n(s); s) = 0.
\end{cases}
\]

- Decomposition by frequency and amplitude (MDFA): for $0 < s \leq \tau$,

\[
\begin{cases}
2i\dot{z}^n_\pm(s) + \alpha z^n_\pm(s) + f_\pm(z^n_+(s), z^n_-(s)) = 0, \\
\varepsilon^2\ddot{r}^n(s) + \left(\alpha + \frac{1}{\varepsilon^2}\right) r^n(s) + f_r(z^n_+(s), z^n_-(s), r^n(s); s) + \varepsilon^2 u^n(s) = 0,
\end{cases}
\]

with $u^n(s) = e^{is/\varepsilon^2}\ddot{z}^n_+(s) + e^{-is/\varepsilon^2}\ddot{z}^n_-(s)$.
In pure power nonlinearity case:

- explicit $f_{\pm}$ and $f_r$.
- $f_{\pm}(z_{\pm}^n(s), z_{-}^n(s)) = g_{\pm}(|z_{\pm}^n(s)|^2, |z_{-}^n(s)|^2) z_{\pm}^n(s)$, where
  $g_{\pm}(\rho_{+}, \rho_{-})$ is a real-valued polynomial.
- Example of cubic nonlinearity ($p=1$): $f(u) = \lambda |u|^2 u$,
  
  $f_{\pm}(z_{+}, z_{-}) = \lambda(|z_{\pm}|^2 + 2|z_{\mp}|^2) z_{\pm}$.

Initial conditions for the decomposed system:

\[
\begin{cases}
  z_{+}^n(0) + \overline{z_{-}^n(0)} + r^n(0) = y(t_n) = \phi_1^n, \\
  \frac{i}{\varepsilon^2} [z_{+}^n(0) - \overline{z_{-}^n(0)}] + \dot{z}_{+}^n(0) + \overline{z_{-}^n(0)} + \dot{r}^n(0) = \dot{y}(t_n) = \frac{\phi_2^n}{\varepsilon^2}.
\end{cases}
\]

freedom to choose.
Choose proper initial conditions,

- MDFA needs $r^n(0), \dot{r}^n(0), z^n_\pm(0)$: choose $r^n(0) = 0$ and by matched asymptotic orders
  - $z^n_\pm(s)$ totally independent of $\varepsilon$.
  - $r^n(s) \sim O(\varepsilon^2)$.

- MDF needs in addition $\dot{z}^n_\pm(0)$: be well-prepared (W. Bao and Y. Cai, *SIAM J. Numer. Anal.* 2012) for the two coupled Schrödinger type equations with wave operators when $0 < \varepsilon \ll 1$:
  \[
  \dot{z}^n_\pm(0) = \frac{i}{2} \left[ \alpha z^n_\pm(0) + f_\pm(z^n_+(0), z^n_- (0)) \right].
  \]

- $z^n_\pm(s)$ weakly oscillatory: $z^n_\pm(s), \dot{z}^n_\pm(s), \ddot{z}^n_\pm(s) \sim O(1)$.
- $r^n(s) \sim O(\varepsilon^2)$.

The decomposed system is easier to handle numerically!
Numerical integrators based on multiscale decompositions.

- **MDF:**
  
  \[
  \begin{align*}
  2i\dot{z}_{\pm}^n(s) + \varepsilon^2\ddot{z}_{\pm}^n(s) + \alpha z_{\pm}^n(s) + f_{\pm}(z_{\pm}^n(s), z_{-\pm}^n(s)) &= 0, \\
  \varepsilon^2\dot{r}^n(s) + \left(\alpha + \frac{1}{\varepsilon^2}\right) r^n(s) + f_r(z_{\pm}^n(s), z_{-\pm}^n(s), r^n(s); s) &= 0.
  \end{align*}
  \]

  - \(z_{\pm}^n(t)\): EWIs—truncation error independent of \(\varepsilon\).
  - \(r^n(t)\): EWIs—numerical solution is \(O(\varepsilon^2)\).

- **MDFA:**
  
  \[
  \begin{align*}
  2i\dot{z}_{\pm}^n(s) + \alpha z_{\pm}^n(s) + g_{\pm}(|z_{\pm}^n(s)|^2, |z_{-\pm}^n(s)|^2) z_{\pm}^n(s) &= 0, \\
  \varepsilon^2\dot{r}^n(s) + \left(\alpha + \frac{1}{\varepsilon^2}\right) r^n(s) + f_r(z_{\pm}^n(s), z_{-\pm}^n(s), r^n(s); s) + \varepsilon^2 u^n(s) &= 0.
  \end{align*}
  \]

  - \(z_{\pm}^n(t)\): Exact integrator.
  - \(r^n(t)\): EWIs—numerical solution is \(O(\varepsilon^2)\).
Solve decomposed problem, get $z_{\pm}(\tau)$, $\dot{z}_{\pm}(\tau)$, $r^n(\tau)$, $\dot{r}^n(\tau)$.

- Reconstruct $y(t_{n+1})$, $\dot{y}(t_{n+1})$ via the fundamental ansatz:

$$y(t_n + s) = e^{is/\varepsilon^2} z_{\pm}^n(s) + e^{-is/\varepsilon^2} \overline{z_{\pm}^n(s)} + r^n(s), \quad 0 \leq s \leq \tau.$$ 

- Each step:

$$y(t_n), \dot{y}(t_n) \xrightarrow{} \text{MDF or MDFA} \xrightarrow{} y(t_{n+1}), \dot{y}(t_{n+1})$$

a decomposition and reconstruction flow.

