Braids and Lie algebras

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For a group $G$ the descending central series

$$G = \Gamma_1 > \Gamma_2 > \cdots > \Gamma_i > \Gamma_{i+1} > \cdots$$

is defined by the formula

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

This series of groups gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $gr^*_\Gamma(G)$

$$gr^i_\Gamma(G) = \Gamma_i/\Gamma_{i+1}.$$
Let $K$ be a commutative ring with unit.

**Definition.** An algebra $L$ over $K$ is called a Lie algebra over $K$ if its multiplication (denoted by $(x, y) \mapsto [x, y]$) verifies identities:

1. $[x, x] = 0$

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z$ in $L$. 

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Braids and Lie algebras
Let $M$ be a topological manifold. Symmetric group $\Sigma_n$ acts on the Cartesian power $M^n$ of $M$ by the standard formula

$$w(y_1, \ldots, y_n) = (y_{w(1)}, \ldots, y_{w(n)}), \quad w \in \Sigma_n.$$  \hfill (1)
We denote by $F(M, n)$ the space of $n$-tuples of pairwise different points in $M$:

$$F(M, n) = \{(p_1, \ldots, p_n) \in M^n : p_i \neq p_j \text{ for } i \neq j\}.$$ 

It is called \textit{ordered configuration space of $n$ points on $M$}. 
The orbit space of this action $B(M, n) = F(M, n)/\Sigma_n$ is the (unordered) configuration space of $n$ points on $M$.

The braid group of the manifold $M \ Br_n(M)$ is the fundamental group of configuration space

$$Br_n(M) = \pi_1(B(M, n)).$$
The fundamental group of the ordered configuration space is called the pure (or colored) braid group of the manifold $M$

$$P_n(M) = \pi_1(F(M, n)).$$
The free action of $\Sigma_n$ on $F(M, n)$ and the projection

$$p : F(M, n) \to F(M, n)/\Sigma_n = B(M, n)$$

defines a covering. The initial segment of the long exact sequence of this covering is as follows:

$$1 \to P_n(M) \xrightarrow{p^*} Br_n(M) \to \Sigma_n \to 1. \quad (2)$$
Define the elements $a_{i,j}$, $1 \leq i < j \leq m$, of $Br_m$ by:

$$a_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{j-1}.$$
Geometrically generator of this type is depicted as follows

\[
\begin{array}{cccccc}
1 & 2 & j - 1 & j & n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Figure: Generator $a_{1,j}$
They satisfy the Burau relations:

\[
\begin{cases}
  a_{i,j}a_{k,l} = a_{k,l}a_{i,j} & \text{for } i < j < k < l \text{ or } i < k < l < j, \\
  a_{i,j}a_{i,k}a_{j,k} = a_{i,k}a_{j,k}a_{i,j} & \text{for } i < j < k, \\
  a_{i,k}a_{j,k}a_{i,j} = a_{j,k}a_{i,j}a_{i,k} & \text{for } i < j < k, \\
  a_{i,k}a_{j,k}a_{j,k}^{-1} = a_{j,k}a_{j,k}^{-1}a_{i,k} & \text{for } i < j < k < l.
\end{cases}
\]
W. Burau proved that this gives a presentation of the pure braid group $P_m$. 
A presentation of the Lie algebra $L(P_n)$ for the pure braid group was given by Toshitake Kohno.

It is the quotient of the free Lie algebra $L[A_{i,j} | 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the following relations.
\[
\begin{align*}
[A_{i,j}, A_{s,t}] &= 0, \text{ if } \{i, j\} \cap \{s, t\} = \emptyset, \\
[A_{i,j}, A_{i,k} + A_{j,k}] &= 0, \text{ if } i < j < k, \\
[A_{i,k}, A_{i,j} + A_{j,k}] &= 0, \text{ if } i < j < k.
\end{align*}
\] (3)

Here, \(A_{i,j}\) is the image of \(a_{i,j}\) in \(L_1(P_n)\).
For a group $G$ we denote by $Aut(\pi)$ the automorphism group of $G$.

