Dropping Polishness

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How do we "generalize" descriptive set theory?

Recall that a space is *Polish* if it is separable and completely metrizable. I can think of three ways...

1. Dropping separability: the prototypical space is κ^ω, where κ has the discrete topology. It is completely metrizable, but not necessarily separable. (Ask Sergey Medvedev...)
2. Dropping everything: the prototypical space is κ^κ with the $<\kappa$-box-topology, where κ has the discrete topology and it satisfies $\kappa^{<\kappa} = \kappa$. (Ask Sy Friedman...)
3. Dropping Polishness: consider questions of "descriptive set-theoretic flavor" in spaces that are separable and metrizable, but not necessarily **completely** metrizable. (Ask Arnie Miller...)

From now on, we will assume that every space is separable and metrizable, but not necessarily Polish.

How do you define complexity then?

Γ will always be one of the following (boldface) pointclasses.

- $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$, where ξ is an ordinal such that $1 \leq \xi < \omega_1$ (these are the *Borel pointclasses*).
- $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$, where n is an ordinal such that $1 \leq n < \omega$ (these are the *projective pointclasses*).

We will assume that the definition of a Γ subset of a Polish space is well-known, and recall that it can be generalized to arbitrary spaces as follows.

Definition

Fix a pointclass Γ. Let X be a space. We will say that $A \subseteq X$ is a Γ *subset of* X if there exists a Polish space T containing X as a subspace such that $A = B \cap X$ for some Γ subset B of T.

In the case of the Borel pointclasses, this is not really necessary, because the usual definition works in arbitrary spaces. But we prefer to give a unified treatment.

The following “reassuring” proposition can be proved by induction on Γ.

Proposition

Fix a pointclass Γ. Let X be a space and $A \subseteq X$. Then the following conditions are equivalent.

- *A is a Γ subset of X.*
- *For every space T containing X as a subspace there exists a Γ subset B of T such that $A = B \cap X$.*

One could also define the so-called *ambiguous pointclasses* as follows.

- Let $\Delta^0_\xi = \Sigma^0_\xi \cap \Pi^0_\xi$ for every ordinal ξ such that $1 \leq \xi < \omega_1$.
- Let $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$ for every ordinal n such that $1 \leq n < \omega$.

However, the above proposition fails for these classes, already at the level $\Delta^0_1 =$ clopen.

Illustrious precedents

Definition

The *Baire order* $\mathrm{ord}(X)$ of a space X is the minimum ordinal α such that $\Sigma^0_\alpha(X) = \mathrm{Borel}(X)$.

Examples:

- $\mathrm{ord}(X) = 1$ if and only if X is discrete.
- $\mathrm{ord}(X) \leq 2$ whenever X is countable, and $\mathrm{ord}(\mathbb{Q}) = 2$.
- $\mathrm{ord}(X) \leq \omega_1$ for every space.
- $\mathrm{ord}(X) = 3$ if X is a Luzin set. (Recall that an uncountable set of reals X is a *Luzin set* if every uncountable subset of X is non-meager. They exist under CH.)
- $\mathrm{ord}(X) = 2$ if X is a Sierpiński set. (Recall that an uncountable set of reals X is a *Sierpiński set* if every uncountable subset of X is non-null. They exist under CH.)

Theorem (Lebesgue)

$\operatorname{ord}(X) = \omega_1$ *for every uncountable Polish space* X*.*

Banach asked whether we can “drop Polishness”...

Conjecture (Banach)

$\operatorname{ord}(X) = \omega_1$ *for every uncountable space* X*.*

Theorem (Miller, 1979)

It is consistent that $\operatorname{ord}(X) = \omega_1$ *for every uncountable space* X*.*

Theorem (Kunen)

Assume CH*. Then for every* α *such that* $1 \leq \alpha < \omega_1$ *there exists a space* X *such that* $\operatorname{ord}(X) = \alpha$*.*

Miller improved this by showing that CH can be weakened to the existence of a Luzin set.

