The topology and geometry of the moment-angle manifolds

Liman Chen
Joint work with Feifei Fan and Xiangjun Wang
(School of Mathematical Sciences, Nankai University)

Aug 17, 2015, National University of Singapore
1. Topology of moment-angle manifolds
2. Positive Ricci curvature of moment-angle manifolds

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**Definition**

*K* is a simplicial $n - 1$ complex. Let $[m] = \{1, \ldots, m\}$ represent the $m$ vertices of the simplicial complex, $\sigma$ be a simplex in the complex $K$, $|\sigma|$ is the number of the vertices of $\sigma$. Define

$$D^{2|\sigma|}_{\sigma} \times \hat{T}_\sigma = \{(z_1, z_2, \cdots, z_m) \in (D^2)^m : |z_j| = 1 \text{ for } j \notin \sigma\}.$$

and define $Z_K$ corresponding to $K$ as

$$Z_K = \bigcup_{\sigma \in K} D^{2|\sigma|}_{\sigma} \times \hat{T}_\sigma \subset (D^2)^m.$$
1. A conjecture and the idea of the proof

We want to study the topology of the moment-angle manifold. Method: considering the change of $Z(K)$ after we do some operation on $K$. One operation is making connected sum of $K$ with $\partial \Delta^n$. 
For this case, S.Gitler and S.Lopez conjecture:

**Conjecture**

Let $K$ be a simplicial $n-1$ sphere with $m$ vertices and $K_\sigma = K \# \sigma \partial \Delta^n$. Let $Z$ and $Z_\sigma$ be the corresponding moment-angle manifolds, then $Z_\sigma$ is homeomorphic to

$$\partial \left[ \left( Z - D_{n+m}^\circ \right) \times D^2 \right] \# \bigg\#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1}).$$

If $K$ is dual to the boundary of a simple polytope, the 'homeomorphism' will be 'diffeomorphism'.
By the definition, the moment-angle complex corresponding to $K$ is:

$$ Z = \bigcup_{\sigma \in K} D_{\sigma}^{2|\sigma|} \times T_{\sigma} \subset (D^2)^m. $$

Then we can express the moment-angle complex corresponding to $K_{\sigma}$ as follows:

$$ Z_{\sigma} = \left( Z \times S^1 - T_{\sigma}^{m-n} \times D_{\sigma}^{2n} \times S^1 \right) \cup T_{\sigma}^{m-n} \times S_{\sigma}^{2n-1} \times D^2 $$

$$ \simeq \partial \left[ (Z - T_{\sigma}^{m-n} \times D_{\sigma}^{2n}) \times D^2 \right] $$
In the case $m < 3n$, S. Gitler and S. López firstly proved that $T^m_{\sigma} - n \times \{0\}$ can be contracted to a point in $Z$. Since $m < 3n$, it is isotopic to a $(m - n)$-torus inside an open disk in $Z$. therefore,

$$Z - T^m_{\sigma} - n \times D_n^2 \cong Z - T^m_{\sigma} - n \times D_n^2 \cong (Z - D^{n+m}) \cup (D^{n+m} - T^m_{\sigma} - n \times D_n^2)$$

and

$$Z_{\sigma} \cong \partial \left[ \left( Z - D^{n+m} \right) \times D^2 \right] \# \partial \left[ \left( S^{m+n} - T^m - n \times D_n^2 \right) \times D^2 \right].$$

Then they prove

$$\partial \left[ \left( S^{m+n} - T^m - n \times D_n^2 \right) \times D^2 \right] \cong \frac{m-n}{\# j=1} \left( \begin{array}{c} m-n \\ j \end{array} \right) \left( S^{j+2} \times S^{m+n-j-1} \right).$$
We prove the conjecture in general case, the main idea is:

- Firstly, we construct an isotopy of $T^{m-n}_\sigma$ in $Z$ to move it to the regular embedding $T^{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq Z$, thus we prove the following:

**Proposition**

$Z_\sigma$ is homeomorphic to

$$\partial \left[ \left( Z - D^{n+m}_n \right) \times D^2 \right] \# \partial \left[ \left( S^{m+n} - T^{m-n} \times D^{2n}_n \right) \times D^2 \right],$$

where $T^{m-n} \times D^{2n}_n$ is the regular embedding in $S^{m+n}$. 

