On the Lie Algebra of Braid Groups
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On the Lie Algebra of Braid Groups

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Brunnian Braid Groups

A braid $\beta \in B_n(M)$ is called **Brunnian** if (1) it is a pure braid and (2) it becomes trivial braid by removing any of its strands. Since the composition of any two Brunnian braids and the inverse of a Brunnian braid are still Brunnian, the set of Brunnian braids is a normal subgroup of the pure braid group which is denoted by $\text{Brun}_n(M)$. For convenient, $\text{Brun}_n(M)$ is denoted by $\text{Brun}_n$ when $M$ is the disc $D^2$. 
For a group $G$, the descending central series

$$G = \Gamma_1(G) \geq \Gamma_2(G) \geq \cdots \geq \Gamma_i(G) \geq \Gamma_{i+1}(G) \geq \cdots$$

is defined by the formulas

$$\Gamma_1(G) = G, \Gamma_{i+1}(G) = [\Gamma_i(G), G] \quad (i \geq 1).$$

The descending central series of a discrete group $G$ gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$:

$$L(G) = \bigoplus_{q=1}^{\infty} \Gamma_q(G)/\Gamma_{q+1}(G).$$
A presentation of the Lie algebra $L(P_n)$ for the pure braid group was done in the work of T.Kohno, and can be described as follows. It is the quotient of the free Lie algebra $L[A_{i,j} | 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the ”infinitesimal braid relations ” or ”horizontal 4T relation” given by the following three relations:

1. $[A_{i,j}, A_{s,t}] = 0$ if $\{i, j\} \cap \{s, t\} = \emptyset$,
2. $[A_{i,j}, A_{i,k} + A_{j,k}] = 0$ if $i < j < k$,
3. $[A_{i,k}, A_{i,j} + A_{j,k}] = 0$ if $i < j < k$.

Where $A_{i,j}$ denote the projections of the $a_{i,j} \in P_n$ to $L(P_n)$.

Reference

Tow Famous Results about the Lie Algebra of Pure Braid Groups

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Reference

Some Basic Definitions
The Relative Lie Algebra \( L^P(\text{Brun}_n) \)
Current Progress about \( L^P(\text{Brun}_n(S^2)) \)

Brunnian Braid Groups
Lie Algebra from Descending Central Series of Groups
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1. \([A_{i,j}, A_{s,t}] = 0 \) if \( \{i, j\} \cap \{s, t\} = \emptyset \),
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Where \( A_{i,j} \) denote the projections of the \( a_{i,j} \in P_n \) to \( L(P_n) \).

Reference


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On the Lie Algebra of Braid Groups
Tow Famous Results about the Lie Algebra of Pure Braid Groups

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1. $[ A_{i,j}, A_{s,t} ] = 0$ if $\{ i, j \} \cap \{ s, t \} = \emptyset$,
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1. $[A_{i,j}, A_{s,t}] = 0$ if $\{i, j\} \cap \{s, t\} = \emptyset$,
2. $[A_{i,j}, A_{i,k} + A_{j,k}] = 0$ if $i < j < k$,
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Where $A_{i,j}$ denote the projections of the $a_{i,j} \in P_n$ to $L(P_n)$.

Reference

Y. Ihara gave a presentation of the Lie algebra $L(P_n(S^2))$ of the pure braid group of a sphere. It is the quotient of the free Lie algebra $L[B_{i,j} \mid 1 \leq i, j \leq n]$ generated by elements $B_{i,j}$ with $1 \leq i, j \leq n$ modulo the following relations:

$$
\begin{cases}
B_{i,j} = B_{j,i} & \text{for } 1 \leq i, j \leq n, \\
B_{i,i} = 0 & \text{for } 1 \leq i \leq n, \\
[B_{i,j}, B_{s,t}] = 0, & \text{if } \{i, j\} \cap \{s, t\} = \emptyset, \\
\sum_{j=1}^{n} B_{i,j} = 0, & \text{for } 1 \leq i \leq n.
\end{cases}
$$

(1)

Where $B_{i,j}$ denote the projections of the $b_{i,j} \in P_n(S^2)$ to $L(P_n)(S^2)$.