Named as multiscale time integrator based on MDF (MTI-F) or MDFA (MTI-FA).
Error bounds of MTIs for $t \in [0, T]$: for $0 < \tau \leq \tau_0$, with $0 < \tau_0 \leq 1$ a constant independent of $\varepsilon$ (W. Bao, X. Dong, X. Zhao, J. Math. Study, 2014),

- **MTI-F:**
  \[
  |e^n| \lesssim \frac{\tau^2}{\varepsilon^2} \quad \text{and} \quad |e^n| \lesssim \tau^2 + \varepsilon^2, \quad 0 \leq n \leq \frac{T}{\tau}.
  \]

- **MTI-FA:**
  \[
  |e^n| \lesssim \frac{\tau^2}{\varepsilon^2} \quad \text{and} \quad |e^n| \lesssim \varepsilon^2, \quad 0 \leq n \leq \frac{T}{\tau}.
  \]

- **Both MTIs:** the minimum of the two error bounds for $\varepsilon \in (0, 1]$ shows
  \[
  |e^n| \lesssim \tau, \quad 0 \leq n \leq \frac{T}{\tau}.
  \]

Thus, $\varepsilon$-scalability improved to $\tau \lesssim 1$. 
A numerical example:

- Choose

\[ \alpha = 2, \quad f(y) = |y|^2 y, \quad \phi_1 = 1, \quad \phi_2 = 1, \]

for the HODE.

- Test the error at time \( T \) for different \( \varepsilon \in (0, 1] \) and \( \tau \):

\[ e^{\varepsilon, \tau}(T) = y(T) - y^M, \]

with \( M = \frac{T}{\tau} \).
**Table:** Error analysis of MTI-F: $e^{\varepsilon, \tau}(T)$ with $T = 4$ and convergence rate.

<table>
<thead>
<tr>
<th>$\varepsilon, \tau(T)$</th>
<th>$\tau_0 = 0.2$</th>
<th>$\tau_0/2^2$</th>
<th>$\tau_0/2^4$</th>
<th>$\tau_0/2^6$</th>
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Table: Error analysis of MTI-FA: $e^{e,\tau}(T)$ with $T = 4$ and convergence rate.

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<th>$e^{e,\tau}(T)$</th>
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Concluding remarks for the studies on the model problem-HODEs:

- MTIs have $\varepsilon$-scalability $\tau \lesssim 1$.
- MTIs work uniformly correctly for $0 < \varepsilon \leq 1$.
- MTIs is much better then FD and EWIs in regime $0 < \varepsilon \ll 1$, and MTI-FA is better than MTI-F.

The methods and results obtained here will be the fundament of further studies to Klein-Gordon equation and others related equations!
Nonlinear Klein-Gordon equation (KGE)

The KGE in $d$ dimensions ($d = 1, 2, 3$) reads

$$\frac{\hbar^2}{mc^2} \partial_{tt} u(x, t) - \frac{\hbar^2}{m} \Delta u + mc^2 u + f(u) = 0, \quad x \in \mathbb{R}^d, \quad t > 0.$$ 

- $c$: speed of light; $\hbar$: Plank constant; $m$: mass of particle.
- describe the spinless particle.
- relativistic version of the Schrödinger equation.

Nondimensionalization: $t \rightarrow \frac{\hbar}{m \varepsilon^2 c^2} t$ and $x \rightarrow \frac{\hbar}{m \varepsilon c} x$,

the dimensionless KGE reads

$$\begin{cases} 
\varepsilon^2 \partial_{tt} u - \Delta u + \frac{1}{\varepsilon^2} u + f(u) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
u(x, 0) = \phi_1(x), \quad \partial_t u(x, 0) = \frac{1}{\varepsilon^2} \phi_2(x), \quad x \in \mathbb{R}^d.
\end{cases}$$ (3)
0 < \varepsilon \leq 1: a dimensionless parameter inversely proportional
to the speed of light c.

\phi_1 \text{ and } \phi_2: \text{ given complex-valued initial data independent of } \varepsilon.

f(u): \mathbb{C} \to \mathbb{C} \text{ independent of } \varepsilon.

- describing the nonlinear interactions.
- satisfying gauge invariance: \( f(e^{is}u) = e^{is}f(u), \forall s \in \mathbb{R} \).
- most application cases: \( f(u) \) is the pure power nonlinearity, i.e.

\[ f(u) = g(|u|^2)u, \text{ with } g(\rho) = \lambda \rho^p, \]

for some \( \lambda \in \mathbb{R}, \ p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

- Energy conservation

\[
E(t) := \int_{\mathbb{R}^d} \left[ \varepsilon^2 |\partial_t u(x, t)|^2 + |\nabla u(x, t)|^2 + \frac{1}{\varepsilon^2} |u(x, t)|^2 + F(|u(x, t)|^2) \right] dx
\]

\[
\equiv \int_{\mathbb{R}^d} \left[ \frac{1}{\varepsilon^2} |\phi_2(x)|^2 + |\nabla \phi_1(x)|^2 + \frac{1}{\varepsilon^2} |\phi_1(x)|^2 + F(|\phi_1(x)|^2) \right] dx = E(0),
\]

where \( F(\rho) = \int_0^\rho g(s) ds \).
**KGE-Background**

- **Fixed** $0 < \varepsilon \leq 1$: well-posedness
  - defocusing case $(F(\rho) \geq 0, \forall \rho \in \mathbb{R})$: globally well-posed (*P. Brenner and W. van Wahl, Math. Z.*, 1981).
  - focusing case $(F(\rho) \leq 0, \forall \rho \in \mathbb{R})$: possible finite time blow up (*D.D. Bainov and E. Minchev, J. Math. Phys.*, 1995).

