Linearisation of algebraic structures via functor calculus

Combinatorial and toric homotopy

Singapore, august 2015

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Reminder: Relations between groups and Lie algebras
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1. Lie groups: Classical equivalence of simply connected Lie groups and Lie algebras \((G \mapsto (T_e(G), [-,-]))\).
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2. The associated graded of arbitrary groups: For any group \(G\) and elements \(x, y \in G\) let \([x, y] = (xy)(yx)^{-1}\). An \(N\)-series of \(G\) is a filtration

\[
\mathcal{N}: G = N_1 \supset N_2 \supset \ldots
\]

of \(G\) by subgroups \(N_n\) such that \([N_i, N_j] \subset N_{i+j}\).
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2. The associated graded of arbitrary groups: For any group \(G\) and elements \(x, y \in G\) let \([x, y] = (xy)(yx)^{-1}\). An \(N\)-series of \(G\) is a filtration

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of \(G\) by subgroups \(N_n\) such that \([N_i, N_j] \subset N_{i+j}\). Then \(\text{Gr}^N_n(G) = N_n/N_{n+1}\) is an abelian group, and

\[ \text{Gr}^N(G) = \sum_{k \geq 1} \text{Gr}^N_k(G) \]

is a graded Lie ring whose bracket is induced by the commutator of \(G\).
Examples of N-series

1. The lower central series

\[ \gamma : G = \gamma_1(G) \supset \gamma_2(G) \supset \ldots \]

where \( \gamma_n(G) = \langle [x_1, \ldots, x_n] \mid x_1, \ldots, x_n \in G \rangle \) with

\[ [x_1, \ldots, x_n] = [x_1, [x_2, \ldots [x_{n-1}, x_n] \ldots] \]

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2. The dimension series: let \( \mathbb{K} \) be a commutative ring. Then the subgroups

\[ D_{n,\mathbb{K}}(G) = G \cap (1 + I^n_{\mathbb{K}}(G)) \]

form an N-series where \( I^n_{\mathbb{K}}(G) \) denotes the \( n \)-th power of the augmentation ideal of the group algebra \( \mathbb{K}(G) \).
3. Mal’cev/Lazard equivalence: There is a canonical equivalence between the categories of radicable $n$-step nilpotent groups and $n$-step nilpotent Lie algebras over $\mathbb{Q}$, based on the Baker-Campbell-Hausdorff formula.
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4. Primitive operations on group algebras: Primitive elements of Hopf algebras (the bialgebra type of group algebras) form a Lie algebra under the usual ring commutator.
Relations between groups and Lie algebras - Summary

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2. The associated graded of arbitrary groups
3. Mal’cev/Lazard equivalence
4. Primitive operations on group algebras
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GOAL:

Given a suitable non-linear “algebraic” structure generalizing groups,

- exhibit a related linear structure = type of algebras (linear operad), generalizing Lie algebras
- generalize the relations 1. to 4. above to this situation.
Approach

- develop and use algebraic functor calculus to construct a suitable notion of commutators and a suitable operad in abelian groups, satisfying relation 2. (basically done)
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- try to generalize relation 1. (dream!)
Framework

Definition [Janelidze, Márki, Tholen 2002]: A category $C$ is called **semi-abelian** if it is pointed, finitely complete and cocomplete, protomodular and Barr-exact.
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1. $C$ is pointed and finitely complete and cocomplete.