Reference

Definition of the Relative Lie Algebra $L^P(Brun_n(M))$

Since $Brun_n(M)$ is the normal subgroup of the pure braid group $P_n(M)$, we have the following descending central series

$$Brun_n(M) = \Gamma_1(P_n(M)) \cap Brun_n(M) \geq \Gamma_2(P_n(M)) \cap Brun_n(M) \geq \cdots$$

and the relative Lie algebra

$$L^P(Brun_n(M)) = \bigoplus_{q=1}^{\infty} \Gamma_q(P_n(M)) \cap Brun_n(M) / \Gamma_{q+1}(P_n(M)) \cap Brun_n(M).$$
Some Basic Definitions
The Relative Lie Algebra $L^P(B_{un})$
Current Progress about $L^P(B_{un}(S^2))$

Property of $L^P(B_{un})$
Free Generators of the Relative Lie Algebra $L^P(B_{un})$
The Symmetric Bracket Sum of Ideals
The Rank of $L^P_q(B_{un})$

Property of $L^P(B_{un})$

**Proposition**

(Proposition 1)

$$L^P(B_{un}) = \bigcap_{k=1}^{n} \ker(d_k : L(P_n) \to L(P_{n-1})).$$

**Remarks.**

**Remark**

The relative Lie algebra $L^P(B_{un})$ has better features: (1) it is freely generated; (2) it is of finite type; (3) it has connection to the theory of Vassiliev invariants.
**Proposition**

(Proposition 1)

\[ L^P(\text{Brun}_n) = \bigcap_{k=1}^{n} \ker(d_k : L(P_n) \rightarrow L(P_{n-1})). \]

**Remarks.**

**Remark**

The relative Lie algebra \( L^P(\text{Brun}_n) \) has better features: (1) it is freely generated; (2) it is of finite type; (3) it has connection to the theory of Vassiliev invariants.
### Definition of $\mathcal{K}(n)_k$

We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_1,n, A_2,n, \cdots, A_{n-1},n\}$.

2) Suppose that $\mathcal{K}(n)_{k+1}$ is defined as a subset of Lie monomials on the letters $A_1,n, A_2,n, \cdots, A_{n-1},n$ with $k < n$. Let

$$A_k = \{ W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries} \}.$$ 

Define

$$\mathcal{K}(n)_k = \{ W' \text{ and } [\cdots [[W', W_1], W_2], \ldots, W_t] \}$$

for $W' \in \mathcal{K}(n)_{k+1} \setminus A_k$ and $W_1, W_2, \ldots, W_t \in A_k$ with $t \geq 1$. 

Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_1,n, A_2,n, \cdots, A_{n-1},n$. 

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**On the Lie Algebra of Braid Groups**
Definition of $\mathcal{K}(n)_k$

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1) Let $\mathcal{K}(n)_n = \{ A_{1,n}, A_{2,n}, \ldots, A_{n-1,n} \}$.

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Define

$$\mathcal{K}(n)_k = \{ W' \text{ and } \cdots [[W', W_1], W_2], \ldots, W_t] \}$$

for $W' \in \mathcal{K}(n)_{k+1} \setminus A_k$ and $W_1, W_2, \ldots, W_t \in A_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$.
Free Generators of Lie Algebra $L^P(Brun_n)$

**Theorem**

(Theorem 2) The relative Lie algebra $L^P(Brun_n)$ is a free Lie algebra generated by $K(n)_1$ as a set of free generators.
Example

Let \( n = 4 \). The set \( \mathcal{K}(4)_1 \) is constructed by the following steps:

1) \( \mathcal{K}(4)_4 = \{ A_{1,4}, A_{2,4}, A_{3,4} \} \).

2) \( \mathcal{K}(4)_3 = \{ [[A_{3,4}, A_{j_1,4}], \cdots , A_{j_t,4}] \mid 1 \leq j_1, \cdots , j_t \leq 2, \ t \geq 0 \} \), where, for \( t = 0 \), \( [[A_{3,4}, A_{j_1,4}], \cdots , A_{j_t,4}] = A_{3,4} \).