Perfect set property: the classical case

Definition

Let X be a space and Γ a pointclass. We will say that X has the *perfect set property for* Γ *subsets* (briefly, the $\mathrm{PSP}(\Gamma)$) if every Γ subset of X is either countable or it contains a copy of 2^ω.

One of the classical problems of descriptive set theory consists in determining for which pointclasses Γ the statement "Every Polish space has the $\mathrm{PSP}(\Gamma)$" holds. The following three famous theorems essentially solve this problem.

Theorem

- *(Suslin) Every Polish space has the* $\mathrm{PSP}(\text{analytic})$.
- *(Gödel) Assume* $\mathrm{V} = \mathrm{L}$. *Then no uncountable Polish space has the* $\mathrm{PSP}(\text{coanalytic})$.
- *(Davis) Assume the axiom of Projective Determinacy. Then every Polish space has the* $\mathrm{PSP}(\text{projective})$.

Perfect set property: the non-Polish case

But what happens in arbitrary (that is, not necessarily Polish) spaces? By the following simple proposition, the problem described above becomes trivial.

Recall that a subset B of an uncountable Polish space T is a *Bernstein set* if $B \cap K \neq \varnothing$ and $(T \setminus B) \cap K \neq \varnothing$ for every copy K of 2^ω in T. It is easy to see that Bernstein sets exist in ZFC. Since $2^\omega \approx 2^\omega \times 2^\omega$, every Bernstein set has size $\mathfrak{c}$.

Proposition

Let X be a Bernstein set in some uncountable Polish space. Then X does not have the $\mathrm{PSP}(\Gamma)$ *for any poinclass Γ.*

Proof: Let Γ be a pointclass. Then X itself is an uncountable Γ subset of X that does not contain any copy of 2^ω.

Thank you for your attention

and good night!

Perfect set property: the non-Polish case, II

Much less trivial, however, is to determine the status of the statement

> *"For every space X, if X has the* PSP(Γ) *then X has the* PSP(Γ')*"*

as Γ, Γ' range over all pointclasses of complexity at most analytic. We will focus on the case "PSP(analytic) vs. PSP(closed)".

Theorem

The following are equivalent.

- *For every space X, if X has the* PSP(closed) *then X has the* PSP(analytic)*.*
- $\mathfrak{b} > \omega_1$.

We will prove the interesting half of the above theorem.

A first attempt (suggested by Kunen)

Assume $\mathfrak{b} = \omega_1$. We want to construct a space X such that

- Every uncountable closed subset of X contains a copy of 2^ω, but
- There exists an uncountable analytic subset of X that contains no copies of 2^ω.

Using $\mathfrak{b} = \omega_1$, fix a family $Z = \{f_\alpha : \alpha \in \omega_1\} \subseteq \omega^\omega$ such that

- Z is unbounded (there exists no $g \in \omega^\omega$ such that $f <^* g$ for all $f \in Z$),
- Z is well-ordered ($f_\alpha <^* f_\beta$ whenever $\alpha < \beta$).

X will be a subspace of $T = (\omega + 1)^\omega \approx 2^\omega$, where $\omega + 1 = \omega \cup \{\omega\}$ is the converging sequence with limit ω. Also define $T_n = \{x \in T : x(n) = \omega\}$ for every $n \in \omega$, so that

$$T = \omega^\omega \cup \bigcup_{n \in \omega} T_n.$$

Define

$$X = Z \cup \bigcup_{n \in \omega} T_n \subseteq (\omega + 1)^\omega$$

Lemma

Let Z be an unbounded family. Then

- $\mathrm{cl}(Z)$ *is not compact, where the closure is taken in* ω^ω,
- $\mathrm{cl}(Z) \not\subseteq \omega^\omega$, *where the closure is taken in* $(\omega + 1)^\omega$.