2. For any morphism $p: X \to Y$ in $C$ admitting a section $s: Y \to X$, $X$ “is generated by the kernel of $p$ and the image of $s$”, that is the morphism $\text{Ker}(p) + Y \to X$ given by the injection of $\text{Ker}(p)$ and $s$ is a cokernel.
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- the category of cocommutative Hopf algebras (Gadjo-Gran-Vercruysen)
- ............
The idea of the categorical (Higgins) commutator calculus

(blackboard)
Basic (algebraic) functor calculus

In the sequel, \( F : \mathcal{C} \to \mathcal{D} \) denotes a functor between categories satisfying

- \( \mathcal{C} \) is pointed and has finite sums (\( = \) coproducts)
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The $n$-th cross-effect of $F$ is defined to be the multifunctor $\text{cr}_n F : \mathcal{C}^n \to \mathcal{D}$ given by

\[
\text{cr}_n F(X_1, \ldots, X_n) = F(X_1 \mid \ldots \mid X_n) = \bigwedge_{k=1}^n \ker \left( F(X_1 + \ldots + X_n) \to F(X_1 + \ldots + \hat{X}_k + \ldots + X_n) \right) \ll F(X_1 + \ldots + X_n).
\]
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$$cr_n F(X_1, \ldots, X_n) = F(X_1|\ldots|X_n) =$$

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    cr_nF(X_1, \ldots, X_n) &= F(X_1| \ldots |X_n) = \\
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    &\triangleleft F(X_1 + \ldots + X_n)
\end{align*}
\]

In particular, \( cr_1F(X) = \ker\left( F(0) : F(X) \to F(0) \right) \).
Basic (algebraic) functor calculus

In the sequel, $F : C \to D$ denotes a functor between categories satisfying
- $C$ is pointed and has finite sums (= coproducts)
- $D$ is semi-abelian.

The $n$-th cross-effect of $F$ is defined to be the multifunctor $cr_n F : C^n \to D$ given by

$$cr_n F (X_1, \ldots, X_n) = \cap_{k=1}^n \text{Ker} \left( F(X_1 + \ldots + X_n) \to F(X_1 + \ldots + \hat{X}_k + \ldots X_n) \right) \triangleleft F(X_1 + \ldots + X_n)$$

In particular, $cr_1 F (X) = \text{Ker} \left( F(0) : F(X) \to F(0) \right)$ and

$$cr_2 F (X, Y) = \text{Ker} \left( r_{12} : F(X + Y) \to F(X) \times F(Y) \right).$$
Examples

- A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories is additive iff $cr_2 F = 0$. 
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- For $T^2 : Ab \rightarrow Ab$, $T^2(A) = A \otimes A$, we have
  $$cr_2 T^2(A, B) = (A \otimes B) \oplus (B \otimes A),$$
  $$cr_n T^2(A, B) = 0 \text{ for } n > 2.$$
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- Let $Gr$ denote the category of groups. Then for groups $X_1, \ldots, X_n$ and elements $x_k \in X_k$, $k = 1, \ldots, n$, we have

  \[ [x_1, \ldots, x_n] \in \text{Id}_{Gr}(X_1|\ldots|X_n). \]

  If $n = 2$ these elements generate $\text{Id}_{Gr}(X_1|X_2)$ (freely if one takes $x_1, x_2 \neq e$).

- Let $Lp$ denote the category of loops. Then for loops $X_1, X_2, X_3$ and elements $x_k \in X_k$ the associator

  \[ A(x_1, x_2, x_3) = (x_1(x_2x_3))\backslash((x_1x_2)x_3) \in \text{Id}_{Lp}(X_1|X_2|X_3). \]
Basic properties of cross-effects

- the multifunctor $cr_n F$ is symmetric and multi-reduced
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- Inductive nature: for a multifunctor $M : C^n \to D$ define its $k$-th derivative $\partial_k M : C^{n+1} \to D$ by

$$\partial_k M(X_1, \ldots, X_{k-1}, - , X_k + 1, \ldots, X_{n+1}) = cr_2 (M(X_1, \ldots, X_{k-1}, - , X_{k+2}, \ldots, X_{n+1})(X_k, X_k + 1))$$

Then there is a natural isomorphism $\partial_k cr_n F \cong cr_{n+1} F$ for all $k = 1, \ldots, n$.