3) For constructing \( \mathcal{K}(4)_2 \), let \( W = [[A_{3,4}, A_{j_1,4}], \cdots , A_{j_t,4}] \in \mathcal{K}(4)_3 \). If \( W \) does not contain \( A_{2,4} \), then \( W = A_{3,4} \) or \( W = [[A_{3,4}, A_{1,4}], \cdots , A_{1,4}] \). Let

\[
\text{ad}^t(b)(a) = [[a, b], \cdots , b]
\]

with \( t \) entries of \( b \), where \( \text{ad}^0(b)(a) = a \). Then \( W \) does not contain \( A_{2,4} \) if and only if

\[
W = \text{ad}^t(A_{1,4})(A_{3,4})
\]

for \( t \geq 0 \).
Example

Let $n = 4$. The set $\mathcal{K}(4)_1$ is constructed by the following steps:

1) $\mathcal{K}(4)_4 = \{A_{1,4}, A_{2,4}, A_{3,4}\}$.

2) $\mathcal{K}(4)_3 = \{[[A_{3,4}, A_{j_1,4}], \cdots , A_{j_t,4}] | 1 \leq j_1, \cdots , j_t \leq 2, \ t \geq 0\}$, where, for $t = 0$, $[[A_{3,4}, A_{j_1,4}], \cdots , A_{j_t,4}] = A_{3,4}$.

3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \cdots , A_{j_t,4}] \in \mathcal{K}(4)_3$. If $W$ does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [[A_{3,4}, A_{1,4}], \cdots , A_{1,4}]$. Let $\text{ad}^t(b)(a) = [[a, b], \cdots , b]$ with $t$ entries of $b$, where $\text{ad}^0(b)(a) = a$. Then $W$ does not contain $A_{2,4}$ if and only if

$W = \text{ad}^t(A_{1,4})(A_{3,4})$

for $t \geq 0$. 

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On the Lie Algebra of Braid Groups
Example

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2) \( \mathcal{K}(4)_3 = \{ [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \mid 1 \leq j_1, \cdots, j_t \leq 2, \ t \geq 0 \} \),
   where, for \( t = 0 \), \( [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] = A_{3,4} \).

3) For constructing \( \mathcal{K}(4)_2 \), let \( W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \in \mathcal{K}(4)_3 \). If \( W \) does not contain \( A_{2,4} \), then \( W = A_{3,4} \) or \( W = [[A_{3,4}, A_{1,4}], \cdots, A_{1,4}] \). Let

   \[
   \text{ad}^t(b)(a) = [[a, b], \cdots, b]
   \]

   with \( t \) entries of \( b \), where \( \text{ad}^0(b)(a) = a \). Then \( W \) does not contain \( A_{2,4} \) if and only if

   \[
   W = \text{ad}^t(A_{1,4})(A_{3,4})
   \]

   for \( t \geq 0 \).
From the definition, $\mathcal{K}(4)_2$ is given by

$$[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}]$$

and

$$[[[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}], \text{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \text{ad}^{s_q}(A_{1,4})(A_{3,4})],$$

where $1 \leq j_1, \cdots, j_t \leq 2$ with at least one $j_i = 2$, $s_1, \cdots, s_q \geq 0$ and $q \geq 1$.

4) For constructing $\mathcal{K}(4)_1$, let $W$ denote

$$[[[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}], \text{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \text{ad}^{s_q}(A_{1,4})(A_{3,4})] \in \mathcal{K}(4)_2,$$

where, for $q = 0$, $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}]$. Then $W$ does not contain $A_{1,4}$ if and only if $q = 0$ and $W = [[[A_{3,4}, A_{2,4}], \cdots, A_{2,4}]]$, namely

$$W = \text{ad}^t(A_{2,4})(A_{3,4})$$

for $t \geq 1$. 
Thus $\mathcal{K}(4)_1$, which is a set of free generators for $L^P(Brun_4)$, is given by

$$W \text{ and } [[W, \text{ad}^{l_1}(A_{2,4})(A_{3,4})], \cdots, \text{ad}^{l_p}(A_{2,4})(A_{3,4})],$$

where $l_i \geq 1$ for $1 \leq i \leq p$ with $p \geq 1$ and

$$W = [[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}], \text{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \text{ad}^{s_q}(A_{1,4})(A_{3,4})],$$

so that each of $A_{2,4}$ and $A_{1,4}$ appears in $W$ at least once.
Let $L$ be a lie algebra and $I_1, \cdots, I_n$ are its ideals. The fat bracket sum $[[I_1, I_2, \cdots, I_n]]$ of these ideals is defined to be the sub Lie ideal of $L$ generated by all of the commutators $\beta^t(a_{i_1}, \cdots, a_{i_t})$, where