Consider the free group $F_n$ generated by $n$ letters $\{x_1, x_2, \cdots, x_n\}$. The kernel of the natural map

$$Aut(F_n) \rightarrow GL(n, \mathbb{Z})$$

is denoted $IA_n$. 
Nielsen, and Magnus gave automorphisms which generate $IA_n$ as a group:

- $\chi_{k,i}$ for $i \neq k$ with $1 \leq i, k \leq n$, and
- $\theta(k; [s, t])$ for $k, s, t$ distinct integers with $1 \leq k, s, t \leq n$ and $s < t$. 
\[ \chi_{k,i}(x_j) = \begin{cases} 
  x_j & \text{if } k \neq j, \\
  (x_i^{-1})(x_k)(x_i) & \text{if } k = j.
\end{cases} \]
The map $\theta(k; [s, t])$ is defined by

$$\theta(k; [s, t])(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_k) \cdot ([x_s, x_t]) & \text{if } k = j. \end{cases}$$

the commutator is given by

$$[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b.$$
Consider the subgroup of $IA_n$ generated by the $\chi_{k,i}$, the group of basis conjugating automorphisms of a free group.

This group is denoted by $P\Sigma_n$. 
It is the pure group of motions of \( n \) unlinked circles in \( S^3 \) and because of this it is known as the "group of loops".

It is also the pure braid-permutation group.
5. Digression: Braid-permutation group

The *Braid-permutation group* $BP_n$ was introduced by R. Fenn, R. Rimányi and C. Rourke.

It is the subgroup of $Aut(F_n)$ generated by $\xi_i$ and $\sigma_i$:

\[
(x_j)\xi_i = \begin{cases} 
  x_{i+1} & j = i, \\
  x_i & j = i + 1, \\
  x_j & \text{otherwise};
\end{cases}
\]

\[
(x_j)\sigma_i = \begin{cases} 
  x_{i+1} & j = i, \\
  x_{i+1}^{-1} x_i x_{i+1} & j = i + 1, \\
  x_j & \text{otherwise}.
\end{cases}
\]
BP\(_n\) is presented by the generators \(\xi_i\) and \(\sigma_i\), \(1 \leq i \leq n - 1\), and relations:

\[
\begin{align*}
\xi_i^2 &= 1, \\
\xi_i \xi_j &= \xi_j \xi_i & |i - j| > 1, \\
\xi_i \xi_{i+1} \xi_i &= \xi_{i+1} \xi_i \xi_{i+1};
\end{align*}
\]  \hspace{1cm} (4)

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i & |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1};
\end{align*}
\]  \hspace{1cm} (5)

\[
\begin{align*}
\xi_i \sigma_j &= \sigma_j \xi_i & |i - j| > 1, \\
\xi_i \xi_{i+1} \sigma_i &= \sigma_{i+1} \xi_i \xi_{i+1}, \\
\sigma_i \sigma_{i+1} \xi_i &= \xi_{i+1} \sigma_i \sigma_{i+1},
\end{align*}
\]  \hspace{1cm} (6)
BP\(_n\) is also characterized as the subgroup of \(\text{Aut}(F_n)\) of automorphism \(\phi \in \text{Aut}(F_n)\) of permutation-conjugacy type:

\[
(x_i)\phi = w_i^{-1}x_{\lambda(i)}w_i
\]

for a word \(w_i \in F_n\) and permutation \(\lambda \in \Sigma_n\).
Theorem

The group $BP_n$ is the semi-direct product of the symmetric group $\Sigma_n$ and the group $P\Sigma_n$ with a split extension

$$1 \longrightarrow P\Sigma_n \longrightarrow BP_n \longrightarrow \Sigma_n \longrightarrow 1.$$
6. Properties of groups $P\Sigma_n$ and $P\Sigma_n^+$

Theorem (McCool 86)

A presentation of $P\Sigma_n$ is given by generators $\chi_{k,j}$ together with the following relations.