- The functor $cr_n : \text{Func}(C, D) \to \text{Func}(C^n, D)$ is exact.
- “Pseudo-right-exactness” [Van der Linden]: If $F$ preserves coequalizers of reflexive parallel pairs of morphisms (reflexive meaning that these morphisms admit a common section) then so does $cr_n F$ in all variables, for any $n$.
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\partial_k M(X_1, \ldots, X_{n+1})
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Preservation of coequalizers of reflexive parallel pairs

A functor $F : \mathcal{C} \to \mathcal{D}$ as before preserves coequalizers of reflexive parallel pairs iff for any right-exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{} 0$$

in $\mathcal{C}$ the sequence

$$F(A) + F(A|B) \xrightarrow{\langle F(a) \rangle_\delta} F(B) \xrightarrow{F(b)} F(C) \xrightarrow{} 0$$

in $\mathcal{D}$ is exact, where

$$\delta : F(A|B) \xrightarrow{F(a|1_B)} F(B|B) \xrightarrow{} F(B + B) \xrightarrow{F(\nabla^2)} F(B) .$$
Operadic structure of cross-effects

- Operadic structure: Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be reduced functors where the category $\mathcal{E}$ is semi-abelian, too.

Denote “multi-objects”, i.e. sequences of objects in $\mathcal{C}$, by $X_j = X_{j,1}, \ldots, X_{j,k_j}$ and concatenation of such by

$X_1 \cup \ldots \cup X_n = X_{1,1}, \ldots, X_{1,k_1}, \ldots, X_{n,1}, \ldots, X_{n,k_n}$.

Then there is a natural transformation

$$cr_n G \left( cr_{k_1} F(X_1), \ldots, cr_{k_n} F(X_n) \right)$$

$$\downarrow$$

$$cr_{k_1 + \ldots + k_n} (G \circ F) (X_1 \cup \ldots \cup X_n)$$

rendering a certain canonical diagram commutative.
Polynomial functors

**Definition:** The functor $F$ is polynomial of degree $\leq n$ if $cr_{n+1}F = 0$. 
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Example 1: A reduced functor between abelian categories is linear (that is, polynomial of degree $\leq 1$) iff it is additive.

Example 2: The $n$-th tensor power functor $T^n : \text{Ab} \to \text{Ab}, T^n(A) = A \otimes^n$, is polynomial of degree $n$. 
Commutators via functor calculus

Let $F : C \to D$ be a reduced functor as before.

For subobjects $x_k : X_k \to X$, $k = 1, \ldots, n$, of an object $X$ of $C$ define the subobject $[X_1, \ldots, X_n]_F$ of $F(X)$ to be the image of the morphism

$$F(X_1|\ldots|X_n) \to F(X_1 + \ldots + X_n) \xrightarrow{F(x_1,\ldots,x_n)} F(X)$$

Note that

$$[X_1, \ldots, X_n]_{Id_D} \leq X,$$

and that

$$[X_1]_F = \text{Im}\left( cr_1 F(X_1) \xrightarrow{\text{Id}} F(X_1) \xrightarrow{F(x_1)} F(X) \right).$$
Examples

1. If $\mathcal{D}$ is the category of groups $\text{Gr}$ then

   - $[X_1, X_2]_{\text{Id}_{\text{Gr}}} = [X_1, X_2]$;
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   - \( [X_1, X_2]_{ld_{Gr}} = [X_1, X_2] \);

   - \( [X_1, X_2, X_3]_{ld_{Gr}} \) is the normal subgroup of \( \langle X_1 \cup X_2 \cup X_3 \rangle \) generated by the product
     \[
     [X_1, [X_2, X_3]].[X_2, [X_3, X_1]].[X_3, [X_1, X_2]].
     \]
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In particular, if $X_1, X_2, X_3$ are normal subgroups of $X$ then $[X_1, X_2, X_3]_{Id_{Gr}}$ is their symmetric commutator.
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   In particular, if $X_1, X_2, X_3$ are normal subgroups of $X$ then $[X_1, X_2, X_3]_{Id_{Gr}}$ is their symmetric commutator.