1) $1 \leq i_s \leq n$;
2) $\{i_1, \cdots, i_t\} = \{1, \cdots, n\}$, that is each integer in $\{1, 2, \cdots, n\}$ appears as at least one of the integers $i_s$;
3) $a_j \in I_j$;
4) $\beta^t$ runs over all of the bracket arrangements of weight $t$ (with $t \geq n$).
The Symmetric Bracket Sum of Ideals

Let $L$ be a Lie algebra and $I_1, \ldots, I_n$ are its ideals. The fat bracket sum $[[I_1, I_2, \ldots, I_n]]$ of these ideals is defined to be the sub Lie ideal of $L$ generated by all of the commutators

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2) $\{i_1, \cdots, i_t\} = \{1, \cdots, n\}$, that is each integer in $\{1, 2, \cdots, n\}$ appears as at least one of the integers $i_s$;

3) $a_j \in I_j$;

4) $\beta^t$ runs over all of the bracket arrangements of weight $t$ (with $t \geq n$).
The symmetric bracket sum of these ideals is defined as

\[ [l_1, \ldots, l_l]_s := \sum_{\sigma \in \Sigma_n} [[l_{\sigma(1)}, l_{\sigma(2)}], \cdots, l_{\sigma(n)}], \]

where \( \Sigma_n \) is the symmetric group of degree \( n \).
The symmetric bracket sum of these ideals is defined as

\[ [l_1, \ldots, l_l]_S := \sum_{\sigma \in \Sigma_n} [[l_{\sigma(1)}, l_{\sigma(2)}], \ldots, l_{\sigma(n)}], \]

where \( \Sigma_n \) is the symmetric group of degree \( n \).

**Lemma**

*(Lemma 3)* Let \( l_j \) be any Lie ideals of a Lie algebra \( L \) with \( 1 \leq j \leq n \). Then

\[ [[l_1, l_2, \cdots, l_n]] = [[[l_1, l_2], \cdots, l_n]]_S. \]
Let us denote the ideal

\[ L[A_{k,n}, \ldots [A_{k,n}, A_{j_1,n}], \ldots , A_{j_m,n}] \mid j_i \neq k, n; i \leq m; m \geq 1 \]

by \( I_k \). Then we have the following theorem.

**Theorem 4**

\[ L^P(Brun_n) = [[I_1, I_2], \ldots , I_{n-1}]_{s} \]
Let us denote the ideal

\[ L[A_{k,n}, [ \ldots [A_{k,n}, A_{j_1,n}], \ldots , A_{j_m,n}] \mid j_i \neq k, n; i \leq m; m \geq 1] \]

by \( I_k \). Then we have the following theorem.

**Theorem**

*Theorem 4*

\[ L^P(\text{Br}u_n) = [[l_1, l_2], \ldots , l_{n-1}]_S. \]
Proof

It is evident that the symmetric bracket sum \([[[l_1, l_2], \cdots, l_{n-1}]]_S\) lies in the kernels of all \(d_i\). On the other hand, from lemma 3 and theorem 2, \(L^P(\text{Brun}_n)\) is given as “fat bracket sum” of \(l_1, \cdots, l_{n-1}\) because each element in \(K(n)_1\) is a Lie monomial containing each of \(A_{1,n}, \cdots, A_{n-1,n}\). we know that

\[
K(n)_1 \subseteq [[[l_1, \cdots, l_{n-1}]]] = [[[l_1, l_2], \cdots, l_{n-1}]]_S.
\]
Proposition

