The fixed point data and equivariant Chern numbers

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- Preliminaries.

- Main Result.

- Proof.
G-manifold and G-map

NOTE: All the mfds and maps are smooth.

- **G-manifold** $M$:
  \[
  \theta : G \times M \to M
  \]
  satisfying:
  1. $e \in G$, $\theta(e, x) = x$,
  2. $g_1, g_2 \in G$, $\theta(g_1 g_2, x) = \theta(g_1, \theta(g_2, x))$.

- **G-map** $f$:
  \[
  f : M \to N
  \]
  satisfying the following square commute,

\[
\begin{array}{ccc}
G \times M \xrightarrow{(id,f)} & G \times N \\
\downarrow \theta & & \downarrow \theta \\
M \xrightarrow{f} & N
\end{array}
\]
**G-(complex) bundle**

Bundle $\xi : E \to M$ satisfied:

1. $M$ is a $G$-manifold,
2. $G$ acts linearly on the fibres (i.e. $g \in G$, $g : E_x \to E_{gx}$ is a $G$-(complex) linear map)

**G-unitary manifold**

$M$ is a $G$-manifold and

$$\tau(M) \oplus \mathbb{R}^l \to M$$

is a $G$-complex bundle for some $l$. Where $\mathbb{R}^l$ is trivial $G$-bundle with trivial $G$-action on the fibres $\mathbb{R}^l$. 
Fixed Point Set

fixed points: \( M^G = \{ m \in M \mid gm = m, \ \forall g \in G \} \).

Lemma

\( M \) is a \( G \)-manifold and \( M^G = \bigsqcup_F F \), where \( F \) be the connected component of the fixed point set:

1. \( F \) is a closed manifold with trivial \( G \) action,
2. \( \nu_{F,M} \) is a \( G \)-bundle without trivial summand.

Lemma

\( M \) is a \( G \)-unitary manifold and \( M^G = \bigsqcup_F F \), where \( F \) be the connected component of the fixed point set:

1. \( F \) is an unitary manifold with trivial \( G \) action,
2. \( \nu_{F,M} \) is a \( G \)-complex bundle without trivial summand.
Fixed Point Data

We focus on the following two cases.

- $G = \mathbb{Z}_2^k$-manifold $M$ and the fixed point set $M^G = \bigsqcup_F F$. \{\nu_{F,M} \to F\} is called the Fixed Point Data of $M$.

- $G = T^k$-unitary manifold $M$ and the fixed point set $M^G = \bigsqcup_F F$. \{\nu_{F,M} \to F\} is called the Fixed Point Data of the unitary manifold $M$. 
Realising fixed point data

Question

For given a family $\mathbb{Z}_2^k$-bundle (or $T^k$-complex bundle)

$$\{\nu_F \to F\},$$

find necessary and sufficient conditions for the existence of a $\mathbb{Z}_2^k$-manifold ($T^k$-unitary manifold) with the given fixed point data.
Bordism Ring

- **Unoriented Bordism Ring:** \( \Omega_*^O = \sum \Omega_n^O \)

\[ \Omega_n^O = \{ \text{n-dim closed mfds} \}/\sim. \]

- **Unitary Bordism Ring:** \( \Omega_*^U = \sum \Omega_n^U \)

\[ \Omega_n^U = \{ \text{n-dim closed unitary mfds} \}/\sim. \]
geometric $G$-equivariant Bordism Ring

For given $G$-mfd $M_1$, $M_2$, we can define $G \bowtie M_1 \times M_2$.

- geometric unoriented $\mathbb{Z}_2^k$-equivariant Bordism Ring:

  $\Omega^{O,\mathbb{Z}_2^k}_* = \sum \Omega^{O,\mathbb{Z}_2^k}_n$

  $\Omega^{O,\mathbb{Z}_2^k}_n = \{n\text{-dim }\mathbb{Z}_2^k \text{ closed mfds}\}/\sim_{\mathbb{Z}_2^k}$.

- geometric unitary $T^k$-equivariant Bordism Ring:

  $\Omega^{U,T^k}_* = \sum \Omega^{U,T^k}_n$

  $\Omega^{U,T^k}_n = \{n\text{-dim }T^k \text{ closed unitary mfds}\}/\sim_{T^k}$. 
Cobordism theory and Characteristic numbers

- $\Omega_*^O \leftrightarrow$ Stiefel-Whitney numbers.
- $\Omega_*^U \leftrightarrow$ Chern numbers.
equivariant Stiefel-Whitney class and number

\[ \pi : EG \to BG \text{ is the universal principal } G\text{-bundles.} \]

The Borel construction gives us \( EG \times_G \tau_M \) over \( EG \times_G M \).