2. If $\mathcal{D}$ is the category of loops $Lp$, then
   - $[X_1, X_2]_{Id_{Lp}}$ is the normal subloop of $\langle X_1 \cup X_2 \rangle$ generated by the elements $[x_2, x_1], A(x_1, y_1, y_2), A(x_1, x_2, y_1), A(x_1, x_2, y_2) A(x_2, x_1, y_2)$, and $A_3(x_1, x_2, x_1, y_2)$ where $x_i, y_i \in X_i$ and

     $[a, b] = ba \backslash ab$

     $A_3(a, b, c, d) = (A(a, b, c)A(a, b, d)) \backslash A(a, b, cd)$. 
Examples - sequel 1

3. If $\mathcal{D}$ is a category of $\omega$-loops then $[X_1, X_2]_{Id_{\mathcal{D}}}$ is the normal subobject of $X_1 \vee X_2$ generated by the elements

$$\theta((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \theta(x_1 y_1, \ldots, x_n y_n)/\theta(x_1, \ldots, x_n)\theta(y_1, \ldots, y_n)$$

where $x_1, \ldots, x_n \in X_1$, $y_1, \ldots, y_n \in X_2$ and $\theta$ is a generating operation of $\mathcal{D}$.

- 3a) If $\mathcal{D} = Groups$ then $[x, y] = y^{-1}x^{-1}yx$ and $[(x_1, x_2), (y_1, y_2)]. = x_1[y_1, x_2]$.

- 3a) If $\mathcal{D} = Loops$ then

$$[(x_1, x_2), (y_1, y_2)]. = ((x_1 y_1)(x_2 y_2))/((x_1 x_2)(y_1 y_2)).$$

In particular, $[(e, x_2), (y_1, e)]. = (y_1 x_2)/(x_2 y_1)$ and $[(x_1, e), (y_1, y_2)]. = ((x_1 y_1)y_2)/(x_1(y_1 y_2))$. 
4. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras $\mathcal{P}$-$\text{Alg}$, then

$$[X_1, \ldots, X_n]_{\mathcal{P}$-$\text{Alg}} = \sum_{p_k \geq 1} \mu_p(X_1^{\otimes p_1} \otimes \ldots \otimes X_n^{\otimes p_n} \otimes \mathcal{P}(p)).$$

where $p = p_1 + \ldots + p_n$. 
Properties

- **Reducedness**: if one of the $X_i = 0$ then
  $[X_1, \ldots, X_n]_F = 0$. 
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- **Removing internal brackets or repetitions** of subobjects enlarges the commutator (the relations below indicated in red color are valid only if $C$ is semi-abelian and $F$ preserves the class of cokernels):

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$$[A, A, B]_F \subset [A, B]_F.$$ 

- **Preservation by morphisms**: For $f : X \to Y$ in $C$,

$$F(f)([X_1, \ldots, X_n]_F) = [f(X_1), \ldots, f(X_n)]_F.$$
Lower central series

For an object $X$ of $\mathcal{D}$ let

$$\gamma^F_n(X) = [X, \ldots, X]_F \leq F(X)$$
Lower central series

For an object $X$ of $\mathcal{D}$ let

$$\gamma_n^F(X) = [X, \ldots, X]_F \leq F(X)$$

Suppose that $F$ is reduced, i.e. that $F(0) = 0$. We then obtain a filtration

$$F(X) = \gamma_1^F(X) \geq \gamma_2^F(X) \geq \ldots$$

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of $F(X)$ which is an \textit{N-series}, that is,

$$[N_{k_1}, \ldots, N_{k_n}]_{Id_{\mathcal{D}}} \subset N_{k_1 + \ldots + k_n}$$

for $N_k = \gamma^F_k(X)$. 
**Lower central series**

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of $F(X)$ which is an $N$-series, that is,

$$[N_{k_1}, \ldots, N_{k_n}]_{Id_D} \subset N_{k_1 + \ldots + k_n}$$

for $N_k = \gamma^F_k(X)$. In particular, taking $F = Id_D$ we obtain the (categorically defined) lower central series (c.l.c.s.) of $X$,

$$X = \gamma^{Id_D}_1(X) \geq \gamma^{Id_D}_2(X) \geq \ldots$$
Examples

1. If \( D \) is the category of groups, then the categorically defined lower central series coincides with the classical l.c.s.