1. $\chi_{i,j} \cdot \chi_{k,j} \cdot \chi_{i,k} = \chi_{i,k} \cdot \chi_{i,j} \cdot \chi_{k,j}$ for $i, j, k$ distinct.
2. $[\chi_{k,j}, \chi_{s,t}] = 1$ if $\{j, k\} \cap \{s, t\} = \emptyset$.
3. $[\chi_{i,j}, \chi_{k,j}] = 1$ for $i, j, k$ distinct.
4. $[\chi_{i,j} \cdot \chi_{k,j}, \chi_{i,k}] = 1$ for $i, j, k$ distinct (redundantly).
The subgroup of $P \Sigma_n$ generated by the $\chi_{k,i}$ for $i < k$ is denoted $P \Sigma_n^+$ and is called the “upper triangular McCool group".
The next several Theorems are from the joint work with Fred Cohen, J. Pakianathan and Jie Wu.
Theorem

The homomorphism

$$\pi : P\Sigma_n \to P\Sigma_{n-1}$$

defined by

$$\pi(\chi_{k,i}) = \begin{cases} 
\chi_{k,i} & \text{if } i < n, \text{ and } k < n, \\
1 & \text{if } i = n \text{ or } k = n.
\end{cases}$$

is an epimorphism. The kernel $K_n$ of $\pi$ is generated by $\chi_{n,i}$ and $\chi_{j,n}$ for $1 \leq i, j \leq n - 1$. This extension is split and the conjugation action of $P\Sigma_{n-1}$ on $H_1(K_n)$ is trivial.
Theorem

The homomorphism \( \pi : P \Sigma_n \to P \Sigma_{n-1} \) restricts to a homomorphism

\[
\pi \big|_{P \Sigma_n^+} : P \Sigma_n^+ \to P \Sigma_{n-1}^+
\]

which is an epimorphism. The kernel \( K_n^+ \) of \( \pi \big|_{P \Sigma_n^+} \) is a free group freely generated by the elements \( \chi_{n,i} \) for \( 1 \leq i \leq n-1 \). This extension is split and the conjugation action of \( P \Sigma_{n-1}^+ \) on \( H_1(K_n^+) \) is trivial.
Theorem

There is a split short exact sequence

$$0 \to gr^*(K_n) \to gr^*(P \Sigma_n) \to gr^*(P \Sigma_{n-1}) \to 0$$

and the following relations in $gr^*(P \Sigma_n)$:

1. If $\{i, j\} \cap \{s, t\} = \emptyset$, then $[\chi_{j,i}, \chi_{s,t}] = 0$.
2. If $i, j, k$ are distinct, then $[\chi_{i,k}, \chi_{i,j} + \chi_{k,j}] = 0$.
3. If $i, j, k$ are distinct, the element $[\chi_{k,i}, \chi_{j,i} + \chi_{j,k}]$ is non-zero.
4. If $i, j, k$ are distinct, $[\chi_{i,j}, \chi_{k,j}] = 0$. 
Theorem

There is a split epimorphism

$$\gamma : K_n \to \bigoplus_{n-1} \mathbb{Z}$$

with kernel denoted $\Lambda_n$ together with a split short exact sequence

$$0 \to \mathbb{L}_n \to \text{gr}^*(K_n) \xrightarrow{\text{gr}^*(\gamma)} \bigoplus_{n-1} \mathbb{Z} \to 0$$

where $\mathbb{L}_n$ is the Lie algebra kernel of $\text{gr}^*(\gamma)$.
Theorem

The Lie algebra \( \text{gr}^*(P \Sigma^+_n) \), is generated by \( \chi_{k,i}, i < k \) subject to the relations

- \([\chi_{k,j}, \chi_{s,t}] = 0 \) if \( \{j, k\} \cap \{s, t\} = \emptyset \),
- \([\chi_{k,j}, \chi_{s,j}] = 0 \) if \( \{s, k\} \cap \{j\} = \emptyset \),
- \([\chi_{i,k}, \chi_{i,j} + \chi_{k,j}] = 0 \) for \( j < k < i \).

Additively it is isomorphic to a direct sum of free sub-Lie algebras

\[ \oplus_{2 \leq k \leq n} L[\chi_{k,1}, \chi_{k,2}, \ldots, \chi_{k,k-1}] . \]
Theorem

1. $H^k(P^+ \Sigma_n)$ is a finitely generated, torsion-free abelian group.

2. If $1 \leq k \leq n$, a basis for $H^k P \Sigma^+_n$ is given by

   $\chi_{i_1,j_1}^* \cdot \chi_{i_2,j_2}^* \cdots \chi_{i_k,j_k}^*$

   where $2 \leq i_1 < i_2 < \cdots < i_k \leq n$, and $1 \leq j_t < i_t$ for all $t$.