(Proposition 5) There is a decomposition

\[ L_q(P_n) = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} d^{i_k} d^{i_{k-1}} \cdots d^{i_1} (L^P_{q}(\text{Brun}_{n-k})) \]

for each \( n \) and \( q \).
Proposition

(Proposition 6) There is a formula

$$\text{rank}(L_q(P_n)) = \sum_{k=0}^{n-1} \binom{n}{k} \text{rank}(L^P_q(\text{Brun}_{n-k}))$$

for each $n$ and $q$. 
Let $b_q(P_n) = \text{rank}(L_q(P_n))$ and $b^P_q(\text{Brun}_n) = \text{rank}(L^P_q(\text{Brun}_n))$. We have

$$
\begin{pmatrix}
  b_q(P_n) \\
  b_q(P_{n-1}) \\
  b_q(P_{n-2}) \\
  \vdots \\
  b_q(P_1)
\end{pmatrix} = 
\begin{pmatrix}
  1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\
  0 & 1 & \binom{n-1}{1} & \cdots & \binom{n-1}{n-2} \\
  0 & 0 & 1 & \cdots & \binom{n-2}{n-3} \\
  \vdots & \vdots & \vdots & \cdots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
  b^P_q(\text{Brun}_n) \\
  b^P_q(\text{Brun}_{n-1}) \\
  b^P_q(\text{Brun}_{n-2}) \\
  \vdots \\
  b^P_q(\text{Brun}_1)
\end{pmatrix}.
$$
Let $A_n$ be the coefficient matrix of the above linear equations. Then

$$A_n^{-1} = \begin{pmatrix}
1 & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \cdots & (-1)^{n-1}\binom{n}{n-1} \\
0 & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \cdots & (-1)^{n-2}\binom{n-2}{n-1} \\
0 & 0 & 1 & -\binom{n-2}{1} & \cdots & (-1)^{n-3}\binom{n-3}{n-2} \\
0 & 0 & 0 & 1 & \cdots & (-1)^{n-4}\binom{n-4}{n-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}$$
Theorem

(Theorem 7)

\[
\text{rank}(L^P_{q}(\text{Brun}_n)) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \text{rank}(L_q(P_{n-k}))
\]

for each \(n\) and \(q\), where \(P_1 = 0\) and, for \(m \geq 2\),

\[
\text{rank}(L_q(P_m)) = \frac{1}{q} \sum_{k=1}^{m-1} \left( \sum_{d \mid q} \mu(d) k^{q/d} \right)
\]

with \(\mu\) the Möbius function.
The removing-strand operation on braids induces an operation

\[ d_k : L(P_n(S^2)) \rightarrow L(P_{n-1}(S^2)) \].

**Proposition**

*(Proposition 8)* There is an inclusion of Lie algebras

\[ L^P(Brun_n(S^2)) \subset \bigcap_{i=1}^{n} \ker (d_i : L(P_n(S^2)) \rightarrow L(P_{n-1}(S^2))). \]
The Homotopy Group of a Lie Algebra

Let \( L = \{L_n\}_{n \geq 0} \) denote a simplicial Lie algebra with faces \( d_i : L_n \rightarrow L_{n-1} \) for \( 0 \leq i \leq n \). The Moore complex \( N(L) = \{N_n(L)\}_{n \geq 0} \) of \( L \) is defined by

\[
N_n(L) = \bigcap_{i=1}^{n} \ker(d_i : L_n \rightarrow L_{n-1}).
\]

Then \( N(L) \) with \( d_0 \) is a chain complex of Lie algebra. The Moore cycle and Moore boundary of \( L \) are defined by

\[
Z_n(L) = \ker(d_0 : N_n(L) \rightarrow N_{n-1}(L)) = \bigcap_{i=0}^{n} \ker(d_i : L_n \rightarrow L_{n-1}),
\]

and

\[
B_n(L) = d_0(N_{n+1}(L))
\]

respectively. The \( n \)th homotopy group is defined to be the quotient of

\[
\pi_n(L) = Z_n(L) / B_n(L).
\]
Lie Algebra $L(\hat{F})$

Let $\hat{F}_{n+1}$ be the quotient of the free group $F(x_0, x_1, \ldots, x_n)$ subject to the single relation $x_0x_1\cdots x_n = 1$. Let $\hat{x}_j$ be the image of $x_j$ in $\hat{F}_{n+1}$. The group $\hat{F}_{n+1}$ is written $\hat{F}(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_n)$ in case the generators $\hat{x}_j$ are used. Clearly