2. If \( D \) is the category of loops, then the categorically defined lower central series coincides with the commutator-associator filtration introduced by Mostovoy.

3. If \( D \) is the category of \( \mathcal{P} \)-algebras \( \mathcal{P}\text{-Alg} \), then

\[
\gamma_{n}^{\text{Id_{\mathcal{P}\text{-Alg}}}}(X) = \sum_{k \geq n} \mu_{k}(X \otimes^{k} \otimes \mathcal{P}(k)).
\]

How to prove this?
Characterisation of the c.l.c.s.

Theorem. Let $\mathcal{X}(X): X = X_1 \geq X_2 \geq \ldots$ be a natural filtration of all objects $X$ in $\mathcal{D}$ by normal subobjects $X_n$ of $X$. 
Characterisation of the c.l.c.s.

**Theorem.** Let $X(X): X = X_1 \geq X_2 \geq \ldots$ be a natural filtration of all objects $X$ in $\mathcal{D}$ by normal subobjects $X_n$ of $X$. Then $X(X)$ coincides with the c.l.c.s. of $X$ for all $X$ if and only if
Characterisation of the c.l.c.s.

Theorem. Let \( \mathcal{X}(X) : X = X_1 \geq X_2 \geq \ldots \) be a natural filtration of all objects \( X \) in \( \mathcal{D} \) by normal subobjects \( X_n \) of \( X \). Then \( \mathcal{X}(X) \) coincides with the c.l.c.s. of \( X \) for all \( X \) if and only if there exist

- multifunctors \( M_n : \mathcal{D}^n \to \mathcal{D} \)
- natural maps \( m_n : M_n(X, \ldots, X) \to X_n \)

such that the following two conditions are satisfied:
Characterisation of the c.l.c.s.

Theorem. Let $\mathcal{X}(X) : X = X_1 \supseteq X_2 \supseteq \ldots$ be a natural filtration of all objects $X$ in $\mathcal{D}$ by normal subobjects $X_n$ of $X$. Then $\mathcal{X}(X)$ coincides with the c.l.c.s. of $X$ for all $X$ if and only if there exist
- multifunctors $M_n : D^n \to D$
- natural maps $m_n : M_n(X, \ldots, X) \to X_n$

such that the following two conditions are satisfied:

1. Factorisations $\overline{m_n}$ exist and are cokernels rendering the following diagrams commutative:

$$
\begin{array}{c}
M_n(X, \ldots, X) \xrightarrow{m_n} X_n \\
\downarrow t_1 \quad \downarrow q_n
\end{array}
$$

$$
\begin{array}{c}
(T_1 M_n)(X, \ldots, X) \xrightarrow{\overline{m_n}} X_n/X_{n+1}
\end{array}
$$
2. The images of the maps

\[ m_k : M_k(X, \ldots, X) \to X_k \hookrightarrow X_n, \]

\( k \geq n \), jointly generate \( X_n \) as a normal subobject of \( X \).
2. The images of the maps

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Proof of the identity $$\gamma_n^{\text{Id}_D}(X) = \gamma_n(X)$$ in $$\mathcal{D} = \text{Groups}$$:
2. The images of the maps

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\( k \geq n \), jointly generate \( X_n \) as a normal subobject of \( X \).