Lemma

Let Z be a **well-ordered** *unbounded family of size* ω_1.
Then, for every uncountable $Y \subseteq Z$,

- $\mathrm{cl}(Y)$ *is not compact, where the closure is taken in* ω^ω,
- $\mathrm{cl}(Y) \not\subseteq \omega^\omega$, *where the closure is taken in* $(\omega + 1)^\omega$.

Since each $T_n \approx (\omega + 1)^\omega$ is compact, Z is a G_δ subset of X. So, by the second lemma, Z witnesses that X does not have the $\mathrm{PSP}(\mathrm{G}_\delta)$. In particular, X does not have the PSP(analytic).

What goes wrong? (And how do we fix it?)

We still have to show that every uncountable closed subset of X contains a copy of 2^ω.

Dream

Is it true that $\mathrm{cl}(Y) \cap \bigcup_{n\in\omega} T_n$ *is uncountable whenever* $Y \subseteq Z$ *is uncountable?*

Reality

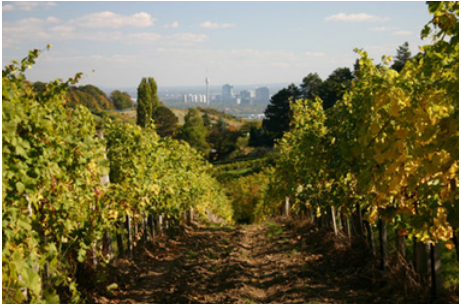
If Z *consists only of increasing functions, then*

$$\mathrm{cl}(Z) \cap \bigcup_{n\in\omega} T_n \subseteq \{s^\frown\langle\omega,\omega,\ldots\rangle : s \in \omega^{<\omega}\},$$

which is countable!

The solution comes from a property of certain subsets of 2^ω. Since the speaker is just a romantic little guy, he decided to name it after the lovely area of Vienna where he lives...

Introducing



the Grinzing property!

The Grinzing property

Definition

We will say that a subset W of 2^ω has the *Grinzing property* (briefly, the GP) if it is uncountable and for every uncountable $Y \subseteq W$ there exist uncountable subsets Y_α of Y for $\alpha \in \omega_1$ such that $\mathrm{cl}(Y_\alpha) \cap \mathrm{cl}(Y_\beta) = \varnothing$ whenever $\alpha \neq \beta$, where the closure is taken in 2^ω.

Notice that an uncountable $W \subseteq 2^\omega$ has the GP if and only if every subset of W of size ω_1 has the GP.
Could the whole 2^ω have the GP?

Theorem

- *Assume* CH. *Then* 2^ω *does not have the* GP.
- *Assume* $\mathrm{MA} + \neg\mathrm{CH}$. *Then* 2^ω *has the* GP.

Why do we care?

As we have just seen, it is consistent that **all** uncountable subsets of 2^ω have the GP. Therefore, it seems likely that there exists at least **one** in ZFC...

Question

Is it possible to prove in ZFC *that there exists a subset of* 2^ω *with the* GP*?*

This would allow us to finish the proof of the main theorem!

Let W be a subset of 2^ω with the GP such that $|W| = \omega_1$.

Fix an injective enumeration $W = \{g_\alpha : \alpha \in \omega_1\}$.

	Old "proof"	**New proof**
T	$(\omega+1)^\omega$	$(\omega+1)^\omega \times 2^\omega$
T_n	$\{x \in (\omega+1)^\omega : x(n) = \omega\}$	$\{x \in (\omega+1)^\omega : x(n) = \omega\} \times 2^\omega$
Z	$\{f_\alpha : \alpha \in \omega_1\}$	$\{\langle f_\alpha, g_\alpha\rangle : \alpha \in \omega_1\}$

Notice that each $T_n \approx (\omega+1)^\omega \times 2^\omega \approx 2^\omega$ as before.