3. A complete set of relations:

   - $\chi_{i,k}^* \cdot \chi_{i,k}^* = 0$ for all $i > k$, and
   - $\chi_{i,j}^* [\chi_{i,k}^* - \chi_{j,k}^*] = 0$ for $k < j < i$. 
9. Pure braid group of a sphere

For a sphere let us introduce the elements $a_{i,j}$ for all $i,j$:

$$\begin{cases}
a_{j,i} = a_{i,j} \text{ for } i < j \leq n,
a_{i,i} = 1.
\end{cases} \quad (8)$$

The pure braid group for the sphere has the generators $a_{i,j}$ which satisfy Burau relations, relations (8), and the following relations:

$$a_{i,i+1}a_{i,i+2}\cdots a_{i,i+n-1} = 1 \text{ for all } i \leq n,$$

where $k + n$ is considered modulo $n$ having in mind (8).
Y. Ihara gave a presentation of the Lie algebra $\text{gr}^*(P_n(S^2))$ of the pure braid group of the sphere.

It is convenient to have conventions like (8).

Lie algebra $\text{gr}^*(P_n(S^2))$ is the quotient of the free Lie algebra $L[B_{i,j} | 1 \leq i, j \leq n]$
Modulo the following relations:

\[
\begin{align*}
  B_{i,j} &= B_{j,i} \text{ for } 1 \leq i, j \leq n, \\
  B_{i,i} &= 0 \text{ for } 1 \leq i \leq n, \\
  [B_{i,j}, B_{s,t}] &= 0, \text{ if } \{i,j\} \cap \{s,t\} = \emptyset, \\
  \sum_{j=1}^{n} B_{i,j} &= 0, \text{ for } 1 \leq i \leq n.
\end{align*}
\]
For surfaces of positive genus $R$. Bezrukavnikov determined the Lie algebra $gr^*(P_n(S_g)) \otimes \mathbb{Q}$. It has the generators $C_{l,i}$, $D_{l,i}$ $1 \leq l \leq g$, $1 \leq i \leq n$, so that $l$ corresponds to the genus of a surface and $i$ corresponds to the number of strings.
Relations are as follows

\[
\begin{align*}
\{ [C_{l,i}, D_{k,j}] &= 0 \text{ for } i \neq j, \ l \neq k, \\
[C_{l,i}, C_{k,j}] &= 0 \text{ for } i \neq j, \\
[D_{l,i}, D_{k,j}] &= 0 \text{ for } i \neq j, \\
[C_{l,i}, D_{l,j}] &= [C_{k,j}, D_{k,i}] \text{ for all } k, l \text{ and } i \neq j,
\end{align*}
\]

(9)

denote it by \(S_{i,j},\)

\[
\sum_{i=1}^{g} [C_{l,i}, D_{l,i}] = -\sum_{j \neq i} S_{i,j},
\]

\[
\begin{align*}
[C_{l,i}, S_{j,k}] &= 0 \text{ for } i \neq j \neq k, \\
[D_{l,i}, S_{j,k}] &= 0 \text{ for } i \neq j \neq k.
\end{align*}
\]
We show that this presentation is true over \( \mathbb{Z} \).

The essential ingredient in the study of the pure braid groups is a natural fibration of configuration spaces and its initial term of the homotopy exact sequence.
We remind that $F(S_g, n)$ is the space of $n$-tuples of pairwise different points in $S_g$, then we have the following fibration:

$$S_g \setminus Q_{n-1} \to F(S_g, n) \to F(S_g, n - 1),$$

where $S_g \setminus Q_{n-1}$ is the surface with $n - 1$ points deleted and $F(S_g, n) \to F(S_g, n - 1)$ is the projection on the first $n - 1$ components of an $n$-tuple; $S_g \setminus Q_{n-1}$ is a fiber of the fibration.
The above fibration generates the exact sequence of groups:

\[ 1 \to \pi_1(S_g \setminus Q_{n-1}) \to P_n(S_g) \to P_{n-1}(S_g) \to 1, \quad (10) \]

where \( \pi_1(S_g \setminus Q_{n-1}) \) is a \((g, n - 1)\)-surface group; it is a free group of rank \(2g + n - 2\) and it has the canonical presentation