**Application:**

Proof of the identity \( \gamma^{\text{Id}_D}(X) = \gamma_n(X) \) in \( D = \text{Groups} \):

take

- \( M_n(X_1, \ldots, X_n) \) to be the free group generated by the set \( X_1 \times \ldots \times X_n \) modulo the normal subgroup generated by the tuples \( (x_1, \ldots, x_n) \) where one of the \( x_k \)'s is trivial

- \( m_n \) to send a basis element \( (x_1, \ldots, x_n) \in X^n \) to \([x_1, \ldots, x_n]\).
Lower central series of the group ring functor

Let $F : \text{Groups} \to \text{Ab}$ be the functor sending a group $G$ to its group ring $\mathbb{Z}[G]$. Then

$$\gamma_n^F(G) = I^n(G)$$

where $I^n(G)$ is the $n$-th power of the augmentation ideal of $\mathbb{Z}[G]$. 
Polynomialization of functors

For any functor $F : C \to D$ as before let

$$T_n F = F / \gamma_n(F) \quad \text{and} \quad t_n : F \longrightarrow T_n F .$$

Then the functor $T_n F$ is polynomial of degree $\leq n$ and $t_n$ is initial among all natural transformations from $F$ to polynomial functors of degree $\leq n$. 
Nilpotency

Define an object $X$ of $\mathcal{D}$ to be $n$-step nilpotent if
\[ γ^{ld_\mathcal{D}}_{n+1}(X) = 0. \]

“Polynomiality subsumes nilpotency”

1. Global statement: All objects of $\mathcal{D}$ are $n$-step nilpotent iff the identity functor of $\mathcal{D}$ is polynomial of degree $\leq n$. For arbitrary $\mathcal{D}$, the “$n$-step nilization” functor $X \mapsto Nil_n(X) = X/γ^{ld_\mathcal{D}}_{n+1}(X)$ equals $T_n ld_\mathcal{D}$.

1. Local statement: A single object $X$ of $\mathcal{D}$ is $n$-step nilpotent iff its “commutator map”
\[ S_{2}^{ld_\mathcal{D}} : ld_\mathcal{D}(X|X) \rightarrow X + X \xrightarrow{\nabla^2} X \]
is polynomial of degree $\leq n - 1$ in both (equivalently any of the two) variables, which by definition means that $S_{2}^{ld_\mathcal{D}}$ factors through the bi-polynomialization
\[ t_{n-1,n-1} : cr_2 ld_\mathcal{D}(X, X) \longrightarrow T_{n-1,n-1}(cr_2 ld_\mathcal{D})(X, X). \]
Example: 2-step nilpotency in groups

Let $\mathcal{D} = Gr$ and $n = 2$. Here

$$cr_2 ld_{Gr}(X, X) = \text{Free}(|X^*| \times |X^*|)$$

$S_{ld_{Gr}}^2 : (x, y) \mapsto [x, y]$ 

$t_{1,1} : (x, y) \mapsto (xX') \otimes (yY')$

$$T_{1,1}(cr_2 ld_{Gr})(X, X) = X_{ab} \otimes_{\mathbb{Z}} X_{ab}$$

Hence $S_{ld_{Gr}}^2$ is polynomial of degree $\leq 1$ in both variables iff the classical commutator map $c : X \times X \to X$, $(x, y) \mapsto [x, y]$, is bi-additive, which indeed is a well-known characterization of the fact that $X$ is 2-nilpotent.
Polynomial functors and nilpotency

Suppose that $F : C \to D$ is polynomial of degree $\leq n$. Then:

1. $F$ takes values in the full subcategory $\text{Nil}_n(D)$ of $n$-step nilpotent objects in $D$.

