Orbit type filtrations of torus manifolds and combinatorics of simplicial posets

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Let $X$ be a closed $2n$-manifold on which the compact $n$-torus $T$ acts in a locally standard manner. The orbit space $X/T$ is a manifold with corners; let $S$ be the simplicial poset dual to $X/T$. Consider a filtration of $X$ by torus invariant subsets $X_i$, the union of all $i$-dimensional orbits, and take the homological spectral sequence associated with this filtration. In the classical case, i.e. when $X$ is a toric or quasitoric manifold, this spectral sequence collapses at the second page, all entries of $E_{pq}^\infty$ are concentrated at the diagonal $p = q$, and their ranks are $h$-numbers of $S$. The same holds when all faces of the orbit space are acyclic. In more general situations the spectral sequence does not collapse at a second page. Nevertheless sometimes it can be described in full.

In combinatorial commutative algebra there are notions of $h'$- and $h''$-numbers of simplicial manifolds which generalize $h$-numbers of spheres. We obtain those numbers as the ranks of certain terms of the spectral sequence. In particular, this gives a topological proof that $h''$-numbers of Buchsbaum posets are nonnegative, which was previously proved algebraically by Novik and Swartz.