\[
\pi_{g,n-1} = \pi_1(S_g \setminus Q_{n-1}) = \left< c_1, d_1, \ldots, c_g, d_g, u_1, \ldots, u_{n-1} \mid \prod_{i=1}^{n-1} u_i \prod_{m=1}^{g} [c_m, d_m] = 1 \right>.
\]
If we consider the descending central series filtration of the groups of this exact sequence and apply the functor of the associated Lie algebras, then the corresponding sequence will be not left exact for $n \geq 3$:

$$\text{gr}^*(\pi_1(S_g \setminus Q_{n-1})) \to \text{gr}^*(P_n(S_g)) \to \text{gr}^*(P_{n-1}(S_g)) \to 1.$$
To fix the situation we use another filtration which was introduced by M. Kaneko and H. Nakamura and H. Tsunogai. The authors call it the *weight filtration*. Roughly speaking the difference with respect to the descending central series filtration is that the elements $u_1, \ldots, u_{n-1}$ are given the grading two instead of one.
Let us denote by $\pi_{g,k}(m)$ and by $P_n(S_g)(m)$ the subgroups of $\pi_{g,k}$ and $P_n(S_g)$ of the weight filtration $m$. For $\pi_{g,0}$ and $\pi_{g,1}$ the weight filtration coincides with the descending central series filtration.

$$\pi_{g,k}(1) = \pi_{g,k},$$

$$\pi_{g,k}(2) = [\pi_{g,k}, \pi_{g,k}] < u_1, \ldots, u_k >,$$

$$\pi_{g,k}(m) = < [\pi_{g,k}(i), \pi_{g,k}(j)] | i + j = m >, m \geq 3.$$
\[ P_n(S_g)(1) = P_n(S_g), \]
\[ P_n(S_g)(2) = [P_n(S_g), P_n(S_g)]_{\pi_g,n-1}(2), \]
\[ P_n(S_g)(m) = \langle [P_n(S_g)(i), P_n(S_g)(j)] \mid i + j = m \rangle, m \geq 3. \]
It is proved by H. Nakamura, N. Takao and R. Ueno that the sequence of Lie algebras associated to the weight filtration and corresponding to the sequence (10)

\[ 1 \rightarrow \text{gr}_{w}^{*}(\pi_{1}(S_{g} \setminus Q_{n-1})) \rightarrow \text{gr}_{w}^{*}(P_{n}(S_{g})) \rightarrow \text{gr}_{w}^{*}(P_{n-1}(S_{g})) \rightarrow 1 \]

is exact (even for a punctured surface).
Theorem (joint with B. Enriquez)

For the pure braid group of a closed surface of the genus $g$, $P_n(S_g)$, the descending central series filtration and the weight filtration coincide. The generators $C_{l,i}, D_{l,i} \ 1 \leq l \leq g, \ 1 \leq i \leq n$, and the Bezrukavnikov relations (9) give a presentation of the graded Lie algebra $\text{gr}^*(P_n(S_g))$. 
Proof.
Let now $\Gamma_i$ be the descending central series filtration on $P_n(S_g)$. We consider the presentation of a surface pure braid group given by Gonçalves and Guaschi. There are $2ng$ generators: $\rho_{i,l}$ and $\tau_{i,l}$, $1 \leq l \leq g$, $1 \leq i \leq n$, and relations of 30 types. We have $\Gamma_1 = P_n(S_g)(1)$, $\Gamma_2 \subset P_n(S_g)(2)$, denote this inclusion by $\psi$ and the corresponding quotient map by $\phi$. 
There is a commutative diagram of the exact sequences

\[
\begin{array}{cccccc}
1 & \rightarrow & \Gamma_2 & \rightarrow & P_n(S_g) & \rightarrow & g_1^1(P_n(S_g)) & \rightarrow & 1 \\
\psi & \downarrow & & \downarrow & \text{Id} & & \phi & \downarrow \\
1 & \rightarrow & P_n(S_g)(2) & \rightarrow & P_n(S_g) & \rightarrow & g_1^1(P_n(S_g)) & \rightarrow & 1.
\end{array}
\]

