On nontriviality of homotopy groups of the 2-sphere

S. O. Ivanov (joint with R. Mikhailov and J. Wu)

IMS, Singapore.
August 2015
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• Composition on $E(S^n)$ and the Leibniz’s rule.
• Conjecture about $\lambda$-filtration on $E(S^n)$.
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Known results

- The following theorem was proved Curtis using the lambda algebra analysing of Adams $d$- and $e$-invariants of the stabilization of either that element or its Hopf image.

**Theorem (Curtis (1969))**

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\begin{align*}
\pi_n(S^2) &\neq 0 \quad \text{if } n \neq 1 \mod 8, \ n \geq 3. \\
\pi_n(S^4) &\neq 0 \quad \text{if } n \geq 4.
\end{align*}
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- The same results on non-vanishing terms of the homotopy groups of spheres were obtained with the help of the composition method by M. Mimura, M. Mori and N. Oda in 1974.

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After proving this result, we became aware of the paper of B. Gray “Unstable families related to the image of J” (1984).

Theorem 12 (e) of this article implies that \( \pi_{8k+1}(S^3) \neq 0 \).

Hence, this theorem was ’proved’ by B. Gray but was not stated.

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- We study \((p)\pi_n(S^3)\) for \(p \neq 2\).

Lemma

For an odd prime \(p\) and \(k \geq 2\)

\[(p)\pi_{2(p-1)k+1}(S^3) \neq 0.\]

- \((3)\pi_{4k+1}(S^3) \neq 0\)
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A **central series** of $G$ is a sequence of normal subgroups
\[ G \supseteq \ldots G_{n-1} \supseteq G_n \supseteq G_{n+1} \supseteq \ldots \]
such that $G_n/G_{n+1} \subseteq \text{Cent}(G/G_{n+1})$.

- A central series is **$p$-central series** if $G_n/G_{n+1}$ is $\mathbb{F}_p$-vector space.
- The **lower central series** of $G$ is given by
  \[ G = \gamma_1 G \supseteq \gamma_2 G \supseteq \gamma_3 G \supseteq \ldots, \]
  where $\gamma_{n+1} G = [G, \gamma_n G]$.
- If $H \triangleleft G$, then $H^p = \langle \{ h^p \mid h \in H \} \rangle^G$.
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A **monad** on a category $\mathcal{C}$ is a triple $(\mathcal{M}, m, e)$, where $\mathcal{M} : \mathcal{C} \to \mathcal{C}$, $m : \mathcal{M}^2 \to \mathcal{M}$, $e : \text{id}_\mathcal{C} \to \mathcal{M}$, such that

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\mathcal{M}^3 & \xrightarrow{\mathcal{M}m} & \mathcal{M}^2 \\
\mathcal{M}m & \downarrow & m \\
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\end{array}
\quad
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{M}e} & \mathcal{M}^2 \quad \mathcal{M}e & \xleftarrow{\mathcal{M}e} & \mathcal{M} \\
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**Kleisli composition:**

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g \otimes f = m_c \circ \mathcal{M}g \circ f
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If $(\mathcal{M}, m, e)$ is a monad on the homotopy category then

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\mathcal{M}^3 & \xrightarrow{\text{m} \circ \mathcal{M}} & \mathcal{M}^2 \\
\downarrow \text{m} \circ \mathcal{M} & & \downarrow \text{m} \\
\mathcal{M}^2 & \xrightarrow{\text{m}} & \mathcal{M} \\
\end{array}
\]

Kleisli composition:

$$
\circ : \mathcal{C}(b, \mathcal{M}c) \times \mathcal{C}(a, \mathcal{M}b) \to \mathcal{C}(a, \mathcal{M}c).
$$

$$
g \circ f = m_c \circ \mathcal{M}g \circ f
$$

If $(\mathcal{M}, m, e)$ is a monad on the homotopy category then

$$
\circ : \pi_n(\mathcal{M}X) \times \pi_m(\mathcal{M}S^n) \to \pi_m(\mathcal{M}X).
$$
Consider the monad

\[ F : \text{Sets}_\ast \longrightarrow \text{Sets}_\ast , \]

where \( FX \) is the group generated by \( X \) with the relation \( \ast = 1 \).