- **\( G \) equivariant Stiefel-Whitney class**

  \[ w^G(M) := w(EG \times_G \tau_M). \]

- **\( G \) equivariant Stiefel-Whitney number**

  The constant map gives \( p! : H^*_G(M, \mathbb{Z}_2) \to H^*(BG, \mathbb{Z}_2) \).

  Then

  \[ w^G_\omega[M] := p!(w^G_\omega(M)). \]
equivariant Chern class and number

\[ \pi : EG \to BG \] is the universal principal G-bundles. The Borel construction gives us \( EG \times_G \tau_M \) over \( EG \times_G M \).

**G equivariant Chern class**

\[ c^G(M) := c(EG \times_G \tau_M). \]

**G equivariant Chern number**

The constant map gives \( p! : H^*_G(M) \to H^*(BG) \). Then

\[ c^G_\omega[M] := p!(c^G_\omega(M)) \]
Equivariant case

- $\Omega^O_*, \mathbb{Z}_2^k \leftrightarrow \mathbb{Z}_2^k$-equivariant Stiefel-Whitney numbers. (tom Dieck in 1971 Inventiones math)

- $\Omega^U_*, T^k \leftrightarrow T^k$-equivariant Chern number. (Guillemin-Ginzburg-Karshon’s conjecture, answered by Lü-Wang)
Unoriented $\mathbb{Z}_2^k$-manifold with only isolated fixed-points

Unoriented $\mathbb{Z}_2^k$-manifold with only isolated fixed-points:

**Theorem (tom Dieck)**

For given $G = \mathbb{Z}_2^k$ representation $W^1, \ldots, W^s$, they are the fixed point data of a closed $G$-manifold if and only if for any symmetric homogeneous polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_2$,

$$\sum_{i=1}^{s} \left( \frac{f(x_1^r, \ldots, x_n^r)}{x_1^r \cdots x_n^r} \right) \in H^*(BG, \mathbb{Z}_2).$$

where $W^r = \bigoplus_{i=1}^{n} W_i^r$ and $x_i^r = w_1^G(W_i^r)$. 
Unoriented $\mathbb{Z}_2$-case:

**Theorem (Stong and Kosniowski)**

*For given $\mathbb{Z}_2$-bundle $\{\nu_{F}^{n-r} \to F^r\}$, they are the fixed point data of a $\mathbb{Z}_2$-manifold, if and only if for any symmetric polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_2$ of degree at most $n$,

$$\sum_{r} \frac{f(1 + y_1, \ldots, 1 + y_{n-r}, z_1, \ldots, z_r)}{\prod(1 + y_i)} [F] = 0,$$

where $w(F^r) = \prod(1 + z_i)$ and $w(\nu_F) = \prod(1 + y_i) \in H^*(F; \mathbb{Z}_2)$.***
Remark

$p$ is an isolated fixed point of the unitary $T^k$-manifold $M$. The normal bundle $\nu_{p,M}$ has two orientations:

1. induced by the orientation of $M$ which comes from the unitary structure
2. induced by the orientation of $\nu_{p,M}$.

Then we can define the sign of the isolated fixed point $p$:

$$\zeta(p) := \begin{cases} +1, & \text{two orientations are same,} \\ -1, & \text{otherwise.} \end{cases}$$
Question (B, P and R in [Toric genera])

For any set of signs $\zeta(x)$ and complex representation $W_x$, and necessary and sufficient conditions for the existence of a tangentially stably complex $T^k$ manifold with the given fixed point data.
Preliminaries.

Main Result.

Proof.
Unitary $G = T^k$-manifold

$n$ is even.

1. isolated fixed points $x$, $n/2$-dim complex representation $W_x$, and the sign $\zeta(x)$.