Define

$$X = Z \cup \bigcup_{n \in \omega} T_n \subseteq (\omega + 1)^{\omega} \times 2^{\omega}$$

The PSP(analytic) still fails

As before, Z is a G_δ subset of X. If Z contained a copy of 2^ω, then $\pi[Z]$ would too, where $\pi : \omega^\omega \times 2^\omega \longrightarrow \omega^\omega$ is the projection on the first coordinate.

Thanks to the GP, we can finally prove the PSP(closed)

As before, we have to prove that $\mathrm{cl}(Y) \cap \bigcup_{n\in\omega} T_n$ is uncountable for every uncountable $Y \subseteq Z$. The difference is that now, it **will** be enough to prove $\mathrm{cl}(Y) \cap \bigcup_{n\in\omega} T_n \neq \varnothing$ for every such Y. In fact, Y will be in the form $\{\langle f_\alpha, g_\alpha \rangle : \alpha \in S\}$ for some uncountable $S \subseteq \omega_1$. Then look at

$$\{g_\alpha : \alpha \in S\} \subseteq W$$

and use the fact that W has the GP.

OK then. Why is it non-empty?

Fix an uncountable $Y \subseteq Z$. We have to prove that $\mathrm{cl}(Y) \cap \bigcup_{n\in\omega} T_n \neq \varnothing$.

Since being a well-ordered unbounded family is preserved by taking uncountable subsets, we can assume that $Y = Z$.

By an old lemma, there exists $f \in (\omega+1)^\omega \setminus \omega^\omega$ and a sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of ω_1 such that

$$\langle f_{\alpha_n} : n \in \omega \rangle \longrightarrow f$$

Since 2^ω is compact, there exists $g \in 2^\omega$ and a subsequence of $\langle g_{\alpha_n} : n \in \omega \rangle$ that converges to g in 2^ω.

It follows that the corresponding subsequence of $\langle \langle f_{\alpha_n}, g_{\alpha_n} \rangle : n \in \omega \rangle$ converges to $\langle f, g \rangle$, which is clearly an element of $\bigcup_{n\in\omega} T_n$.

A set with the Grinzing property in ZFC

Theorem (Todorčević)

There exists a subtree $\mathcal{T}$ of $\omega^{<\omega_1}$ and a system $\langle K_s : s \in \mathcal{T}\rangle$ of perfect subsets of 2^ω satisfying the following properties.

- *Each level of $\mathcal{T}$ is countable and non-empty.*
- $K_t \subsetneq K_s$ *whenever* $t \supsetneq s$.
- $K_s \cap K_t = \varnothing$ *whenever* $s \perp t$.

Since there are no strictly descending ω_1-sequences of closed subsets of 2^ω, any tree $\mathcal{T}$ as above must be Aronszajn.
Miller found a mistake in my "proof" of the following result, and kindly supplied a new one, based on the above theorem.

Corollary (Miller)

There exists a subset of 2^ω with the GP.

Proof of Miller's result

Let $\mathcal{T}$ and $\langle K_s : s \in \mathcal{T}\rangle$ be given by Todorčević's theorem. Let $W = \{w_s : s \in \mathcal{T}\}$ where each $w_s \in K_s$ and they are distinct. We will show that W has the GP.

Fix an uncountable $Y \subseteq W$ and let $\mathcal{S}$ be the subtree of $\mathcal{T}$ generated by $\{s \in \mathcal{T} : w_s \in Y\}$. Assume without loss of generality that $\{t \in \mathcal{S} : t \supseteq s\}$ is uncountable for every $s \in \mathcal{S}$.

Notice that $\mathcal{S}$ cannot be Souslin, otherwise forcing with $\mathcal{S}$ would yield a strictly descending ω_1-sequence of closed subsets of 2^ω, contradicting the fact that ω_1 is preserved.

So we can fix an uncountable antichain $\langle s_\alpha : \alpha \in \omega_1\rangle$ in $\mathcal{S}$. It is easy to check that setting $Y_\alpha = Y \cap K_{s_\alpha}$ for $\alpha \in \omega_1$ yields uncountable subsets of Y with pairwise disjoint closures in 2^ω.