If \( X \) is a pointed simplicial set we define \( FX \) dimensionwise.

\[ F : \text{sSets}_\ast \longrightarrow \text{sSets}_\ast . \]

\( FX \) is called Milnor’s construction of \( X \). \( |FX| \simeq \Omega \Sigma |X| \).

Similarly we define the monad

\[ \mathcal{L} : \text{sSets}_\ast \longrightarrow \text{sSets}_\ast , \]

where \( \mathcal{L}X \) is the restricted Lie algebra over \( \mathbb{F}_p \) generated by \( X \) with the relation \( \ast = 0 \).

\[ \bigoplus_i \gamma_i^{[p]} FX / \gamma_{i+1}^{[p]} FX \simeq \mathcal{L}X. \]
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$p$-ALCS-spectral sequence

- Consider the monad

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- If $X$ is a pointed simplicial set we define $FX$ dimensionwise.

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$$\bigoplus_i \gamma_i^{[p]} FX / \gamma_{i+1}^{[p]} FX \simeq \mathcal{L}X.$$
For a simplicial set $X$ we denote
\[ \pi_i(X, p) = \pi_i(X) / (\text{torsion prime to } p). \]


If $X$ is a reduced simplicial set, then there is a spectral sequence called mod $p$ accelerated lower spectral sequence of $X$:

\[ E(X) \Rightarrow \pi_*(\Sigma X, p), \quad E^1(X) = \pi_* \left( \frac{\gamma_{p^*[p]} F X}{\gamma_{p^{*+1}} F X} \right). \]

\[ E^1(S^{2n}) = \pi_* (L_{p^*} S^{2n}), \quad E(S^{2n}) \Rightarrow \pi_*(S^{2n+1}, p). \]

There is a natural suspension homomorphism on the level of the spectral sequences: $\sigma : E(X) \rightarrow E(\Sigma X)$. 
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\[ p-\text{ALCS-spectral sequence} \]
Kleisli composition and Leibniz’s rule

- The structure of a monad on $\mathcal{L}$ induces the composition

\[ \circlearrowleft: \pi_n(\mathcal{L}^p_i S^k) \times \pi_m(\mathcal{L}^p_j S^m) \rightarrow \pi_m(\mathcal{L}^{p+i} S^k) \]

- If we define $E(S^*) = \bigoplus E(S^n)$, then we get

\[ \circlearrowleft: E^1(S^*) \times E^1(S^*) \rightarrow E^1(S^*) \]

Theorem (R. Mikhailov, J. Wu, - (2015))

Then the Kleisli composition $\circlearrowleft$ induces a well-defined map

\[ \circlearrowleft: E^r(S^*) \times E^r(S^*) \rightarrow E^r(S^*) \]

and for elements $a, b \in E^r(S^*)$ the following derivation-like rule holds

\[ d^r(a \circlearrowleft \sigma b) = \pm d^r a \circlearrowleft b + a \circlearrowleft d^r \sigma b. \]
The structure of a monad on $L$ induces the composition

$$\odot : \pi_n(L_{p^i} S^k) \times \pi_m(L_{p^j} S^m) \rightarrow \pi_m(L_{p^{i+j}} S^k)$$

If we define $E(S^*) = \bigoplus E(S^n)$, then we get

$$\odot : E^1(S^*) \times E^1(S^*) \rightarrow E^1(S^*).$$

**Theorem (R. Mikhailov, J. Wu, - (2015))**

Then the Kleisli composition $\odot$ induces a well-defined map

$$\odot : E^r(S^*) \times E^r(S^*) \rightarrow E^r(S^*),$$

and for elements $a, b \in E^r(S^*)$ the following derivation-like rule holds

$$d^r(a \odot \sigma b) = \pm d^r a \odot b + a \odot d^r \sigma b.$$
Kleisli composition and Leibniz’s rule

- The structure of a monad on \( \mathcal{L} \) induces the composition

\[
\odot : \pi_n(\mathcal{L}_{p^i} S^k) \times \pi_m(\mathcal{L}_{p^j} S^n) \rightarrow \pi_m(\mathcal{L}_{p^{i+j}} S^k)
\]

