Invariants of 3-manifolds from intersecting kernels of Heegaard splittings

Fengling Li

(Joint with Fengchun Lei and Jie Wu)
Dalian University of Technology
dutlf1@163.com

Young Topologist Seminar
11-19 August 2015
1. Brief review on Heegaard splittings
2. Intersecting Kernels of Heegaard Splitting
3. Main results
A handlebody of genus $n \geq 1$ is the boundary connected sums of $n$ copies of $D^2 \times S^1$ (the solid torus). We may regard a 3-ball as a handlebody of genus 0.
A handlebody of genus $n \geq 1$ is the boundary connected sums of $n$ copies of $D^2 \times S^1$ (the solid torus). We may regard a 3-ball as a handlebody of genus 0.

$M$: closed orientable connected 3-mfd, A Heegaard splitting (H.S.) of $M$ is a decomposition $M = V \cup_S W$ with

- $M = V \cup W$
- $V$ and $W$ are both handlebodies
- $V \cap W = S = \partial V = \partial W$

$S$: Heegaard surface of $M$. 
A handlebody of genus $n \geq 1$ is the boundary connected sums of $n$ copies of $D^2 \times S^1$ (the solid torus). We may regard a 3-ball as a handlebody of genus 0.

$M$: closed orientable connected 3-mfd, A Heegaard splitting (H.S.) of $M$ is a decomposition $M = V \cup S \cup W$ with

- $M = V \cup W$
- $V$ and $W$ are both handlebodies
- $V \cap W = S = \partial V = \partial W$

$S$: Heegaard surface of $M$.

Fact: Any closed orientable connected 3-mfd admits a H.S..
$V \cup_{S} W$: a H.S. for $M$.

$V \cup_{S} W$ is reducible if $\exists$ essential disks $D_1 \subset V$ and $D_2 \subset W$ s.t. $\partial D_1 = \partial D_2$. Otherwise, $V \cup_{S} W$ is irreducible.
$V \cup_{S} W$: a H.S. for $M$.

$V \cup_{S} W$ is reducible if $\exists$ essential disks $D_{1} \subset V$ and $D_{2} \subset W$ s.t. $\partial D_{1} = \partial D_{2}$. Otherwise, $V \cup_{S} W$ is irreducible.

$V \cup_{S} W$ is stabilized if $\exists$ essential disks $D_{1} \subset V$ and $D_{2} \subset W$ s.t. $|\partial D_{1} \cap \partial D_{2}| = 1$. Otherwise, $V \cup_{S} W$ is unstabilized.
Reducible (stabilized) H.S.s

\[ V \cup_S W: \text{ a H.S. for } M. \]

\[ V \cup_S W \text{ is reducible} \text{ if } \exists \text{ essential disks } D_1 \subset V \text{ and } D_2 \subset W \text{ s.t. } \partial D_1 = \partial D_2. \] Otherwise, \( V \cup_S W \text{ is irreducible.} \)

\[ V \cup_S W \text{ is stabilized} \text{ if } \exists \text{ essential disks } D_1 \subset V \text{ and } D_2 \subset W \text{ s.t. } |\partial D_1 \cap \partial D_2| = 1. \] Otherwise, \( V \cup_S W \text{ is unstabilized.} \)

Clearly, a stabilized H.S. of genus \( g \geq 2 \) is reducible.
A stabilized H.S. $V \cup_S W$ can be viewed as a connected sum of a H.S. $V' \cup_{S'} W'$ (with genus $g(S) - 1$) and a genus 1 H.S. of $S^3$. $V \cup_S W$ is called an \textit{elementary stabilization} of $V' \cup_{S'} W'$. 

---

Fengling Li

Invariants of 3-manifolds from Heegaard splittings
A stabilized H.S. $V \cup_S W$ can be viewed as a connected sum of a H.S. $V' \cup_{S'} W'$ (with genus $g(S) - 1$) and a genus 1 H.S. of $S^3$. $V \cup_S W$ is called an elementary stabilization of $V' \cup_{S'} W'$.

A H.S. $V \cup_S W$ is called a stabilization of a H.S. $V'' \cup_{S''} W''$ if $V \cup_S W$ can be obtained from $V'' \cup_{S''} W''$ by a finite number of elementary stabilizations.
Let $V \cup_S W$ and $V' \cup_{S'} W'$ be two H.S.s for 3-mfd $M$. They are called equivalent if $S$ and $S'$ are isotopic in $M$, and stably equivalent if they have a common stabilization up to equivalence.

The following is a classical theorem on the stabilizations of H.S.s:

**Theorem (Reidemeister-Singer, 1935)**

Any two H.S.s $V \cup_S W$ and $V' \cup_{S'} W'$ for 3-manifold $M$ are stably equivalent.
Let $H_g$ be an oriented handlebody of genus $g$, $H'_g = \tau(H_g)$ a copy of $H_g$, with the induced orientation.