The map \(\phi\) is surjective and it follows from the work of Gonçalves and J. Guaschi that the rank of \(g_1^1(P_n(S_g))\) is equal to \(2gn\). The rank of \(g_1^1(P_n(S_g))\) can not be bigger than \(2gn\), so it is an isomorphism. Hence \(\psi\) is also an isomorphism. From the definition of \(P_n(S_g)(m)\) it follows that \(\Gamma_m = P_n(S_g)(m)\).
The Lie algebra for torus

It has the generators $C_i, D_i$, $1 \leq i \leq n$, $i$ corresponds to the number of a string. Relations are as follows:

\[
\begin{align*}
[C_i, C_j] &= 0 \text{ for } i \neq j, \\
[D_i, D_j] &= 0 \text{ for } i \neq j, \\
[C_i, D_j] &= [C_j, D_i] \text{ for all } i \neq j, \text{ denote it by } S_{i,j}, \\
[C_i, D_i] &= -\sum_{j \neq i} S_{i,j}, \\
[C_i, S_{j,k}] &= 0 \text{ for } i \neq j \neq k, \\
[D_i, S_{j,k}] &= 0 \text{ for } i \neq j \neq k.
\end{align*}
\]
A braid is Brunnian if it becomes trivial after removing any one of its strands.

We consider operations

$$d_i : Br_n(M) \rightarrow Br_{n-1}(M)$$

which are obtained by forgetting the $i$-th strand, $1 \leq i \leq n$. 
We can interpret a Brunnian braid $\beta \in Br_n(M)$ as a solution of system of $n$ equations

$$\begin{cases} 
    d_1(\beta) = 1, \\
    \ldots \\
    d_n(\beta) = 1. 
\end{cases} \tag{11}$$

Let $Brun_n(M)$ denote the set of the $n$-strand Brunnian braids. Then $Brun_n(M)$ forms a normal subgroup of $Br_n(M)$. 
A typical example of a 3-strand Brunnian braid on a disk is the braid given by the expression $(\sigma_1^{-1} \sigma_2)^3$, where $\sigma_1$ and $\sigma_2$ are the standard generators of the 3-strand braid group $\langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$. 
The next results are from the joint work with Jingyan Li and Jie Wu.
We consider the restriction \( \{ \Gamma_q(P_n) \cap \text{Brun}_n \} \) of the descending central series of \( P_n \) to \( \text{Brun}_n \). This gives a relative Lie algebra

\[
L^P(\text{Brun}_n) = \bigoplus_{q=1}^{\infty} \frac{\Gamma_q(P_n) \cap \text{Brun}_n}{\Gamma_{q+1}(P_n) \cap \text{Brun}_n}.
\]
Proposition

$L^P(\text{Brun}_n)$ is a Lie algebra, it is a two-sided Lie ideal in $L(P_n)$. 
The removing-strand operation on braids induces an operation

\[ d_k : L(P_n) \rightarrow L(P_{n-1}) \]

formulated by

\[
d_k(A_{i,j}) = \begin{cases} 
A_{i,j} & \text{if } i < j < k \\
0 & \text{if } k = j \\
A_{i,j-1} & \text{if } i < k < j \\
0 & \text{if } k = i \\
A_{i-1,j-1} & \text{if } k < i < j.
\end{cases}
\]

(13)
A sequence of sets $S = \{S_n\}_{n \geq 0}$ is called a bi-$\Delta$-set if there are faces $d_j : S_n \to S_{n-1}$ and co-faces $d^j : S_{n-1} \to S_n$ for $0 \leq j \leq n$ such that the following identities hold:

1. $d_j d_i = d_i d_{j+1}$ for $j \geq i$;
2. $d^j d^i = d^{i+1} d^j$ for $j \leq i$;
3. $d_j d^i = \begin{cases} d^{i-1} d_j & \text{if } j < i, \\ id & \text{if } j = i, \\ d^i d_{j-1} & \text{if } j > i. \end{cases}$
A sequence of groups $G$ is called a bi-$\Delta$-group if $G$ is a bi-$\Delta$-set such that all faces and co-faces are group homomorphism.
Let $\mathbb{P}_n = P_{n+1}$. The sequence of groups $\mathbb{P} = \{\mathbb{P}_n\}_{n \geq 0}$ with faces relabeled as $\{d_0, d_1, \ldots\}$ and co-faces relabeled as $\{d^0, d^1, \ldots\}$ forms a bi-$\Delta$-group structure. Where the face operation $d_i : \mathbb{P}_n \to \mathbb{P}_{n-1} = d_{i+1} : P_{n+1} \to P_n$ is obtained by deleting the $i + 1$st string, the co-face operation $d^i : \mathbb{P}_n \to \mathbb{P}_{n+1}$ is obtained by adding a trivial $i + 1$st string in front of the other strings ($i = 0, 1, 2, \cdots, n$).
Proposition

