Computable structure theory and Polish group actions.

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(Joint work with Alexander Melnikov)
A generalization

What is the connection between the following two theorems?

(A) Thm [M. 14] A structure is uniformly computably categorical on a cone \(\iff\) it has a \(\Pi^0_2\) Scott Sentence.

(B) Thm [Effros 65] Let \(G\) be a Polish group acting continuously on a Polish space \(X\), and let \(x\) be a point in \(X\). The map \(g \mapsto g \cdot x: G \to X\) is open \(\iff\) the orbit of \(x\) is \(G\delta\).

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We analyze the following theorems:

3. [McCoy 02] Proper finite dimension does not relativize.
4. [Knight et al. 90’s] No degree spectrum is the union of two cones.
5. [Goncharov 80’s] $\Delta^0_2$- but not $\Delta^0_1$-isomorphic structures have $\infty$ dim.
Part 1:

Background on Polish group actions.
The space of structures

Definition

Let $\text{Mod}(L)$ be the set of all $L$-structures with domain $\omega$.

We give $\text{Mod}(L)$ the topology generated by the basic open sets $[\phi] = \{ A \in \text{Mod}(L) : A|\cdot = \phi \}$ where $\phi$ is an atomic ($L \cup \text{Constants } N$)-sentence and $\text{Constants } N = \{ 0, 1, 2, ... \}$.

Equivalently:

Let $D : \text{Mod}(L) \to 2^\omega$ map $A \in \text{Mod}(L)$ to its atomic diagram $D(A) \in 2^\omega$.

The topology of $\text{Mod}(L)$ is so that $\text{Mod}(L)$ is homeomorphic to its image.
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\( \text{Mod}(L) \) is an effective Polish space

**Definition**

A topological space \( X \) is *Polish* if

It has a countable dense subset \( \{x_0, x_1, x_2, \ldots \} \), and it admits a complete metric \( d: X \times X \to \mathbb{R} \geq 0 \).

\( X \) is *effectively Polish* if also \( d \) is computable on \( \{x_0, x_1, \ldots \} \), i.e., the questions \( d(x_i, x_j) < q \) and \( d(x_i, x_j) \leq q \) are decidable.

**Obs:** For a computable vocabulary \( L \), \( \text{Mod}(L) \) is effectively Polish.

We represent points in \( X \) by fast Cauchy sequences from \( \{ x_0, x_1, \ldots \} \).

**Def:** A point is computable if the sequence is computable and fast approaching.

**Fact:** \( F: X \to Y \) is continuous \( \iff \) it is computable relative to some oracle.
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For $A \in \text{Mod}(L)$, $f \in S_\infty$, $f \cdot A$ is the structure $B$ such that

$$(n_1, \ldots, n_k) \in R^A \iff (f(n_1), \ldots, f(n_k)) \in R^B.$$
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**Obs:** This action, $: S_\infty \times \text{Mod}(L) \to \text{Mod}(L)$, is computable.
Effective Polish group actions

Throughout the rest of the talk

- $G$ is an effective Polish group,
- $X$ is an effective Polish space, and
- $G$ acts on $X$ computably.

Definition

For $x, y \in X$, we let $x \equiv y \iff (\exists g \in G) g \cdot x = y$.

We let the $G$-orbit of $x$ be $\{y \in X : y \equiv x\} = G \cdot x$.

Note: In the case of $S_\infty$ acting on $\text{Mod}(L)$, $A \equiv B \iff A \sim B$.
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Note: In the case of $S_\infty$ acting on $Mod(L)$, $A \equiv B \iff A \cong B$. 
Other examples of computable Polish group actions

The following are examples of computable Polish group actions:

- $GL_n$ acting on $\mathbb{R}^n$.
- Any computable Polish group acting on itself by conjugation.
- $Hom^+[0, 1]$ acting on $C[0, 1]$ by right composition (using sup norm).
Part 2:
Theorems from computable structure theory.
Theorem (((2) [Scott 65; Lopez-Escobar 65; Goncharov 75; M. 14])

For a structure $\mathcal{A}$, the following are equivalent:

1. The set $\{ \mathcal{B} \in \text{Mod}(L) : \mathcal{B} \cong \mathcal{A} \}$ is $\Sigma^0_3$.
2. $\mathcal{A}$ is computably categorical on a cone.
3. $\mathcal{A}$ has a Scott family of $\exists$-formulas with parameters.
4. $\mathcal{A}$ has a $\Sigma^\infty_3$ Scott sentence.
Computable categoricity

**Definition** A structure $\mathcal{A}$ is *computably categorical (c.c.)* if

every computable $\mathcal{B}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. 