$M$ is obtained by gluing $H_g$ and $H'_g$ together via a diffeomorphism $\partial H_g \to \partial H'_g$.

The gluing map determines an element $\phi \in \text{MCG}(S_g)$, we denote the H.S. of $M$ by $H_g \cup_{\phi} H'_g$. The H.S. is also denoted by $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$. 
A canonical extension of a mapping class

Note that if a H.S.s $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$ is reducible, and $
\{C_1, \cdots, C_k\}$ a collection of pairwise disjoint essential circles on $S_g$ each of which bounds disks in both of $H_g$ and $H'_g$, then we may choose $\phi$ in $\mathcal{M}$ s.t. $\phi(C_i) = C_i$ for each $1 \leq i \leq k$.

In particular, in an elementary stabilization $\mathcal{M}' = (M; H_{g+1}, H'_{g+1}; S_{g+1}; \phi')$ of $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$, the sewing map $\phi'$ can be described as a canonical extension of $\phi$. 
Definition

\( M \): a closed orientable 3-mfd,
\( \mathcal{M} = (M; H, H'; S) \): a H.S. for \( M \),
\( i : S \hookrightarrow H \) and \( i' : S \hookrightarrow H' \): the inclusions,
\( i_* : \pi_1(S) \to \pi_1(H) \), \( i'_* : \pi_1(S) \to \pi_1(H') \): the induced homomorphisms.

Then \( \text{Ker} i_* \cap \text{Ker} i'_* \) is called the \textit{intersecting kernel} of the H.S. \( \mathcal{M} \),
and is denoted by \( K(\mathcal{M}) \).
$\mathcal{M} = (S^3; H_1, H'_1; T)$: a genus 1 H.S. for $S^3$

$a, b$: two essential s.c.c. on the torus $T$ s.t. $a$ bounds a disk in $H_1$, $b$ bounds a disk in $H'_1$, and $a \cap b = P$, which we choose as a base point.

Then $\{[a], [b]\}$ is a basis for the free abelian group $\pi_1(T)$.

Clearly,

$$\text{Ker}(i_* : \pi_1(T) \to \pi_1(H_1)) = \{n[a] : n \in \mathbb{Z}\},$$

$$\text{Ker}(i'_* : \pi_1(T) \to \pi_1(H'_1)) = \{n[b] : n \in \mathbb{Z}\}.$$

Thus $K(\mathcal{M}) = \{0\}$.
Proposition

A H.S. $\mathcal{M} = (M; V, W; S)$ is reducible if and only if $\exists$ an essential s.c.c. $C$ in $S$ s.t. $[C] \in K(\mathcal{M})$. 

Fengling Li

Invariants of 3-manifolds from Heegaard splittings
Theorem (Lei-Wu, 2012)

\[ \mathcal{M}_1 = (M_1; V_1, W_1; S_1), \mathcal{M}_2 = (M_2; V_2, W_2; S_2): \text{two H.S.s}, \]
\[ \mathcal{M} = \mathcal{M}_1 \# S^2 \mathcal{M}_2 = (M; V, W; S). \]
Then there is a short exact sequence of groups

\[ 1 \to \langle [C] \rangle^N \to K(\mathcal{M}) \to K(\mathcal{M}_1) \ast K(\mathcal{M}_2) \to 1, \]
where \( C = S^2 \cap S. \)
Applying above theorem to a stabilized H.S., we have

**Corollary (Lei-Wu, 2012)**

Let $\mathcal{M}' = (M; V', W'; S')$ be an elementary stabilization of the H.S. $\mathcal{M} = (M; V, W; S)$. Then there is a short exact sequence of groups

$$1 \rightarrow \langle [C] \rangle^N \rightarrow K(\mathcal{M}') \rightarrow K(\mathcal{M}) \rightarrow 1,$$

where $C = S^2 \cap S'$. 
Proposition

The intersecting kernel is an invariant of equivalent Heegaard splittings.
Proposition
The intersecting kernel is an invariant of equivalent Heegaard splittings.

Question: Is it an invariant of equivalent manifolds?
Let \( \mathcal{M}' = (S^3; V, W; S) \) be a Heegaard splitting of genus 2 for \( S^3 \), from a direct consequence of the corollary above,

\[
K(\mathcal{M}') \cong \langle [C] \rangle^N,
\]

where \( C \) is a s.c.c. on \( S \), s.t.
\( C \) cuts \( S \) into two once punctured tori and \( C \) bounds disks in both \( V \) and \( W \).
Let $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$ be a H.S. for a closed orientable 3-mfd, and $K = K(\mathcal{M})$ the intersecting kernel.
Let $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$ be a H.S. for a closed orientable 3-mfd, and $K = K(\mathcal{M})$ the intersecting kernel.

