The Homotopy Type of Configuration Spaces

Don Stanley

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June 2, 2012
Configuration Spaces

The rational case $Q_2$

Models for $F(M, 3)$

Throughout $M$, $M'$ are closed oriented manifolds of dimension $n$.

Then the configuration space of $k$ points in $M$ is $F(M, k) = \{ (x_1, \ldots, x_k) \in M^k | x_i = x_j \Rightarrow i = j \}$

(Yes you could define this for any manifold, for example $M = \mathbb{R}^n$ or even any space, but that would be a bit extreme.)

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Alternatively if $M$ and $M'$ are rationally homotopy equivalent then are $F(M, k)$ and $F(M', k)$ rationally homotopy equivalent?
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Alternatively if $M$ and $M'$ are rationally homotopy equivalent then are $F(M, k)$ and $F(M', k)$ rationally homotopy equivalent?

(I will say more about Sullivan models soon.)
Answers to Q1 (The good news)

Theorem (Levitt)

Suppose $M, M'$ be a simply connected

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then for every $k$, $\Omega F(M, k) \simeq \Omega F(M', k)$
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There exist lens spaces $L(p, q), L(p, q')$ such that $L(p, q), \simeq L(p, q')$ but
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$F(L(p, q), 2) \not\sim F(L(p, q'), 2)$
Rational homotopy theory
From now on $A$ will be a CDGA

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A CDGA is an algebra and a chain complex with compatible structures
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We will say $A$ is a model of $X$, if it is the CDGA corresponding to $X$.

Fact: $H(A) \simeq H^*(X)$ in a functorial way.
Rational Poincare Duality Models

Definition
A CDGA $A$ is PDCDGA (Poincare duality CDGA) if the underlying algebra of $A$ satisfies Poincare duality.

Theorem (Lambrechts-S)
If $B$ is a CDGA such that $H(B)$ satisfies Poincare duality then there is a quasi-isomorphic PDCDGA $A$. 

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Models for configuration spaces of projective manifolds

Definition
Suppose $A$ is a PDCDGA, let $F(A, k) = A^k[\delta_{ij}]_{1 \leq i < j \leq k} \cong d(\delta_{ij}) = \Delta_{ij}$

This is an exterior algebra on the $\delta_{ij}$ modulo symmetry and Arnold relations and $\Delta_{ij}$ is a diagonal element in the $i$th and $j$th factor of $A^k$. 

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Known models for configuration spaces

Theorem (Fulton-MacPherson, Kriz)
Suppose that $M$ is a projective algebraic complex manifold. Then $F((H^* M), k)$ is a model for $F(M, k)$.

Note that by a result of Deligne-Griffiths-Morgan-Sullivan, $M$ is formal and so $H^*(M)$ is a model for $M$.

Theorem (Lambrechts-S)
If $M$ is 2-connected and $A$ is a PDCDGA model of $M$ then $F(A, 2)$ is a model of $F(M, 2)$.
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If $M$ is 2-connected and $A$ is a PDCDGA model of $M$ then $F(A, 2)$ is a model of $F(M, 2)$.
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If $M$ is 4-connected and $A$ is a PDCDGA model of $M$ then $F(M, 3)$ is a model of $F(M, 3)$. 
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Ideas from the proofs 2-points
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The embedding
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**Theorem (Lambrechts-S)**

*If $M$ is 4-connected and $A$ is a PDCDGA model of $M$ then $F(A, 3)$ is a model of $F(M, 3)$.*

The embedding $M \xrightarrow{\Delta} M \times M$
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The embedding $M \xrightarrow{\Delta} M \times M$

has high enough codimension that the homotopy class of the embedding determines the isotopy class and hence
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**Theorem (Lambrechts-S)**

*If M is 4-connected and A is a PDCDGA model of M then \( F(A, 3) \) is a model of \( F(M, 3) \).*

The embedding \( M \xrightarrow{\Delta} M \times M \)

has high enough codimension that the homotopy class of the embedding determines the isotopy class and hence determines \( M \times M \setminus \Delta = F(M, 2) \).
Ideas from the proofs 2-points rational

If \( A \) is a model for \( M \) then the multiplication \( \phi: A \otimes A \to A \) is a model for \( M \). Also there is a diagonal map \( \Delta: \text{sn} A \to A \times A \) that is a shriek map for \( \phi \). This implies (using general results of Lambrechts-S) that

\[
A \otimes A \oplus \Delta \text{sn} A - 1 = A^2 \bigg[ g_{12} \bigg] / \simeq
\]

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If $A$ is a model for $M$ then the multiplication $\phi: A \otimes A \to A$

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Ideas from the proofs 3-points I
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Notation: $\Delta_{ij} = \{(x_1, x_2, x_3) \in M^3 | x_i = x_j\}$
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so $F(M, 3) = M^3 \setminus \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}$. 
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so $F(M, 3) = M^3 \setminus \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}$.

**Lemma (*)**

![Diagram]

Is a pullback and a homotopy pullback.
Ideas from the proofs 3-points II

Is $3^n - 4$ connected.
The natural map $F(M, 3)$ into the holim of

\[
\begin{align*}
M^3 \setminus \Delta_{12} \cup \Delta_{13} & \rightarrow M^3 \setminus \Delta_{12} \leftarrow M^3 \setminus \Delta_{12} \cup \Delta_{23} \\
M^3 \setminus \Delta_{12} & \rightarrow M^3 \setminus \Delta_{123} \\
M^3 \setminus \Delta_{123} & \rightarrow M^3 \setminus \Delta_{13} \\
M^3 \setminus \Delta_{13} & \rightarrow M^3 \setminus \Delta_{13} \cup \Delta_{23} \\
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M^3 \setminus \Delta_{12} \cup \Delta_{23} & \rightarrow M^3 \setminus \Delta_{12} \leftarrow M^3 \setminus \Delta_{12} \cup \Delta_{23}
\end{align*}
\]

Is $3n - 4$ connected.
Ideas from the proofs 3-points III

Lemma (*) allows us to compare diagrams of the form

\[ M^3 \setminus \Delta_{12} \cup \Delta_{13} \rightarrow M^3 \setminus \Delta_{12} \leftarrow M^3 \setminus \Delta_{12} \cup \Delta_{23} \]

But still need to extend over whole diagrams.
Ideas from the proofs 3-points III

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\[ M^3 \downarrow \downarrow M^3 \]

\[ M^3 \setminus \Delta_{13} \leftarrow M^3 \setminus \Delta_{23} \]

\[ M^3 \setminus \Delta_{13} \cup \Delta_{23} \]

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