The relative Lie algebra $L^P(Bun_n)$ is the Lie subalgebra
$\bigcap_{i=1}^{n} \ker(d_i : L(P_n) \to L(P_{n-1}))$. 
Our next step is to determine a set of generators for the Lie algebra $L^P(Brun_n)$. 
The following fact is a Lie algebra analogue of the theorem proved by A. A. Markov for the pure braid group.

**Proposition**

The kernel of the homomorphism $d_n : L(P_n) \rightarrow L(P_{n-1})$ is a free Lie algebra, generated by the free generators $A_{i,n}$, for $1 \leq i \leq n-1$.

$$\text{Ker}(d_n : L(P_n) \rightarrow L(P_{n-1})) = L[A_{1,n}, \ldots, A_{n-1,n}].$$
For a set $Z$, let $L[Z]$ denote the free Lie algebra freely generated by $Z$. Let $X$ and $Y$ be non-empty sets with $X \cap Y = \emptyset$, $X \cup Y = Z$. Let $\pi$ be the Lie homomorphism

$$\pi : L[Z] \longrightarrow L[Y]$$

such that $\pi(x) = 0$ for $x \in X$ and $\pi(y) = y$ for $y \in Y$. 

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Braids and Lie algebras
Proposition

The kernel of $\pi$ is a free Lie algebra, generated by the following family of free generators:

$$x, [\cdots [x, y_1], \cdots, y_t]$$

(14)

for $x \in X, y_i \in Y$ for $1 \leq i \leq t$. 
Proposition

The intersection of the kernels of the homomorphisms $d_n$ and $d_k$, $k \neq n$, is a free Lie algebra, generated by the following infinite family of free generators:

$$A_{k,n}, \ldots [A_{k,n}, A_{j_1,n}], \ldots , A_{j_m,n}$$

for $j_i \neq k, n; j_i \leq n - 1; i \leq m; m \geq 1$: 

(15)
Another set of free generators of $\text{Ker}(d_n) \cap \text{Ker}(d_k)$ can be obtained using Hall bases.
We suppose that all Lie monomials on $B_1, \ldots, B_k$ are ordered lexicographically. Lie monomials $B_1, \ldots, B_k$ are the standard monomials of degree 1.

If we have defined standard monomials of degrees 1, $\ldots$, $n - 1$, then $[u, v]$ is a standard monomial if both of the following conditions hold:

(1) $u$ and $v$ are standard monomials and $u > v$.

(2) If $u = [x, y]$ is the form of the standard monomial $u$, then $v \geq y$.

Standard monomials form the Hall basis of a free Lie algebra (also over $\mathbb{Z}$).
Examples of standard monomials are the products of the type:

$$\cdots [B_{j_1}, B_{j_2}, B_{j_3}, \ldots, B_{j_t}], \ j_1 > j_2 \leq j_3 \leq \cdots \leq j_t. \quad (16)$$
Proposition

The intersection $\ker(d_n) \cap \ker(d_k)$, $k \neq n$, is a free Lie algebra, generated by the standard monomials on $A_{i,n}$ where the letter $A_{k,n}$ has only one enter. In other words the free generators are standard monomials which are products of monomials of type (16) where only one such monomial contains one copy of $A_{k,n}$. 
We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_1,n, A_2,n, \ldots, A_{n-1},n\}$.