- If we define \( E(S^*) = \bigoplus E(S^n) \), then we get

\[
\odot : E^1(S^*) \times E^1(S^*) \rightarrow E^1(S^*) .
\]

**Theorem (R. Mikhailov, J. Wu, - (2015))**

Then the Kleisli composition \( \odot \) induces a well-defined map

\[
\odot : E^r(S^*) \times E^r(S^*) \rightarrow E^r(S^*),
\]

and for elements \( a, b \in E^r(S^*) \) the following derivation-like rule holds

\[
d^r(a \odot \sigma b) = \pm d^r a \odot b + a \odot d^r \sigma b.
\]
The structure of a monad on $\mathcal{L}$ induces the composition

$$\odot : \pi_n(\mathcal{L}^i p S^k) \times \pi_m(\mathcal{L}^j p S^m) \longrightarrow \pi_m(\mathcal{L}^i p_{i+j} S^k)$$

If we define $E(S^*) = \bigoplus E(S^n)$, then we get

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Lambda algebra

- Let \( p \) be a fixed odd prime.
- \([p] \Lambda = \Lambda\) is an \( \mathbb{F}_p\)-algebra.
- Generators: \( \lambda_1, \lambda_2, \lambda_3, \ldots, \) and \( \mu_0, \mu_1, \mu_2, \ldots \).
- \( a(k, j) := (-1)^j (p-1)^{(k-j)-1} (k-j)/j, \quad b(k, j) := (-1)^j (p-1)^{(k-j)} (k-j)/j \in \mathbb{F}_p. \)
- Relations:
  \[
  \lambda_i \lambda_{pi+k} = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{a(k, j)}{p^j} \lambda_{i+k-j} \lambda_{pi+j},
  \]
  \[
  \lambda_i \mu_{pi+k} = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{a(k, j)}{p^j} \lambda_{i+k-j} \mu_{pi+j} + \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{b(k, j)}{p^j} \mu_{i+k-j} \lambda_{pi+j},
  \]
  \[
  \mu_i \lambda_{pi+k+1} = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{a(k, j)}{p^j} \mu_{i+k-j} \lambda_{pi+j+1},
  \]
  \[
  \mu_i \mu_{pi+k+1} = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{a(k, j)}{p^j} \mu_{i+k-j} \mu_{pi+j+1}.
  \]
- \( \nu_i \in \{ \lambda_i, \mu_i \}. \)
- A monomial \( \nu_{i_1} \ldots \nu_{i_l} \) is admissible if \( i_{k+1} \leq pi_k - 1 \) whenever \( \nu_{i_k} = \lambda_{i_k} \) and if \( i_{k+1} \leq pi_k \) whenever \( \nu_{i_k} = \mu_{i_k} \).
- The set of admissible monomials is a basis of \( \Lambda \).
Let $p$ be a fixed odd prime.

$[p] \Lambda = \Lambda$ is an $\mathbb{F}_p$-algebra.

Generators: $\lambda_1, \lambda_2, \lambda_3, \ldots$, and $\mu_0, \mu_1, \mu_2, \ldots$.

$a(k, j) := (-1)^{j+1} \binom{p-1}{k-j}^{-1}$, $b(k, j) := (-1)^j \binom{p-1}{k-j}^{-1} \in \mathbb{F}_p$.

Relations:

\[
\lambda_i \lambda_{pi+k} = \sum_{j=0}^{\left\lfloor \frac{k}{p} - \frac{k+1}{p} \right\rfloor} a(k, j) \lambda_{i+k-j} \lambda_{pi+j},
\]

\[
\lambda_i \mu_{pi+k} = \sum_{j=0}^{\left\lfloor \frac{k}{p} - \frac{k+1}{p} \right\rfloor} a(k, j) \lambda_{i+k-j} \mu_{pi+j} + \sum_{j=0}^{\left\lfloor \frac{k}{p} \right\rfloor} b(k, j) \mu_{i+k-j} \lambda_{pi+j},
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\[
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\]

$\nu_i \in \{\lambda_i, \mu_i\}$.