2. If $C$ is semi-abelian and $F$ preserves coequalizers of reflexive parallel pairs then $F$ factors through the $n$-step nilization functor $\text{Nil}_n : C \to \text{Nil}_n(C)$. Hence $F$ factors as

$$
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\downarrow \text{Nil}_n & & \uparrow \\
\text{Nil}_n(C) & \cong & \text{Nil}_n(D)
\end{array}
$$

Abbreviating “pre” for “pseudo-right exact” we obtain an equivalence of functor categories

$$
\text{Pol}_{\leq n}(C, D)_{\text{pre}} \cong \text{Pol}_{\leq n}(\text{Nil}_n(C), \text{Nil}_n(D))_{\text{pre}}.
$$
Associated graded object of an $N$-series

Let $\mathcal{N} : X = N_1 \geq N_2 \geq \ldots$ be an $N$-series of an object $X$ of $\mathcal{D}$. Then each $N_n$ is normal in $X$, and the quotient

$$\text{Gr}^\mathcal{N}_n(X) = N_n/N_{n+1}$$

is abelian,

so

$$\text{Gr}^\mathcal{N}(X) = \bigoplus_{n \geq 1} \text{Gr}^\mathcal{N}_n(X)$$

is a graded object in $\text{Ab}(\mathcal{D})$. 
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**QUESTION:** Does $\text{Gr}^\mathcal{N}(X)$ carry a natural global multilinear “multiplicative” structure relating its various components?
YES!!!!
MAIN THEOREM. Let $\mathcal{N} : X = N_1 \geq N_2 \geq \ldots$ be an $N$-series of an object $X$ of $\mathcal{D}$. Then $\text{Gr}^\mathcal{N}(X)$ has a natural structure of graded algebra over
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- a multilinear functor operad $\text{LinOp}(\mathcal{D})$ on $\text{Ab}(\mathcal{D})$ whose underlying functors are $T_1(cr_n \text{Id}_\mathcal{D})$ (which in fact preserve all colimits in all variables, in particular are right-exact), for general semi-abelian categories $\mathcal{D}$;
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- an operad in abelian groups $\text{AbOp}(\mathcal{D})$ if $\mathcal{D}$ is a (semi-abelian) algebraic category, that is the category of models of an algebraic theory in the sense of Lawvere; here

$$\text{AbOp}(\mathcal{D})(n) = cr_n(U_{\text{Ab}} \circ \text{Gr}^{\gamma_n} \circ L)(1, \ldots, 1)$$

where $U_{\text{Ab}}: \text{Ab}(\mathcal{D}) \to \text{Ab}$ is the forgetful functor and $L: \text{FinSet} \to \mathcal{D}$ is the functor assigning to the finite set $k = \{1, \ldots, k\} \cong 1^+ k$ the canonical free object of rank $k$. 
Examples

1. If $\mathcal{D}$ is the category of groups, then $\text{AbOp}(\mathcal{D}) \otimes \mathbb{Q}$ is the Lie operad.
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1. If $\mathcal{D}$ is the category of groups, then $\text{AbOp}(\mathcal{D}) \otimes \mathbb{Q}$ is the Lie operad.

2. If $\mathcal{D}$ is the category of loops, then $\text{AbOp}(\mathcal{D}) \otimes \mathbb{Q}$ is the Sabinin operad.

3. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras then $\text{AbOp}(\mathcal{D}) = \mathcal{P}$. 
Work in progress

Let $\mathcal{D}$ be a semi-abelian algebraic category. A set operad $\mathcal{P}_{\mathcal{D}}$ is defined by $\mathcal{P}_{\mathcal{D}}(n) = \mathcal{D}(L(1), L(n))$ and operadic composition induced by composition in the category $\mathcal{D}$. 
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For $\mathbb{K}$ a field of characteristic 0, the set operad $\mathcal{P}_\mathcal{D}$ gives rise to an operad in $\mathbb{K}$-vector spaces $\mathbb{K}[\mathcal{P}_\mathcal{D}]$, by taking $\mathbb{K}[\mathcal{P}_\mathcal{D}](n)$ to be the vector space with basis $\mathcal{P}_\mathcal{D}(n)$. 
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For an object $X$ of $\mathcal{C}$ the vector space $\mathbb{K}[|X|]$ with basis $|X|$ has the structure of an algebra over $\mathbb{K}[\mathcal{P}_\mathcal{D}]$; e.g. if $\mathcal{D} = Gr$, this is the structure of group algebra (including the antipode). Let $\mathbb{K}[X]$ be this “object algebra” of $X$. 
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For $K$ a field of characteristic 0, the set operad $\mathcal{P}_\mathcal{D}$ gives rise to an operad in $K$-vector spaces $K[\mathcal{P}_\mathcal{D}]$, by taking $K[\mathcal{P}_\mathcal{D}](n)$ to be the vector space with basis $\mathcal{P}_\mathcal{D}(n)$.