2) Suppose that $\mathcal{K}(n)_{k+1}$ is defined as a subset of Lie monomials on the letters

$$A_1,n, A_2,n, \ldots, A_{n-1},n$$

with $k < n$. Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries} \}.$$ 

3) Define

$$\mathcal{K}(n)_k = \{W' \text{ and } [\cdots [[W', W_1], W_2], \ldots, W_t]\}$$

for $W' \in \mathcal{K}(n)_{k+1} \setminus \mathcal{A}_k$ and $W_1, W_2, \ldots, W_t \in \mathcal{A}_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_1,n, A_2,n, \ldots, A_{n-1},n$. 
Example

Let $n = 3$. The set $\mathcal{K}(3)_1$ is constructed by the following steps:

1) $\mathcal{K}(3)_3 = \{A_{1,3}, A_{2,3}\}$.

2) $A_2 = \{A_{1,2}\},$

$$\mathcal{K}(3)_2 = \{A_{2,3}, [A_{2,3}, A_{1,3}], \ldots, A_{1,3}\}.$$ 

3) $A_1 = \{A_{2,3}\},$

$$\mathcal{K}(3)_1 = \{[\cdots [A_{2,3}, A_{1,3}], \ldots, A_{1,3}], A_{2,3}, \ldots, A_{2,3}]\}.$$
Theorem

The Lie algebra $L^P(B_{	ext{Br}} n)$ is a free Lie algebra generated by $K(n)_1$ as a set of free generators.
The Rank of $L^P_q(\text{Brun}_n)$

Observe that the Lie algebra $L(P)$ is of finite type in the sense that each homogeneous component $L_k(P_n)$ is a free abelian group of finite rank. Thus the subgroup

$$L^P(P_n) \cap L_k(P_n)$$

is a free abelian group of finite rank. We give now a formula for the rank of $L^P_q(\text{Brun}_n)$
A decomposition formula on bi-$\Delta$-groups

Let $G = \{G_n\}_{n \geq 0}$ be a bi-$\Delta$-group. Define

$$Z_n(G) = \bigcap_{i=0}^{n} \ker(d_i: G_n \to G_{n-1}).$$

Theorem (Decomposition Theorem of bi-$\Delta$-groups)

Let $G = \{G_n\}_{n \geq 0}$ be a bi-$\Delta$-group. Then $G_n$ is the (iterated) semi-direct product the subgroups

$$d^{i_k} d^{i_{k-1}} \cdots d^{i_1}(Z_{n-k}(G)),$$

for $0 \leq i_1 < \cdots < i_k \leq n$, $0 \leq k \leq n$, with lexicographic from right.
Corollary

Let $\mathcal{G} = \{ G_n \}_{n \geq 0}$ be a bi-$\Delta$-group such that each $G_n$ is an abelian group. Then there is direct sum decomposition

$$G_n = \bigoplus d^{i_k} d^{i_{k-1}} \cdots d^{i_1} (\mathbb{Z}_{n-k}(\mathcal{G}))$$

for each $n$.

$$0 \leq i_1 < \cdots < i_k \leq n$$

$$0 \leq k \leq n$$
Let $\mathcal{G} = L_q(\mathbb{P})$. Then $\mathcal{Z}_n(L_q(\mathbb{P})) = L_q^{P}(\text{Brun}_{n+1})$. 

$d^i = d^{i-1} : \mathbb{P}_{n-1} = P_n \rightarrow \mathbb{P}_n = P_{n+1}$ is obtained by adding a trivial $i$th string in front of the other strings ($i = 1, 2, \ldots, n$).
Proposition

There is a decomposition

\[ L_q(P_n) = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} d^{i_k} d^{i_{k-1}} \cdots d^{i_1}(L_q^P(\text{Brun}_{n-k})) \]

for each \( n \) and \( q \).

\( \square \)
**Corollary**

There is a formula

\[
\text{rank}(L_q(P_n)) = \sum_{k=0}^{n-1} \binom{n}{k} \text{rank}(L_q^P(Brun_{n-k}))
\]

for each \(n\) and \(q\).
Theorem

\[ \text{rank}(L^P_q(\text{Brun}_n)) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \text{rank}(L_q(P_{n-k})) \]

for each \( n \) and \( q \), where \( P_1 = 0 \) and, for \( m \geq 2 \),

\[ \text{rank}(L_q(P_m)) = \frac{1}{q} \sum_{k=1}^{m-1} \sum_{d \mid q} \mu(d) k^{q/d} \]

with \( \mu \) the Möbius function.