Consider the normal subgroup

$$\Lambda(K) = \langle [\alpha] \in K : \alpha \text{ is an essential simple closed curve on } S_g \rangle^N$$

of $K$. We call $\Lambda(K)$ the SCC subgroup of $K$. 
Let $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$ be a H.S. for a closed orientable 3-mfd, and $K = K(\mathcal{M})$ the intersecting kernel.

Consider the normal subgroup

$$\Lambda(K) = \langle [\alpha] \in K : \alpha \text{ is an essential simple closed curve on } S_g \rangle^N$$

of $K$. We call $\Lambda(K)$ the **SCC subgroup** of $K$.

H.S. $\mathcal{M}$ is reducible if and only if $\Lambda(K)$ is non-trivial.
Let $\mathcal{M}' = (M; H_{g+1}, H'_{g+1}; S_{g+1}; \phi')$ be an elementary stabilization of H.S. $\mathcal{M} = (M; H_g, H'_g; S_g; \phi)$ for $M$, where $\phi'$ is a canonical extension of $\phi$. By the previous corollary, there is a surjective homomorphism $h : K(\mathcal{M}') \twoheadrightarrow K(\mathcal{M})$. 
Let $\mathcal{M}' = (M; H_{g+1}, H'_{g+1}; S_{g+1}; \phi')$ be an elementary stabilization of H.S. $\mathcal{M} = (M; H, H'_g; S_g; \phi)$ for $M$, where $\phi'$ is a canonical extension of $\phi$. By the previous corollary, there is a surjective homomorphism $h : K(\mathcal{M}') \rightarrow K(\mathcal{M})$.

Since $\Lambda(K(\mathcal{M})) \subset \Lambda(K(\mathcal{M}'))$, there exists a commutative diagram

\[
\begin{array}{ccc}
K(\mathcal{M}') & \xrightarrow{h} & K(\mathcal{M}) \\
\downarrow{q'} & & \downarrow{q} \\
K(\mathcal{M}')/\Lambda(K(\mathcal{M}')) & \xleftarrow{\rho} & K(\mathcal{M})/\Lambda(K(\mathcal{M}))
\end{array}
\]
In general, set 
\( M^{(0)} = M, \ M^{(1)} = M', \ldots \ldots, \ M^{(n)} = (M^{(n-1)})', \ n \in \mathbb{N}, \)
and \( \rho_i : K(M^{(i)})/\Lambda(K(M^{(i)})) \to K(M^{(i+1)})/\Lambda(K(M^{(i+1)})), \)
\( i = 0, 1, 2, \ldots. \) We have a sequence of surjective homomorphisms

\[
K(M^{(0)})/\Lambda(K(M^{(0)})) \to K(M^{(1)})/\Lambda(K(M^{(1)})) \to \cdots \\
\cdots \to K(M^{(n)})/\Lambda(K(M^{(n)})) \to \cdots
\]

The direct limit \( \lim_{n \to \infty} K(M^{(n)})/\Lambda(K(M^{(n)})) \) is a (in general, non-trivial) group, which is denoted by \( Q_k(M). \)
In general, set
\[ M^{(0)} = M, \ M^{(1)} = M', \ \ldots \ldots, \ M^{(n)} = (M^{(n-1)})', \ n \in \mathbb{N}, \]
and \( \rho_i : K(M^{(i)})/\Lambda(K(M^{(i)})) \rightarrow K(M^{(i+1)})/\Lambda(K(M^{(i+1)})), \)
i = 0, 1, 2, \ldots. We have a sequence of surjective homomorphisms
\[
K(M^{(0)})/\Lambda(K(M^{(0)})) \rightarrow K(M^{(1)})/\Lambda(K(M^{(1)})) \rightarrow \ldots \]
\[
\ldots \rightarrow K(M^{(n)})/\Lambda(K(M^{(n)})) \rightarrow \ldots
\]

The direct limit \( \varinjlim_{n \in \mathbb{N}} K(M^{(n)})/\Lambda(K(M^{(n)})) \) is a (in general, non-trivial) group, which is denoted by \( QK(M) \).
Theorem (Lei-L-Wu, 2015)

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be any two H.S.s of a closed orientable 3-mfd $M$. Then

$$QK(\mathcal{M}_1) \cong QK(\mathcal{M}_2).$$
Main theorem

**Theorem (Lei-L-Wu, 2015)**

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be any two H.S.s of a closed orientable 3-mfd $M$. Then

$$QK(\mathcal{M}_1) \cong QK(\mathcal{M}_2).$$

**Remark**: By the above theorem, for any H.S. $\mathcal{M}$ of a closed orientable 3-manifold $M$, we may define $QK(\mathcal{M})$ to be $QK(\mathcal{M})$. $QK(\mathcal{M})$ is independent on the H.S.s of $M$, therefore defines an invariant of $M$. 

Fengling Li

*Invariants of 3-manifolds from Heegaard splittings*
Further discussion

- Relation with other 3-manifold invariants
- Detect manifold
THANKS FOR YOUR ATTENTION!