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We construct the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$ as follows:

$S^1 \subseteq D^2 \subseteq \mathbb{R}^2$

assume that

$T^{i-1} \subseteq D^i \subseteq \mathbb{R}^i$

$T^{i-1} \times S^1 \subseteq D^i \times S^1 \subseteq D^i \times S^1 \cup S^{i-1} \times D^2 - \{\ast\} \cong \mathbb{R}^{i+1}$

The key lemma in the construction of the isotopy is:

**Lemma**

There are two embedding of the torus $T^k$ into $D^{k+2}$:

1. $T^k \subseteq D^{k+1} \subseteq D^{k+2}$
2. $T^k = T^{k-1} \times S^1 \subseteq D^k \times D^2 = D^{k+2}$

The two embeddings are isotopic to each other in $D^{k+2}$.
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Then we prove the following by induction:

**Proposition**

\[ \partial \left[ \left( S^{m+n} - T^{m-n} \times D^{2n} \right) \times D^2 \right] \text{ is diffeomorphic to } \]

\[ \bigotimes_{j=1}^{m-n} \left( \binom{m-n}{j} \left( S^{j+2} \times S^{m+n-j-1} \right) \right) \]

where \( T^{m-n} \times D^{2n} \) is the regular embedding in \( S^{m+n} \).

Combining these two propositions, the conjecture is proved.
Until now, we have proved the conjecture. However, we can prove the following:

**Theorem**

Let $K_1, K_2$ are two simplicial $(n - 1)$-spheres. $Z_{K_1}$ and $Z_{K_2}$ are the moment-angle manifolds corresponding to $K_1$ and $K_2$. Let $K = K_1 \# K_2$ be their connected sum at some maximal simplex $\sigma_1$ and $\sigma_2$. The moment-angle manifold corresponding to $K$ is homeomorphic to

$$G^{m_2-n}(Z_{K_1}) \# G^{m_1-n}(Z_{K_2}) \# M$$

$$M \cong \bigotimes_{j=1}^{m_1+m_2-2n-1} \left[ (m_1+m_2-2n) - (m_1-n) - (m_2-n) \right] S^{j+2} \times S^{m_1+m_2-j-2}.$$

The operation $G$ on an manifold $M^k$ is defined as:

$$G(M^k) = \partial[(M^k - D^k) \times \mathbb{D}^2].$$

$G^p(M^k)$ means doing the operation $G$ on $M^k$ $p$ times.

If $K_1, K_2$ are dual to the boundary of simple polytopes $P_{n_1}^n, P_{n_2}^n$, the 'homeomorphism' will be 'diffeomorphism'.
By the definition, the moment-angle complex corresponding to $K$ is:

$$Z_K = \bigcup_{\tau \in K_1 \text{ or } \tau \in K_2} D^2_{\tau} \times T^{m_1+m_2-n-|\tau|}. $$

which can be expressed as

$$(Z_{K_1} - T^{m_1-n} \times D^2_{\sigma_1}) \times T^{m_2-n} \cup (Z_{K_2} - T^{m_2-n} \times D^2_{\sigma_1}) \times T^{m_1-n}$$

where the two pieces are glued along

$$T^{m_1-n} \times S^2_{\sigma_1} \times T^{m_2-n} \cong T^{m_1-n} \times S^2_{\sigma_2} \times T^{m_2-n},$$

using the identification of $\sigma_1$ and $\sigma_2$. 
Idea of the proof: 
Firstly, using the method above, we prove that 

\[(Z_{K_1} - T^{m_1-n} \times D_{\sigma_1}^{2n}) \times T^{m_2-n} \cup (Z_{K_2} - T^{m_2-n} \times D_{\sigma_2}^{2n}) \times T^{m_1-n}\]

is homeomorphic to 

\[[Z_{K_1} \#(S^{m_1+n} - T^{m_1-n} \times D_{\sigma_1}^{2n})] \times T^{m_2-n} \cup [Z_{K_2} \#(S^{m_2+n} - T^{m_2-n} \times D_{\sigma_2}^{2n})] \times T^{m_1-n},\]