Generalizing the Grinzing property

Definition

Fix cardinals κ, λ such that $\omega_1 \leq \kappa \leq \mathfrak{c}$ and $\lambda \leq \kappa$. We will say that a subset W of 2^ω has the (κ, λ)-*Grinzing property* (briefly, the (κ, λ)-GP) if $|W| \geq \kappa$ and for every $Y \subseteq W$ such that $|Y| \geq \kappa$ there exist subsets Y_α of Y for $\alpha \in \lambda$ such that $|Y_\alpha| \geq \kappa$ for each α and $\mathrm{cl}(Y_\alpha) \cap \mathrm{cl}(Y_\beta) = \varnothing$ whenever $\alpha \neq \beta$, where the closure is taken in 2^ω.

Just like in the case of the ordinary GP, a subset W of 2^ω of size at least κ has the (κ, λ)-GP if and only if every subset of W of size κ has the (κ, λ)-GP.
Also, it is clear that the (κ, λ)-GP gets stronger as λ gets bigger.
Furthermore, the (ω_1, ω_1)-GP is simply the GP.

It is easy to show that 2^ω has the (κ,ω)-GP for every cardinal $\kappa\leq\mathfrak{c}$ of uncountable cofinality. The following proposition shows that the restriction on the cofinality is really necessary.

Proposition

Let κ be a cardinal of countable cofinality such that $\omega_1<\kappa<\mathfrak{c}$. Then no subset of 2^ω has the $(\kappa,2)$-GP.

Some of the “old” results generalize in a straightforward way.

Theorem

- *Assume* MA. *Then 2^ω has the (κ,κ)-GP for every $\kappa<\mathfrak{c}$ of uncountable cofinality.*
- *Assume $\mathfrak{b}=\kappa$. Then 2^ω does not have the (κ,ω_1)-GP.*

In particular, as we have already seen, the statement

"2^ω has the $(\mathfrak{c}, \mathfrak{c})$-GP"

is false under CH. Can it be consistently true?

Theorem (Miller, 1983)

It is consistent that for every $Y \subseteq 2^\omega$ of size $\mathfrak{c}$ there exists a continuous function $f : 2^\omega \longrightarrow 2^\omega$ such that $f[Y] = 2^\omega$.

Corollary

It is consistent that 2^ω has the $(\mathfrak{c}, \mathfrak{c})$-GP.

The following fundamental question remains open.

Question

*For which cardinals κ, λ such that $\omega_1 \leq \lambda \leq \kappa \leq \mathfrak{c}$ and κ has uncountable cofinality is it possible to prove in ZFC that there exists a **subset** of 2^ω with the (κ, λ)-GP?*

Generalizing the perfect set property

Definition

Fix an uncountable cardinal κ. Let X be a space and Γ a pointclass. We will say that X has the *κ-perfect set property for Γ subsets* (briefly, the κ-PSP(Γ)) if for every Γ subset A of X either $|A| < \kappa$ or A contains a copy of 2^ω.

Notice that the κ-PSP(Γ) gets stronger as κ gets smaller and as Γ gets bigger.
Also, it is clear that the ω_1-PSP(Γ) is simply the PSP(Γ).
As an example, we can rephrase a classical theorem using this terminology.

Theorem (Sierpiński)

Every Polish space has the ω_2-PSP(Σ^1_2).

The Holy Grail

The following is the most general question that we can imagine on this subject.

Question

What is the status of the statement

> *"For every space X, if X has the* κ-PSP(Γ) *then X has the* κ'-PSP(Γ')*"*

as κ, κ' range over all uncountable cardinals and Γ, Γ' range over all pointclasses?

Generalizing the main result to arbitrary κ

Assuming $\mathfrak{b} > \kappa$, it is easy to see that the κ-PSP(closed) implies the κ-PSP(analytic). But we do not know the answer to the following question.