- **As** $\varepsilon \to 0$:
  - corresponding to the speed of light goes to infinity, thus known as the nonrelativistic limit.
  - solution of the KGE $u(x, t)$ propagates waves with amplitude at $O(1)$ and wavelength at $O(\varepsilon^2)$ and $O(1)$ in time and space, respectively, when $0 < \varepsilon \ll 1$.

$$u(x, t) = O(1), \ \partial_t u(x, t) = O\left(\frac{1}{\varepsilon^2}\right), \ \partial_{tt} u(x, t) = O\left(\frac{1}{\varepsilon^4}\right), \ldots$$
Example: $d = 1$, $f(u) = |u|^2u$, $\phi_1(x) = e^{-x^2/2}$, $\phi_2(x) = \frac{3}{2}\phi_1(x)$.

**Figure:** The solution of the KGE for different $\varepsilon$.

- the solution of the KGE is **highly oscillatory in time**!
- extend the methods for the HODEs to KGE.
For numerical perspectives, truncate the KGE to a bounded interval with periodic boundary conditions. In 1D,

\[ \varepsilon^2 \partial_{tt} u(x, t) - \partial_{xx} u(x, t) + \frac{1}{\varepsilon^2} u(x, t) + f(u(x, t)) = 0, \quad a < x < b, \quad t > 0, \]

\[ u(a, t) = u(b, t), \quad \partial_x u(a, t) = \partial_x u(b, t), \quad t \geq 0, \]

\[ u(x, 0) = \phi_1(x), \quad \partial_t u(x, 0) = \frac{1}{\varepsilon^2} \phi_2(x), \quad a \leq x \leq b. \]

- periodic B.C.
  - solution decays fast at far field for a fixed \( t \).
  - widely used as numerical setup in literature.
  - can be replaced by zero B.C.

- consider 1D case for simplicity of notation.

- discretization: mesh size \( h = (b - a)/M \) with even positive integer \( M \); time step \( \tau \); grid points
  \[ x_j = a + jh, \quad t_n = n\tau (n = 0, 1, \ldots, j = 0, \ldots, M). \]
Finite difference time domain methods: denote
\[
\delta^2_t u^n_j = \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{\tau^2}, \quad \delta^2_x u^n_j = \frac{u^{n+1}_{j+1} - 2u^n_{j+1} + u^{n-1}_{j+1}}{h^2}, \quad \delta^+_x u^n_j = \frac{u^{n+1}_{j+1} - u^n_{j+1}}{h}.
\]

- energy conservative finite difference (ECFD) method: \(1 \leq j \leq M,
\[
\varepsilon^2 \delta^2_t u^n_j - \delta^2_x u^n_j + \frac{1}{\varepsilon^2} \frac{u^{n+1}_j + u^{n-1}_j}{2} + G(u^{n+1}_j, u^{n-1}_j) = 0,
\]
with B.C. \(u^n_0 = u^n_M, \quad u^n_{M+1} = u^n_1,\) and \(G(v, w) = \frac{F(|v|^2) - F(|w|^2)}{|v|^2 - |w|^2} \cdot \frac{v+w}{2}.

- implicit; nonlinear solver; time consuming.
- unconditionally stable.
- conserve a discrete energy

\[
E^n = \varepsilon^2 \|u^n\|_{l_2}^2 + \frac{\|\delta^+_x u^{n+1}\|_{l_2}^2 + \|\delta^+_x u^n\|_{l_2}^2}{2} + \frac{\|u^{n+1}\|_{l_2}^2 + \|u^n\|_{l_2}^2}{2\varepsilon^2} + h \sum_{j=0}^{M-1} \frac{F(|u^{n+1}_j|^2) + F(|u^n_j|^2)}{2} \equiv E^0.
\]
semi-implicit finite difference (SIFD) method:

\[ \varepsilon^2 \delta_t^2 u_j^n - \delta_x^2 \frac{u_j^{n+1} + u_j^{n-1}}{2} + \frac{1}{\varepsilon^2} \frac{u_j^{n+1} + u_j^{n-1}}{2} + f(u_j^n) = 0, \quad 0 \leq j \leq M, \]

\[ u_0^n = u_M^n, \quad u_{M+1}^n = u_1^n. \]

- linear solver.
- unconditionally stability when \(0 < \varepsilon \ll 1\).

explicit finite difference (EXFD) method:

\[ \varepsilon^2 \delta_t^2 u_j^n - \delta_x^2 u_j^n + \frac{1}{\varepsilon^2} u_j^n + f(u_j^n) = 0, \quad 0 \leq j \leq M, \]

\[ u_0^n = u_M^n, \quad u_{M+1}^n = u_1^n. \]