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$K[|X|]$ also is a cocommutative and coassociative coalgebra defined by putting $\Delta(x) = x \otimes x$ for $x \in |X|$. 
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$\mathbb{K}[|X|]$ also is a cocommutative and coassociative coalgebra defined by putting $\Delta(x) = x \otimes x$ for $x \in |X|$. Thus $\mathbb{K}[X]$ is a (generalized) bialgebra which we call the “object bialgebra” of $X$. 
Conjectures

1. Primitives conjecture. The triple

\[
\left( K[P_D], Com_K, \text{AbOp}(\mathcal{D}) \otimes K \right)
\]

is a good triple of operads in the sense of Loday. This in particular means that the subspace of primitive elements of the graded object bialgebras \( \text{Gr}(K[X]) \) has a canonical structure of an algebra over \( \text{AbOp}(\mathcal{D}) \otimes K \).
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2. Conjectural generalized Jennings theorem. The \( n \)-th dimension subobject \( D_{n,\mathbb{K}}(X) = X \cap (1_X + l^n_\mathbb{K}(X)) \) is
\[
D_{n,\mathbb{K}}(X) = \sqrt{\gamma_n(X)},
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for suitably (already) defined notions of augmentation filtration \( l^n_\mathbb{K}(X) \) of \( \mathbb{K}[X] \) and of isolator \( \sqrt{S} \) of \( S \leq X \).
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3. Conjectural generalized Quillen theorem. There is a natural isomorphism of graded $\mathbb{K}[\mathcal{P}_D]$-algebras

$$\text{Gr}(\mathbb{K}[X]) \cong \text{U}(\text{Gr}^{\gamma}(X) \otimes \mathbb{K}).$$
Conjectures-II


Suppose that $\mathcal{D}$ is

- $n$-step nilpotent, which by definition means that the identity functor of $\mathcal{D}$ is polynomial of degree $\leq n$, or equivalently, that all objects of $\mathcal{D}$ are $n$-step nilpotent;
- $n$-radicable, which by definition means that the abelian group $\text{End}_{\mathcal{D}}(L(1)^{ab})$ is a $\mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{n}]$-module.

Then there is a (canonical?) equivalence of categories

$$\mathcal{D} \cong \text{Alg}(\text{AbOp}(\mathcal{D}))$$

This equivalence would also induce a generalized Baker-Campbell-Hausdorff formula (actually, one for each $n \geq 1$), expressing operations of arity $n$ in $\mathcal{D}$ in terms of the operations given by the operad $\text{AbOp}(\mathcal{D})$. 
APPROACH

Use the theory of polynomial functors from $\mathcal{D}$ to $\text{Ab}$ which encodes them by kind of a “DNA”; the latter involves intricate both algebraic and combinatorical structures (e.g. non-linear pseudo-Mackey functors).

So far, this program is completely achieved only for $n = 2$ and all $\mathcal{D}$ [H., Vespa; Defourneau];

for all $n \geq 2$, the necessary polynomial functor theory is achieved only for $\mathcal{D} = \text{Groups}$ and $\mathcal{D} = \text{Loops}$ (actually, also for $\mathcal{D} = \text{free finitely generated free algebras over a set-operad}$) [H., Pirashvili, Vespa].