A monomial $\nu_{i_1} \ldots \nu_{i_l}$ is admissible if $i_{k+1} \leq pi_k - 1$ whenever $\nu_{i_k} = \lambda_{i_k}$ and if $i_{k+1} \leq pi_k$ whenever $\nu_{i_k} = \mu_{i_k}$.

The set of admissible monomials is a basis of $\Lambda$. 
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Relations:

$\lambda_i \lambda_{pi+k} = \sum_{j=0}^{\left\lfloor \frac{k+1}{p} \right\rfloor} a(k, j) \lambda_{i+k-j} \lambda_{pi+j}$,

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Sergei O. Ivanov  
Homotopy groups of the 2-sphere
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\]

Relations:

\[
\begin{align*}
\lambda_i \lambda_{pi+k} &= \sum_{j=0}^{[k-\frac{k+1}{p}]} a(k, j) \lambda_{i+k-j} \lambda_{pi+j}, \\
\lambda_i \mu_{pi+k} &= \sum_{j=0}^{[k-\frac{k+1}{p}]} a(k, j) \lambda_{i+k-j} \mu_{pi+j} + \sum_{j=0}^{[k-\frac{k}{p}]} b(k, j) \mu_{i+k-j} \lambda_{pi+j}, \\
\mu_i \lambda_{pi+k+1} &= \sum_{j=0}^{[k-\frac{k+1}{p}]} a(k, j) \mu_{i+k-j} \lambda_{pi+j+1}, \\
\mu_i \mu_{pi+k+1} &= \sum_{j=0}^{[k-\frac{k+1}{p}]} a(k, j) \mu_{i+k-j} \mu_{pi+j+1}.
\end{align*}
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$\nu_i \in \{\lambda_i, \mu_i\}$.

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$
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Relations:

\[
\lambda_i \lambda_{p_i + k} = \sum_{j=0}^{\left\lfloor \frac{k-\frac{k+1}{p}}{p} \right\rfloor} a(k, j) \lambda_{i+k-j} \lambda_{p_i+j},
\]

\[
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\]

$\nu_i \in \{\lambda_i, \mu_i\}$.

A monomial $\nu_{i_1} \ldots \nu_{i_l}$ is **admissible** if $i_{k+1} \leq p i_k - 1$ whenever $\nu_{i_k} = \lambda_{i_k}$ and if $i_{k+1} \leq p i_k$ whenever $\nu_{i_k} = \mu_{i_k}$.

The set of admissible monomials is a basis of $\Lambda$. 
Let $p$ be a fixed **odd** prime.

$[p] \Lambda = \Lambda$ is an $\mathbb{F}_p$-algebra.

Generators: $\lambda_1, \lambda_2, \lambda_3, \ldots$, and $\mu_0, \mu_1, \mu_2, \ldots$.

$a(k, j) := (-1)^{j+1}(p^{-1}(k-j)-1), \quad b(k, j) := (-1)^{j}(p^{-1}(k-j)) \in \mathbb{F}_p$.

Relations:

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\begin{align*}
\lambda_i \lambda_{pi+k} &= \sum_{j=0}^{\left\lfloor \frac{k-k+1}{p} \right\rfloor} a(k, j) \lambda_{i+k-j} \lambda_{pi+j}, \\
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- There is an isomorphism
  \[ E^1(S^{2n}) \cong \Lambda(n) \]
  and
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  \[ E^r = F^0_\lambda E^r \supseteq F^1_\lambda E^r \supseteq F^2_\lambda E^r \supseteq \ldots \]
- Conjecture: $d^r(F^i_\lambda E^r) \subseteq F^{i+1}_\lambda E^r$.

Lemma

The conjecture about $\lambda$-filtration implies that
\[ \alpha_k(3) \circ \alpha_1(2(p - 1)k + 2) \neq 0 \in \pi_{2(p-1)(k+1)+1}(S^3). \]
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- There is a very good analysis of cycles $Z(\Lambda) = \text{Ker}(\partial)$ of $\Lambda$ in the following article:


- Theorem 6.1 of [4] implies $Z(\Lambda) \subseteq \Lambda\lambda$.

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Thank you!