- fully explicit.
- stable under condition \(\tau \lesssim \varepsilon^2\), as \(0 < \varepsilon \ll 1\).
Error bounds of FD methods for $t \in [0, T]$: 

$$e^n_j := u(x_j, t^n) - u^n_j, \ 0 \leq j < M, \ 0 \leq n \leq \frac{T}{\tau}.$$ 

(W. Bao and X. Dong, Numer. Math., 2012) For CNFD, SIFD and EXFD, when $0 < \tau \leq \tau_0$, $0 < h \leq h_0$ where $\tau_0$ and $h_0$ are a constant independent of $\varepsilon$, under condition $\tau \lesssim \varepsilon^3$: 

$$\|e^n\|_2 + \|\delta_x^+ e^n\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^6}, \quad 0 \leq n \leq \frac{T}{\tau}.$$

- To get good approximations, one need to use $h \lesssim 1$ but $\tau \lesssim \varepsilon^3$.
- Too much numerical burden when $0 < \varepsilon \ll 1$. 
Exponential wave integrators (EWIs) in time + FD or spectral method in space

For spectral spatial discretization: taking Fourier transform on both sides of the KGE

\[ \varepsilon^2 \hat{u}_l''(t) + \left( \mu_l^2 + \frac{1}{\varepsilon^2} \right) \hat{u}_l(t) + \left( \hat{f}(u) \right)_l(t) = 0, \quad \mu_l = \frac{2\pi l}{b-a}, \quad l \in \mathbb{N}. \]

EWI: based on the variation of constant formula of the ODE for \( t \in [t_{n-1}, t_{n+1}] \),

\[
\hat{u}_l(t_{n+1}) = -\hat{u}_l(t_{n-1}) + 2 \cos(\omega_l \tau) \hat{u}_l(t_n) - \int_{\tau}^{0} \frac{\sin(\omega_l(\tau - \theta))}{\varepsilon^2 \omega_l} \left[ \hat{f}^n_l(\theta) + \hat{f}^n_l(-\theta) \right] d\theta,
\]

where \( \omega_l = \frac{\sqrt{1+\varepsilon^2 \mu_l^2}}{\varepsilon^2} \), \( \hat{f}^n_l(\theta) = \left( \hat{f}(u) \right)_l(t_n + \theta) \).
Gaustchi’s type method:

$$\hat{u}_l(t_{n+1}) \approx - \hat{u}_l(t_{n-1}) + 2 \cos (\omega_l \tau) \hat{u}_l(t_n) - 2 \frac{1 - \cos (\omega_l \tau)}{\varepsilon^2 \omega_l^2} \hat{f}_l^n(0).$$

Deuflhard’s type method:

$$\hat{u}_l(t_{n+1}) \approx - \hat{u}_l(t_{n-1}) + 2 \cos (\omega_l \tau) \hat{u}_l(t_n) - \frac{\tau \sin (\omega_l \tau)}{\varepsilon^2 \omega_l} \hat{f}_l^n(0).$$

Replacing the Fourier transform by pseudospectral approximation in practice for

- Gaustchi’s type EWI Fourier pseudospectral (GIFP) method.
- Deuflhard’s type EWI Fourier pseudospectral (DIFP) method.
In addition, if $\hat{u}'(t_{n+1})$ is of interest in EWIs
\[
\hat{u}'(t_{n+1}) = -\hat{u}'(t_{n-1}) + 2 \cos(\omega l \tau) \hat{u}'(t_n)
- \int_0^\tau \frac{\cos(\omega l (\tau - \theta))}{\varepsilon^2} \left[ \hat{f}_n^+(\theta) - \hat{f}_n^-(\theta) \right] d\theta
\approx -\hat{u}'(t_{n-1}) + 2 \cos(\omega l \tau) \hat{u}'(t_n) - \frac{\tau}{2\varepsilon^2} \left[ \hat{f}_n^+(\tau) - \hat{f}_n^-(\tau) \right]
\]

Interesting finding (X. Dong and X. Zhao, Commun. Comput. Phys., 2014): DIFP with the above is equivalent to solve KGE (system form)
\[
\begin{cases}
\partial_t u(x, t) - v(x, t) = 0, & a < x < b, \ t > 0, \\
\varepsilon^2 \partial_t v(x, t) - \partial_{xx} u(x, t) + \frac{1}{\varepsilon^2} u(x, t) + f(u(x, t)) = 0,
\end{cases}
\]
by a strang splitting FP method as
\[
u(x, t_{n+1}) \approx e^{\frac{1}{2} \varepsilon A \tau} e^{B \tau} e^{\frac{1}{2} A \tau} u(x, t_n)
\]
\[
A : \begin{cases}
\partial_t u = 0, \\
\partial_t v + \frac{1}{\varepsilon^2} f(u) = 0,
\end{cases} \quad B : \begin{cases}
\partial_t u - v = 0, \\
\partial_t v - \frac{1}{\varepsilon^2} \partial_{xx} u + \frac{1}{\varepsilon^4} u = 0,
\end{cases}
\]
Systems A, B are solved exactly in time direction.

Meaning of this finding:
- Shows some relations between EWIs and time splitting method.
- Offers some easier way to prove error estimates of time splitting method.

**Error bounds** of the EWI spectral methods:
$m_0 > 0$ dependents on the regularity of the solution;
$e^n(x) := u(x, t_n) - I_M(u^n)(x)$.