where the two pieces are glued along 

\[T^{m_1-n} \times S^{2n-1} \times T^{m_2-n} \overset{id}{\cong} T^{m_1-n} \times S^{2n-1} \times T^{m_2-n}.\]
Secondly, we construct an isotopy of $\{\ast\} \times T^{m_2-n}(\{\ast\} \times T^{m_1-n})$ in

$$(S^{m_1+n}-T^{m_1-n} \times D^{2n}) \times T^{m_2-n} \cup (S^{m_2+n}-T^{m_2-n} \times D^{2n}) \times T^{m_1-n}$$

\[\text{to move it to the regular embedding} \]

\[T^{m_2-n} \subseteq D^{m_1+m_2}(T^{m_1-n} \subseteq D^{m_1+m_2}). \text{ Thus we prove that} \]

$$[Z_{K_1} \# (S^{m_1+n}-T^{m_1-n} \times D^{2n})] \times T^{m_2-n} \cup [Z_{K_2} \# (S^{m_2+n}-T^{m_2-n} \times D^{2n})] \times T^{m_1-n}$$

\[\text{is homeomorphic to} \]

$$(Z_{K_1} \times T^{m_2-n} \#_{T^{m_2-n}} S^{m_1+m_2}) \# (Z_{K_2} \times T^{m_1-n} \#_{T^{m_1-n}} S^{m_1+m_2}) \# M.$$
Finally, we inductively prove that

1. \( Z_{K_1} \times T^{m_2-n} \# T^{m_2-n} S^{m_1+m_2} \simeq G^{m_2-n}(Z_{K_1}) \).
2. \( Z_{K_2} \times T^{m_1-n} \# T^{m_1-n} S^{m_1+m_2} \simeq G^{m_1-n}(Z_{K_2}) \).
3. \( S^{m_1+n} \times T^{m_2-n} \# T^{m_1-n} \times T^{m_2-n} S^{m_2+n} \times T^{m_1-n} \)

is homeomorphic to

\[
\# \sum_{j=1}^{m_1+m_2-2n-1} \left( \binom{m_1 + m_2 - 2n}{j+1} - \binom{m_1 - n}{j+1} - \binom{m_2 - n}{j+1} \right) S^{j+2} \times S^{m_1+m_2-j-2}.
\]
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$(D^2, S^1)$: moment-angle manifolds.
$(D^{k+1}, S^k), k \geq 2$: generalised moment-angle manifolds.

Similarly,

$$
\overline{Z}_\sigma \approx \partial \left[ \left( \overline{Z} - D^{n+km} \right) \times D^{k+1} \right] \# \#_{j=1}^{m-n} \binom{m-n}{j} \left( S^k(j+1)+1 \times S^k(m-j)+n-1 \right).
$$

From this, we know that if $\overline{Z}$ has a metrics of positive Ricci curvature, then $\overline{Z}_\sigma$ also has a metrics of positive Ricci curvature. We ask: When $Z(P)$ has a metrics of positive Ricci curvature?
Conjecture

For a simple polytope,

1. If $Z(D^k, S^{k-1})$ has a metrics of positive Ricci curvature, then $Z(D^{k+1}, S^k)$ also has a metrics of positive Ricci curvature.
2. $Z(D^{k+1}, S^k)$ has a metrics of positive Ricci curvature, for $k \geq 2$. 
Definition

Let $Q$ be a simplicial convex polytope in $\mathbb{R}^n$ whose vertices are primitive lattice vectors $(\in \mathbb{Z}^n)$, and which contains 0 in the interior. If $a_1, \ldots, a_n$ are the vertices of a facet of $Q$, we suppose $\det(a_1, \ldots, a_n) = \pm 1$ for every facet. Then we call $Q$ a Fano polytope.

The dual of $Q$:

$$P = \{ u \in \mathbb{R}^n | \langle u, v \rangle \leq 1, \forall v \in Q \}$$

is a simple polytope. We claim that the moment-angle manifold corresponding to $P$ has a metrics of positive Ricci curvature.
In fact, we can construct a smooth toric Fano variety $X_P$ from the polytope $P$. According to the Calabi-Yau’s theorem, it has a metrics of positive Ricci curvature. However, the moment-angle manifold $Z(P)$ is a principal $T^{m-n}$ bundle of $X_P$, by a theorem of Gilkey, we know that $Z(P)$ has a metrics of positive Ricci curvature.
Thank you very much!