2. $G = T^k$ complex bundle $\nu_F \to F$ and $dim F = r > 0$, $\nu_F$ is $(n - r)/2$-dim $G$-complex bundle over $F$.
   ($l$ is large enough to stabilize all the $\tau_F$.)
**Theorem**

They are the fixed point data of an unitary $G$-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}$ in $(n + l)/2$ variables,

$$
\sum_{F} \frac{f(y, z)}{\prod y_i} [F] + \sum_{x} \zeta(x) \frac{f(u)}{\prod u_i} \in H^{*}(BG; \mathbb{Z}),
$$

where $c^{G}(F) = \prod(1 + z_i)$, $c^{G}(\nu_{F}) = \prod(1 + y_j)$ and $c^{G}(W_{x}) = \prod(1 + u_{i})$. 
unitary $T^k$-manifold with only isolated points

$n$ is even, $W^1, \ldots, W^s$ are the $n/2$-dim complex representations and $\zeta(r)$ is the sign of $W^r$.

**Remark**

They are the fixed point data of an unitary $G$-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}$ in $n/2$ variables,

$$
\sum_{i=1}^{s} \zeta(r) \frac{f(x_1^r, \ldots, x_n^r)}{x_1^r \cdots x_n^r} \in H^*(BG),
$$

where $W^r = \bigoplus_{i=1}^{n} W_i^r$, and $x_i^r = c_1^G(W_i^r)$. 
unoriented $G = \mathbb{Z}_2^k$-equivariant case

For given $\{\nu_F \rightarrow F\}$,

**Theorem**

They are the fixed point data of a $G$-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}_2$

$$
\sum_{F} \frac{f(y, z)}{\prod y_i} [F] \in H^*(BG; \mathbb{Z}_2),
$$

where

$$
G^*(F) = \prod (1 + z_i) \in H^*_G(F; \mathbb{Z}_2),
$$

$$
G^*(\nu_F) = \prod (1 + y_i) \in H^*_G(F; \mathbb{Z}_2).
$$
Preliminaries.

Main Result.

Proof.
Recall $G = \mathbb{Z}_2^k$ case

Theorem (tom Dieck)

\[ \Omega_*^{O, G} \xrightarrow{PT} MO_G^* \xrightarrow{\alpha} MO^*(BG) \xrightarrow{B} H_G^*(BG) \otimes \mathbb{Z}_2[[a]]. \]

All the maps are injective.
Thom class and Euler class

\( \xi : E \to X \) is \( G \)-vector bundle. There is the \( G \)-equivariant classifying map:

\[
\begin{array}{ccc}
E & \to & EO_G(n) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & BO_G(n)
\end{array}
\]

- **Thom class:**

\[
t(\xi) := \lim \{ Th(f) : Th(\xi) \to MO_G(n) \} \in MO^n_G(Th(\xi))
\]

- **Euler class:**

\[
e(\xi) := s^*(t(\xi)) \in MU^n_G(X)
\]
Theorem (tom Dieck)

\[ G = \mathbb{Z}_2^k \text{ equivariant cobordism theory:} \]

\[
\begin{array}{ccccccc}
\Omega_{*}^{O,G} & \xrightarrow{PT} & MO_{G}^{*} & \xrightarrow{\alpha} & MO^{*}(BG) & \xrightarrow{B} & H^{*}(BG) \otimes \mathbb{Z}_2[[a]] \\
\downarrow{\Lambda} & & \downarrow{\Lambda'} & & \downarrow{\Lambda''} & & \downarrow{\Lambda'''} \\
\Gamma^{*} & \xrightarrow{\Psi \circ \iota} & S^{-1}MO_{G}^{*} & \xrightarrow{S^{-1}\alpha} & S^{-1}MO^{*}(BG) & \xrightarrow{S^{-1}B} & S^{-1}(H^{*}(BG) \otimes \mathbb{Z}_2[[a]]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
coker\Lambda & \xrightarrow{\phantom{\text{S^{-1}B}}} & coker\Lambda' & \xrightarrow{\phantom{\text{S^{-1}B}}} & coker\Lambda'' & \xrightarrow{\phantom{\text{S^{-1}B}}} & coker\Lambda''' \\
\end{array}
\]

\[ \Lambda, \ \Lambda', \ \Lambda'', \ \Lambda''' \text{ and all the horizontal maps are injective.} \]
\[ G = T^k \text{ equivariant cobordism} \]

**Theorem (tom Dieck, Sinha, Hanke)**

\[
\begin{array}{cccc}
\Omega^U,G & \xrightarrow{PT} & MU_G^* & \xrightarrow{\alpha} MU^*(BG) \\
\downarrow \Lambda & & \downarrow \Lambda' & \\
\Gamma^* & \xrightarrow{\Psi \circ \iota} & S^{-1}MU_G^* & \xrightarrow{S^{-1}\alpha} S^{-1}MU^*(BG) \\
\end{array}
\]