Question

Does $\mathfrak{b} = \kappa$ imply that there exists a space with the κ-PSP(closed) but not the κ-PSP(analytic)?

The proof that we have given in the case $\kappa = \omega_1$ would go through under the following assumption.

"There exists a subset of 2^ω with the (κ, ω_1)-GP"

We do not know whether the above holds for every κ of uncountable cofinality (or even regular uncountable).

A new proof of a theorem of Todorčević?

Given an infinite cardinal κ, recall that a subset D of $\mathbb{R}$ is *κ-dense* if $|U \cap D| = \kappa$ for every non-empty open $U \subseteq \mathbb{R}$.
Let BA_κ denote the following statement.

> *Whenever D, E are κ-dense subsets of $\mathbb{R}$, there exists a homeomorphism $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $h[D] = E$.*

Theorem (Cantor, 1895)

BA_ω *holds.*

Theorem (Baumgartner, 1984)

Assume PFA. *Then* BA_{ω_1} *holds.*

Theorem (Todorčević, 1988)

$\mathsf{BA}_{\mathfrak{b}}$ *fails.*

Assume that we could prove the following.

"There exists a subset of 2^ω *with the* $(\mathfrak{b}, \omega_1)$-GP*"*

Then we could give a new proof of the result of Todorčević. The new "proof" would actually give two $\mathfrak{b}$-dense sets which are distinguished by a concrete property.

A $\mathfrak{b}$-dense $D \subseteq \mathbb{R}$ with the $(\mathfrak{b}, \omega_1)$-GP.
Fix a base $\{U_n : n \in \omega\}$ for $\mathbb{R}$ and $K_n \subseteq U_n$ for $n \in \omega$ such that $K_n \approx 2^\omega$. Let W_n be a subset of K_n of size $\mathfrak{b}$ with the $(\mathfrak{b}, \omega_1)$-GP. One can easily check that $D = \bigcup_{n \in \omega} W_n$ is $\mathfrak{b}$-dense in $\mathbb{R}$ and has the $(\mathfrak{b}, \omega_1)$-GP.

A $\mathfrak{b}$-dense $E \subseteq \mathbb{R}$ without the $(\mathfrak{b}, \omega_1)$-GP.
Fix a nowhere dense $K \subseteq \mathbb{R}$ such that $K \approx 2^\omega$. By a theorem from a previous slide, we can fix $Z \subseteq K$ such that $|Z| = \mathfrak{b}$ and Z does not have the $(\mathfrak{b}, \omega_1)$-GP. Extend Z to a $\mathfrak{b}$-dense subset E of $\mathbb{R}$. It is clear that E does not have the $(\mathfrak{b}, \omega_1)$-GP.

Ultrafilters and the perfect set property

From now on, we assume that every filter $\mathcal{F}$ is on ω.

In particular, every filter $\mathcal{F}$ can be identified with a subspace $\mathcal{F} \subseteq 2^\omega$ through characteristic functions.

We also assume $\{\text{cofinite sets}\} \subseteq \mathcal{F}$.

Given any coinfinite $z \in \mathcal{F}$, notice that $z\uparrow = \{x \in 2^\omega : z \subseteq x\}$ is a copy of 2^ω contained in $\mathcal{F}$.

Similarly, one sees that every filter has the PSP(open).

Theorem (Medini and Milovich, 2012)

- *There exists an ultrafilter without the* PSP(closed)*.*
- *Assume* MA(countable)*. Then there exists an ultrafilter with the* PSP(analytic)*.*

Conjecture

For ultrafilters, the PSP(analytic) *and the* PSP(closed) *are equivalent.*

Question (Medini and Milovich, 2012)

For ultrafilters, is the PSP(analytic) *equivalent to being a* P*-point?*

Theorem (He and Zhang, 2014)

- *Every* P*-point has the* PSP^+(analytic)*.*
- *Assume* $\mathrm{MA} + \neg\mathrm{CH}$*. Then there exists a non-*P*-point with the* PSP^+(analytic)*.*

The "$+$" denotes the following strong version of the perfect set property, where we require K to be "bounded".