For GIFP (W. Bao and X. Dong, *Numer. Math.*, 2012): under condition $0 < \tau \lesssim \varepsilon^2$,

$$\| e^n \|_{H^1} \lesssim h^{m_0} + \frac{\tau^2}{\varepsilon^4}, \quad 0 \leq n \leq \frac{T}{\tau}. $$

- Rigorous and optimal.
- Outside regime $\tau \lesssim \varepsilon^2$: $O(1)$ error and no convergence.

under condition $0 < \tau \lesssim \varepsilon^2$,

\[
\|e^n\|_{H^1} \lesssim h^{m_0} + \frac{\tau^2}{\varepsilon^2}, \quad 0 \leq n \leq \frac{T}{\tau}.
\]

Numerical observation.

Outside regime $\tau \lesssim \varepsilon^2$: no convergence.

Figure: Temporal error versus $\varepsilon$ (log-scale) for different $\tau$ at $t = 1$ under $h = 1/8$, with

\[
u(x, 0) = \frac{3 \sin(x)}{e^{0.5x^2} + e^{-0.5x^2}}, \quad v(x, 0) = \frac{2e^{-x^2}}{\sqrt{\pi \varepsilon^2}}, \quad f(u) = u^3.
\]
In all, classical FD and EWI good enough for $\varepsilon = O(1)$; unsatisfactory for $0 < \varepsilon \ll 1$.

Recent study combine numerical methods with tools from applied math: an asymptotic preserving method (E. Faou and K. Schratz, Numer. Math., 2014) is proposed with an error bound

$$\| e^n \|_{H^1} \lesssim h^{m_0} + \tau^2 + \varepsilon^2.$$

- good when $0 < \varepsilon \ll 1$.
- large error for $\varepsilon = O(1)$; unsatisfactory approximation in intermediate regime.

We want to further improve the $\varepsilon$-scalability of classical methods, and look for numerical integrator works uniformly well for all $0 < \varepsilon \leq 1$. (Recent work: P. Chartier, N. Crouseilles, M. Lemou and F. Méhats, Numer. Math., 2014.)

We use the MTIs to the problem!
Starting point

- our two multiscale integrators (MTI-F&MTI-FA) for some highly oscillatory second order ODEs with uniform convergence.
- now try to make it work for the KGE.

Fundamental ansatz (multiscale expansion) (N. Masmoudi and K. Nakanishi, Math. Ann., 2002): from $t_n$ to $t_{n+1}$, let

$$u(x, t_n + s) = e^{is/\varepsilon^2} z^+_n(x, s) + e^{-is/\varepsilon^2} z^-_n(x, s) + r^n(x, s), \quad 0 \leq s \leq \tau.$$ 

Plugging the ansatz into the KGE:

$$e^{is/\varepsilon^2} \left[ \varepsilon^2 \partial_{ss} z^+_n(x, s) + 2i \partial_s z^+_n(x, s) - \Delta z^+_n(x, s) \right] + \varepsilon^2 \partial_{ss} r^n(x, s) + \Delta r^n(x, s)$$

$$+ e^{-is/\varepsilon^2} \left[ \varepsilon^2 \partial_{ss} z^-_n(x, s) - 2i \partial_s z^-_n(x, s) - \Delta z^-_n(x, s) \right] + \frac{r^n(x, s)}{\varepsilon^2}$$

$$+ f(u(x, t_n + s)) = 0.$$
KGE-Multiscale decomposition

- Decompose the nonlinearity similarly:
  \[ f_{\pm}(z_+, z_-) = \frac{1}{2\pi} \int_0^{2\pi} f(z_+ e^{i\theta} z_-) d\theta, \]
  \[ f_r(z_+, z_-, r; s) = f(e^{is/\varepsilon^2} z_+ + e^{-is/\varepsilon^2} z_- + r) - f_+(z_+, z_-) e^{is/\varepsilon^2} - f_-(z_+, z_-) e^{-is/\varepsilon^2}. \]
- In pure power nonlinearity case: explicit \( f_{\pm} \) and \( f_r \).
- Example of cubic nonlinearity: \( f(u) = \lambda |u|^2 u \),
  \[ f_{\pm}(z_+, z_-) = \lambda(|z_+|^2 + 2|z_+|^2)z_\pm. \]

- Multiscale decomposition by frequency and amplitude (MDFA): for \( 0 < s \leq \tau \),
  \[
  \begin{cases}
  2i\partial_s z^n_\pm - \Delta z^n_\pm + f_{\pm}(z^n_+, z^n_-) = 0, \\
  \varepsilon^2 \partial_{ss} r^n - \Delta r^n + \frac{1}{\varepsilon^2} r^n + f_r(z^n_+, z^n_-, r^n; s) + \varepsilon^2 z^n(x, s) = 0.
  \end{cases}
  \]
  with \( z^n(x, s) = e^{is/\varepsilon^2} \partial_{ss} z^n_+(x, s) + e^{-is/\varepsilon^2} \partial_{ss} z^n_-(x, s). \)
- nonlinear Schrödinger equation no longer exactly integrable.
- server stability constraint.