$$\hat{F}_n \cong F(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{n-1})$$

is a free group of rank $n$. Define the faces $\hat{d}_i : \hat{F}_{n+1} \to \hat{F}_n$ and degeneracies $\hat{s}_i : \hat{F}_n \to \hat{F}_{n+1}$ on $\hat{F} = \{\hat{F}_{n+1}\}_{n \geq 0}$ as follows:

\[
\begin{align*}
\hat{d}_i\hat{x}_j &= \begin{cases} 
\hat{x}_j, & \text{if } j < i, \\
1, & \text{if } j = i, \\
\hat{x}_{j-1}, & \text{if } j > i.
\end{cases} \\
\hat{s}_i\hat{x}_j &= \begin{cases} 
\hat{x}_j, & \text{if } j < i, \\
\hat{x}_j\hat{x}_{j+1}, & \text{if } j = i, \\
\hat{x}_{j+1}, & \text{if } j > i.
\end{cases}
\end{align*}
\]
It is straightforward to check that the sequence of groups \( \hat{F} = \{ \hat{F}_{n+1} \}_{n \geq 0} \) is simplicial group under \( \hat{d}_i \) and \( \hat{s}_i \) defined as above.

Let \( L(\hat{F}) = \{ L(\hat{F}_{n+1}) \}_{n \geq 0} \) denote the free simplicial Lie algebra generated by \( \hat{F} \).
The Intersection of the Kernel
\[ d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2)) \]

**Proposition**

*(Proposition 9)* The intersection of the kernel
\[ d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2)) \] is the Moore cycle of \( L(\hat{F}) \), i.e.

\[
\bigcap_{i=1}^{n+1} \ker(d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2))) = \bigcap_{i=0}^{n-1} \ker(\hat{d}_i : L(\hat{F}_n) \to L(\hat{F}_{n-1})).
\]
Let us denote the ideal

\[ L[B_{k,n+1}, \cdots [B_{k,n+1}, B_{j_1,n+1}], \cdots , B_{j_m,n+1}] | j_i \neq k, n+1; i \leq m; m \geq 1 \]

by \( J_k \). Then we have the following theorem.

**Proposition**

(Proposition 10) For \( n \geq 4 \), there is an isomorphism of groups:

\[
\bigcap_{i=1}^{n+1} \ker(d_i : L(P_{n+1}(S^2)) \rightarrow L(P_n(S^2)))/[[J_1, J_2], \cdots , J_{n-1}] = \pi_{n-1}(L(\widehat{F})) \cong \pi_{n-1}(L(F[S^1])) \cong \pi_{n-1}(L(G(S^2))).
\]

**Remarks.**

**Remark**

\( \pi_{n-1}(L(G(S^2))) \) can be computed by using \( \wedge \)-algebra.
Let us denote the ideal

\[ L[B_{k,n+1}, \ldots [B_{k,n+1}, B_{j_1,n+1}], \ldots , B_{j_m,n+1}] | j_i \neq k, n+1; i \leq m; m \geq 1 \]

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\cong \pi_{n-1}(L(\hat{F})) \cong \pi_{n-1}(L(F[S^1])) \cong \pi_{n-1}(L(G(S^2))).
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**Proposition**

(Proposition 10) For \( n \geq 4 \), there is an isomorphism of groups:

\[ \bigcap_{i=1}^{n+1} \ker(d_i : L(P_{n+1}(S^2))) \to L(P_n(S^2))/[[J_1, J_2], \cdots, J_{n-1}]S \]

\[ \cong \pi_{n-1}(L(\hat{F})) \cong \pi_{n-1}(L(F[S^1]) \cong \pi_{n-1}(L(G(S^2)). \]

**Remarks.**

**Remark**

\( \pi_*(L(G(S^2)) \) can be computed by using \( \wedge \)-algebra.
Thanks for your attention!