Definition

Fix a pointclass Γ. A filter $\mathcal{F}$ has the *strong perfect set property for* Γ *subsets* (briefly, the $\mathrm{PSP}^+(\Gamma)$) if every Γ subset of $\mathcal{F}$ is either countable or it contains a copy K of 2^ω such that $K \subseteq z\!\uparrow$ for some $z \in \mathcal{F}$.

Sacks measurability and indestructibility

Definition (Marczewski)

A space $X \subseteq 2^\omega$ has *property* (s) if for every perfect $K \subseteq 2^\omega$ there exists a perfect $K' \subseteq K$ such that $K' \subseteq X$ or $K' \subseteq 2^\omega \setminus X$.

Definition

Let $\mathcal{U}$ be an ultrafilter in V. We say that $\mathcal{U}$ is *preserved* in $W \supseteq V$ if $\mathcal{U}$ generates an ultrafilter in W.

Theorem (Miller, 2009)

Let $\mathcal{U}$ be an ultrafilter. Then the following are equivalent.

1. *$\mathcal{U}$ is preserved by Sacks forcing.*
2. *$\mathcal{U}$ is preserved in some extension that adds a new real.*
3. *For every perfect subset K of 2^ω there exists a perfect $K' \subseteq K$ and $z \in \mathcal{U}$ such that $K' \subseteq z\!\uparrow$ or $K' \subseteq (\omega \setminus z)\!\downarrow$.*

We will call *property* $(\mathrm{s})^+$ the condition that appears in (3).

Towards a complete picture

Conjecture

The following implications (and their obvious consequences) are the only ones that are provable in ZFC.

$$
\begin{tikzcd}
 & \text{P-point} \arrow[d] & \\
 & \mathrm{PSP}^{+}(\text{analytic}) \arrow[dl] \arrow[dr] & \\
\mathrm{PSP}(\text{analytic}) \arrow[dr] & & \text{Property (s)}^{+} \arrow[dl] \\
 & \text{Property (s)} &
\end{tikzcd}
$$

What we know so far

Theorem (Miller, 2009)

Assume MA(countable)*. Then there exists an ultrafilter that has property* (s) *but not property* $(\mathrm{s})^+$*.*

Using the method of Miller, it is possible to prove the following.

Theorem

Assume that CH *plus the following conditions hold.*

- *There exists a* Π^1_1 *set without the perfect set property.*
- *Every* Σ^1_2 *set has the property of Baire.*

Then there exists an ultrafilter that has property (s) *but not the* PSP(closed)*.*

Together with the result of He and Zhang, these are the only known counterexamples.

Four existential questions

Shelah showed that it is consistent that there are no P-points. But for each one of the other four notions, the corresponding problem remains open.

Question (Steprāns)

Is there an ultrafilter with property (s) *in* ZFC*?*

Question (Miller, 2009)

Is there an ultrafilter with property $(\mathrm{s})^{+}$ *in* ZFC*?*

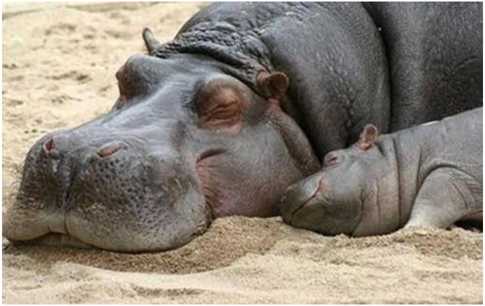
Question (Medini and Milovich, 2012)

Is there an ultrafilter with the PSP(analytic) *in* ZFC*?*

Question

Is there an ultrafilter with the PSP^{+}(analytic) *in* ZFC*?*

Thank you for your attention



and good night!