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Multiscale decomposition by frequency (MDF): for $0 < s \leq \tau$,

\[
\begin{align*}
\varepsilon^2 \partial_{ss} z_{\pm}^n + 2i \partial_s z_{\pm}^n - \Delta z_{\pm}^n + f_{\pm} (z_{\pm}^n, z_{\mp}^n) &= 0, \\
\varepsilon^2 \partial_{ss} r^n - \Delta r^n + \frac{1}{\varepsilon^2} r^n + f_r (z_{\pm}^n, z_{\mp}^n, r^n; s) &= 0.
\end{align*}
\]

- Everything is still fine.
- MTI can be proposed in a similar way as the MTI-F with a proper spatial discretization.

Thus, we do **MDF instead of MDFA!**
Initial conditions for the decomposed system

\[
\begin{align*}
    z_+^n(x, 0) + \overline{z_-^n(x, 0)} + r^n(x, 0) &= \phi_1^n(x), \\
    \frac{i}{\varepsilon^2} \left[ z_+^n(x, 0) - \overline{z_-^n(x, 0)} \right] + \partial_s z_+^n(x, 0) + \partial_s \overline{z_-^n(x, 0)} + \partial_s r^n(x, 0) &= \frac{\phi_2^n(x)}{\varepsilon^2},
\end{align*}
\]

specify the 6 unknowns from the 3 equations: freedom to choose.

- equate \( O \left( \frac{1}{\varepsilon^2} \right) \) and \( O(1) \) terms: \( z_+^n(x, 0) - \overline{z_-^n(x, 0)} = \phi_2^n(x) \).
- make \( r^n \) small: \( r^n(x, 0) = 0 \).
- be well-prepared (W. Bao and Y. Cai, SIAM J. Numer. Anal., 2012) for the two coupled Schrödinger equations with wave operators when \( 0 < \varepsilon \ll 1 \):

\[
    \partial_s z_\pm^n(x, 0) = \frac{i}{2} \left[ -\Delta z_\pm^n(x, 0) + f_\pm \left( z_\pm^n(x, 0) \right) \right].
\]

- solve to get all initial data.
Property of the MDF system with the chosen initial data:

- \( z^n_{\pm} \) is weakly oscillatory:
  \[
  z^n_{\pm}(\cdot, s), \partial_s z^n_{\pm}(\cdot, s), \partial_{ss} z^n_{\pm}(\cdot, s) \sim O(1).
  \]

- \( r^n \) is kept small:
  \[
  r^n(\cdot, s) \sim O(\varepsilon^2).
  \]

The decomposed system is easier to handle numerically!

Based on the decomposed system MDF, we propose numerical approximations to the KGE.
Truncate the IVP of MDF to a bounded domain with periodic B.C.

The IBVP of the MDF in 1D:

\[
\begin{aligned}
\varepsilon^2 \partial_{ss} z^\pm_n + 2i \partial_s z^\pm_n - \partial_{xx} z^\pm_n + f_\pm (z^\pm_n, z^\pm_n) &= 0, \\
\varepsilon^2 \partial_{ss} r^n - \partial_{xx} r^n + \frac{1}{\varepsilon^2} r^n + f_r (z^+_n, z^-_n, r^n; s) &= 0, \quad a < x < b, \quad 0 < s \leq \tau; \\
z^n_\pm(a, s) &= z^n_\pm(b, s), \quad \partial_x z^n_\pm(a, s) = \partial_x z^n_\pm(b, s), \\
r^n(a, s) &= r^n(b, s), \quad \partial_x r^n(a, s) = \partial_x r^n(b, s), \quad 0 \leq s \leq \tau;
\end{aligned}
\]

with initial data

\[
\begin{aligned}
z^+_n(x, 0) &= \frac{1}{2} \left[ \phi_1^n(x) - i \phi_2^n(x) \right], \\
r^n(x, 0) &= 0, \\
\partial_s z^n_\pm(x, 0) &= \frac{i}{2} \left[ -\Delta z^n_\pm(x, 0) + f_\pm (z^+_n(x, 0), z^-_n(x, 0)) \right], \\
\left( \widehat{(\dot{z}_\pm)}_l(0) \right) &= \frac{i}{2} \left[ \frac{1}{\tau} \sin \left( \mu^2_l \tau \right) (\widehat{z}_0)_l(0) + (\widehat{f}_\pm)_l^n(0) \right], \\
\end{aligned}
\]

Used a filter \( \frac{1}{\tau} \sin \left( \mu^2_l \tau \right) \) instead of \( \mu^2_l \) to avoid CFL condition.
KGE-Multiscale method

- Spatial discretization: in Fourier space,

\[
\begin{cases}
\varepsilon^2 (\hat{z}_\pm^n)_l''(s) + 2i(\hat{z}_\pm^n)_l'(s) + \mu_l^2 (\hat{z}_\pm^n)_l(s) + (\hat{f}_\pm^n)_l(s) = 0, & 0 < s \leq \tau, \\
\varepsilon^2 (\hat{r}_l^n)'(s) + \left( \mu_l^2 + \frac{1}{\varepsilon^2} \right) (\hat{r}_l^n)_l(s) + (\hat{f}_r^n)_l(s) = 0, & l \in \mathbb{N}.
\end{cases}
\]