*The diagram is a pull back square. And*

\[
S^{-1}MU^*_G \cong MU_*(B) \otimes \mathbb{Z}[e_1(V)^{-1}, e_1(V)]
\]

\[
\Gamma = MU_*(B) \otimes \mathbb{Z}[e_1(V)^{-1}] \subset S^{-1}MU^*_G.
\]
• \( \eta : E \to F \) is a \( G \) equivariant complex bundle where \( F \) is compact unitary manifold without boundary with trivial \( G \) action.

By using Segal’s theorem:

\[
\Gamma(F, \eta) \in \Gamma_*. 
\]

• For \([M]_G \in \Omega_{*,G}^{U,G} \), \( \{ \nu_{F,M} \to F \} \) is the fixed point data of \( M \).

\[
\Lambda([M]_G) := \sum_F \Gamma(F, \nu_{F,M}) \in \Gamma_.
\]
Unitary \( G = T^k \)-equivariant cobordism theory:

**Theorem**

\[
\begin{align*}
\Omega^*_{U,G} & \xrightarrow{PT} M^*_U \\
\Lambda & \downarrow{} \\
\Gamma^* & \xrightarrow{\Psi\circ\iota} S^{-1} M^*_U \\
\Lambda & \downarrow{} \\
\text{coker}\Lambda & \rightarrow \\
\end{align*}
\]

\[
\begin{align*}
M^*_G & \xrightarrow{\alpha} M^*(BG) \\
\Lambda' & \downarrow{} \\
S^{-1} M^*_G & \xrightarrow{S^{-1}\alpha} S^{-1} M^*(BG) \\
\Lambda'' & \downarrow{} \\
S^{-1}(H^*(BG) \otimes \mathbb{Z}[[a]]) & \rightarrow \\
\Lambda''' & \downarrow{} \\
\text{coker}\Lambda & \rightarrow \text{coker}\Lambda' \rightarrow \text{coker}\Lambda'' \rightarrow \text{coker}\Lambda''' \rightarrow
\end{align*}
\]

\( \Lambda, \Lambda', \Lambda'', \Lambda''' \) and all the horizontal maps are injective.
$T^k$ equivariant Chern numbers

\[ \Omega^U_G \xrightarrow{PT} MU_G^* \xrightarrow{\alpha} MU^*(BG) \xrightarrow{B} H^*(BG) \otimes \mathbb{Z}[[a]] \]

Proposition

Denote $c^G(M) = \prod (1 + x_i)$ then

\[ B \cdot \alpha \cdot PT([M]_G) = \sum_{\omega} S_{\omega} [M] b^\omega, \]

where

\[ (1 + b_1 t + b_2 t^2 + \cdots) \cdot (1 + a_1 t + a_2 t^2 + \cdots) = 1. \]
Theorem

\( \nu_F \to F \) as above,

\[ S^{-1}(B \circ \alpha) \circ \Psi \circ \iota(\Gamma(F, \nu_F)) = \frac{\overline{v}(\nu_F) \cdot \overline{v}(\tau_F)}{e^G(\nu_F)}. \]
Then there must be an unitary $G$-manifold $M$ satisfying

$$\Lambda([M]_G) = \sum_F \Gamma(F, \nu_F).$$
If $\nu_F \to F$ and $\nu_{F'} \to F'$ are $r$-dim complex $G$ bundle, 

$$\Gamma(F, \nu_F) = -\Gamma(F', \nu_{F'}) \in \Gamma_*$$

and $\dim F = \dim F' = s$.

Lemma

There must be a $(s + 2r)$-dim unitary $G$-manifold $M_F$ which $M_F \sim_G 0$, and $\{\nu_F \to F, \nu_{F'} \to F'\}$ are the fixed point data of $M_F$. 
Theorem

\{ \nu_F \to F \} is the fixed point data of an unitary \( G \)-manifold if and only if

\[
\sum_{F} \frac{\overline{\nu}(\nu_F) \cdot \overline{\nu}(\tau_F)}{e^G(\nu_F)} [F] \in H^*(BG) \otimes \mathbb{Z}[[a]].
\]
Thank You!