- Integral form: by variation-of-constant formula, for \( 0 \leq s \leq \tau \)

\[
\begin{cases}
(\hat{z}_\pm^n)_l(s) = a_l(s)(\hat{z}_\pm^n)_l(0) + \varepsilon^2 b_l(s)(\hat{z}_\pm^n)_l'(0) - \int_0^s b_l(s - \theta)(\hat{f}_\pm^n)_l(\theta)d\theta, \\
(\hat{r}_l^n)_l(s) = \frac{\sin(\omega_l s)}{\omega_l} (\hat{r}_l^n)_l(0) - \int_0^s \frac{\sin (\omega_l(s - \theta))}{\varepsilon^2 \omega_l} (\hat{f}_r^n)_l(\theta)d\theta, \\
(\hat{z}_\pm^n)_l'(s) = a'_l(s)(\hat{z}_\pm^n)_l(0) + \varepsilon^2 b'_l(s)(\hat{z}_\pm^n)_l'(0) - \int_0^s b'_l(s - \theta)(\hat{f}_\pm^n)_l(\theta)d\theta, \\
(\hat{r}_l^n)'_l(s) = \cos(\omega_l s)(\hat{r}_l^n)'_l(0) - \int_0^s \frac{\cos (\omega_l(s - \theta))}{\varepsilon^2} (\hat{f}_r^n)_l(\theta)d\theta.
\end{cases}
\]
KGE-Multiscale method

Coefficient:

\[
a_i(s) := \frac{\lambda_i^+ e^{is\lambda_i^-} - \lambda_i^- e^{is\lambda_i^+}}{\lambda_i^+ - \lambda_i^-}, \quad b_i(s) := i \frac{e^{is\lambda_i^+} - e^{is\lambda_i^-}}{\varepsilon^2(\lambda_i^- - \lambda_i^+)}, \quad \lambda_i^\pm = -\frac{1 \pm \sqrt{1 + \mu_i^2 \varepsilon^2}}{\varepsilon^2}.
\]

- Approximate the integrals by either the Gautschi’s type quadrature or the Deuflhard’s properly.

- truncation error of \( z_\pm^n \) and \( \partial_s z_\pm^n \) independent of \( \varepsilon \).

- numerical approximation of \( r^n \) is \( O(\varepsilon^2) \) as well.

- Replace Fourier transform by pseudospectral interpolations in practise.
KGE-Multiscale method

Solve the decomposed problem to get numerical approximations of $z^n_{\pm}(x, \tau), r^n(x, \tau), \partial_s z^n_{\pm}(x, \tau), \partial_s r^n(x, \tau)$

- Reconstruct $u(x, t_{n+1}), \partial_t u(x, t_{n+1})$ by via the fundamental ansatz:

$$u(x, t_{n+s}) = e^{i s/\varepsilon^2} z^n_{+}(x, s) + e^{-i s/\varepsilon^2} z^n_{-}(x, s) + r^n(x, s), \ 0 \leq s \leq \tau.$$  

- Each step:

$$u(x, t_n), \partial_t u(x, t_n) \rightarrow \text{MDF} \rightarrow u(x, t_{n+1}), \partial_t u(x, t_{n+1})$$

- A decomposition and reconstruction flow.
- Named as multiscale time integrator with Fourier pseudospectral discretization based on MDF (MTIFP).
KGE-Multiscale method

Error estimates of MTIFP for \( t \in [0, T], x \in \Omega, \varepsilon \in (0, 1) \) (W. Bao, Y. Cai, X. Zhao, SIAM J. NUMER. ANAL. 2014)

- \( e^n(x) := u(x, t_n) - u^n_I(x), \dot{e}^n(x) := \partial_t u(x, t_n) - \dot{u}^n_I(x) \)

- assumption:

\[
\begin{align*}
  u &\in C^1([0, T]; H^{m_0+4}_p(\Omega)), \\ 
  \|u\|_{L^\infty([0, T]; H^{m_0+4}_p(\Omega))} + \varepsilon^2 \|\partial_t u\|_{L^\infty([0, T]; H^{m_0+4}_p)} \lesssim 1
\end{align*}
\]

- for \( 0 < \tau \leq \tau_0, 0 < h \leq h_0 \), with \( 0 < \tau_0 \leq 1 \) and \( 0 < h_0 \leq 1 \) two constants independent of \( \varepsilon, \tau, h \)

Then rigorous two independent error bounds of MTIFP:

\[
\begin{align*}
  \|e^n\|_{H^2} + \varepsilon^2 \|\dot{e}^n\|_{H^2} \lesssim h^{m_0} + \frac{\tau^2}{\varepsilon^2}, \\ 
  \|e^n\|_{H^2} + \varepsilon^2 \|\dot{e}^n\|_{H^2} \lesssim h^{m_0} + \tau^2 + \varepsilon^2
\end{align*}
\]

Minimum of the two error bounds:

\[
\|e^n\|_{H^2} + \varepsilon^2 \|\dot{e}^n\|_{H^2} \lesssim h^{m_0} + \tau
\]

Thus, \( \varepsilon \)-scalability improved to \( \tau \lesssim 1, h \lesssim 1 ! \)
A numerical example:

- Choose

\[ f(u) = |u|^2 u, \quad u(x, 0) = (1 + i)e^{-x^2/2}, \quad \partial_t u(x, 0) = \frac{3e^{-x^2/2}}{2\epsilon^2}, \]

for the KGE.

- Test the error at time \( T \) for different \( \epsilon \in (0, 1] \) and \( \tau \):

\[ e^{\tau, h}(T) := \| u(\cdot, T) - u^N_1 \|_{H^2}, \quad e^{\tau, h}_\infty(T) := \max_{\epsilon} \left\{ e^{\tau, h}_\epsilon(T) \right\}. \]

with \( n = N = \frac{T}{\tau} \).
Table: Spatial error of MTIFP: $e_{\varepsilon, h}^\tau(T = 1)$ with $\tau = 5 \times 10^{-6}$ for different $\varepsilon$ and $h$.

<table>
<thead>
<tr>
<th>$e_{\varepsilon, h}^\tau(T)$</th>
<th>$h_0 = 1$</th>
<th>$h_0/2$</th>
<th>$h_0/4$</th>
<th>$h_0/8$</th>
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<tr>
<td>$\varepsilon_0 = 0.5$</td>
<td>1.65E−1</td>
<td>3.60E−3</td>
<td>1.03E−6</td>
<td>7.34E−11</td>
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<td>5.49E−11</td>
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</table>
**Table:** Temporal error of MTIFP: $e_{\varepsilon}^{\tau,h}(T = 1)$ with $h = 1/8$ for different $\varepsilon$ and $\tau$.

<table>
<thead>
<tr>
<th>$e_{\varepsilon}^{\tau,h}(T)$</th>
<th>$\tau_0 = 0.2$</th>
<th>$\tau_0/2^2$</th>
<th>$\tau_0/2^4$</th>
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<td>2.01</td>
<td>2.20</td>
</tr>
<tr>
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<tr>
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<td>1.91</td>
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**KGE-Numerical results for comparisons**

<table>
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Concluding remarks for KGE:

- We managed to extend methods and results from our HODE model problem to KGE.
- We find DIFP is equivalent to TSFP, and has smaller error bound than GIFP.
- MTI based on MDF is proposed, but that for MDFA fails.
  - MTIFP has $\varepsilon$-scalability $\tau \lesssim 1$, $h \lesssim 1$.
  - MTIFP works uniformly correctly for all $0 < \varepsilon \leq 1$.
  - MTIFP is much better than FD and EWIs in temporal approximations, especially in regime $0 < \varepsilon \ll 1$. 

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Applications to KGZ

With proper nondimensionlization and simplification, the Klein-Gordon-Zakharov (KGZ) system reads

$$
\varepsilon^2 \partial_{tt} \psi(x, t) - \Delta \psi(x, t) + \frac{1}{\varepsilon^2} \psi(x, t) + \psi(x, t) \phi(x, t) = 0,
$$

$$
\gamma^2 \partial_{tt} \phi(x, t) - \Delta \phi(x, t) - \Delta (\psi^2(x, t)) = 0, \quad x \in \mathbb{R}^d, \quad t > 0,
$$

- Governing equations for interaction between Langmuir waves and ion sound waves in plasma.
- $0 < \varepsilon \leq 1$: inversely proportional to the plasma frequency.
- $0 < \gamma \leq 1$: inversely proportional to the speed of sound.
- Two interesting limit regimes:
  - $\varepsilon \to 0, \gamma = O(1)$: high plasma frequency limit.
  - $\varepsilon, \gamma \to 0$ under $\varepsilon \lesssim \gamma$: simultaneous high-plasma-frequency and subsonic limit.
Applications to KGZ

KGZ conserves the energy

\[ E(t) := \int_{\mathbb{R}^d} \left[ \varepsilon^2 (\partial_t \psi)^2 + |\nabla \psi|^2 + \frac{1}{\varepsilon^2} \psi^2 + \frac{\gamma^2}{2} |\nabla \varphi|^2 + \frac{1}{2} \phi^2 + \phi \psi^2 \right] dx \equiv E(0), \]

where \( \varphi \) is defined via \( \Delta \varphi = \partial_t \phi \) with \( \lim_{|x| \to \infty} \varphi = 0. \)

- **Difficulties:**
  - \( \psi \) and \( \phi \) stay in different energy spaces. When small parameters \( \varepsilon \) and \( \gamma \) step in, hard to do error estimates.
  - design MTIs for the simultaneous high-plasma-frequency and subsonic limit.

- **We managed to:**
  - extend EWIs to KGZ in the simultaneous high-plasma-frequency and subsonic limit (W. Bao, X. Zhao, *SIAM J. SCI. COMPUT.* 2013).
  - apply the MTIFP to KGZ in single high plasma frequency limit (Preprint 2014).

Their performance is similar as that in KGE.
Conclusions of whole work:

- We proposed and analyzed two MTIs for solving the model problem HODEs with uniform convergence, and through comparisons we identify the best one.

- We extended all numerical methods to KGE in the nonrelativistic limit regime, and established the rigorous uniform convergence theorem of a MTIFP.

- We successfully applied all the methods to KGZ in single high plasma frequency limit or simultaneous high-plasma-frequency and subsonic limit.
Future work:

- Establish the optimal error bound for DIFP in the highly oscillatory regime.
- Work the rigorous error estimate results of EWIs and the MTI for solving the KGZ in the highly oscillatory regime.
- Apply and extend the multiscale method to solve other equations like Klein-Gordon-Schrödinger equations, Zakharov-Rubinchik system et al. in some limit regimes.
- Increase the uniform convergence order.
That is all. Thank you!