Theorem ([Downey, Kach, Lempp, Lewis, Montalbán, Turetsky 12]) There is no nice characterization of computably categorical structures.

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Nice characterizations exist if we relativize to all oracles on a cone.
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**Definition** A structure $\mathcal{A}$ is **computably categorical (c.c.) on a cone** if, there is a $C \in 2^\omega$ such that for very $Z \geq_T C$, every $Z$-computable $\mathcal{B}$ isomorphic to $\mathcal{A}$ is $Z$-computably isomorphic to $\mathcal{A}$.

- A linear ordering is c.c. $\iff$ it has finitely many adjacencies [Dzgoev, Goncharov 80].
- A Boolean algebras is c.c. $\iff$ it has finitely many atoms [Goncharov][La Roche 78].
- A ordered abelian group is c.c. $\iff$ it has finite rank [Goncharov, Lempp, and Solomon 03].
- A tree of finite height is c.c. $\iff$ it is of finite type [Lempp, McCoy, R. Miller, Solomon 05].
- A torsion-free abelian group is c.c. $\iff$ it has finite rank [Nurtazin 74].
- A computable p-group is c.c. $\iff$ it can be written in one of the following forms: (i) $(\mathbb{Z}(p^{-\infty}))^\ell \oplus G$ for $\ell \in \omega \cup \{\infty\}$ and $G$ finite, or (ii) $(\mathbb{Z}(p^{-\infty}))^n \oplus (\mathbb{Z}(p^k)^\infty \oplus G$ where $G$ is finite, and $n, k \in \omega$ [Goncharov 80][Smith81].

**Theorem** ([Downey, Kach, Lempp, Lewis, Montalbán, Turetsky 12])

There is no nice characterization of computably categorical structures. The set of indices of computably categorical structures is $\Pi^1_1$-complete.

Nice characterizations exist if we relativize to all oracles on a cone.
Categoricity on group actions

Recall: A computable structure $\mathcal{A} \in \text{Mod}(L)$ is \textit{computably categorical} if for every computable $\mathcal{B} \cong \mathcal{A}$, there is a computable $g \in S_\infty$ with $g \cdot \mathcal{A} = \mathcal{B}$.

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Recall that we have a computable Polish group action of $G$ on $\mathcal{X}$.

- $x^2$ and $\sin(4\pi x)$ are computably categorical.
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### Example
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Theorem ((2) [Lopez-Escobar 65; Scott 65; Goncharov 75; M. 14])

For a structure $A$, the following are equivalent:

1. $A$ is computably categorical on a cone.
2. $A$ has a Scott family of $\exists$-formulas with parameters.
3. $A$ has a $\Sigma^0_3$ Scott sentence.
4. The set $\{B \in \text{Mod}(L) : B \cong A\}$ is $\Sigma^0_3$.

Question: If we have a computable Polish action of $G$ on $X$, do we have that $x \in X$ is computably categorical on a cone $\iff$ its orbit is $\Sigma^0_3$.
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A simpler question

**Theorem ((1) [Lopez-Escobar 65; Scott 65; Goncharov 75; Ventsov 93; M. 14])**

For a structure $A$, the following are equivalent:

1. $A$ is uniformly computably categorical on a cone.
2. $A$ has a Scott family without parameters.
3. $A$ has a $\Pi^1_2$ Scott sentence.
4. The set $\{B \in \text{Mod}(L) : B \cong A\}$ is $\Pi^0_2$. 

**Definition:** A point $x \in X$ is uniformly computably categorical if there is a computable operator $\Phi$ that, given a fast Cauchy sequence for $y \equiv x$, outputs $g \in G$ with $g \cdot x = y$. 

**Question:** If we have a computable Polish action of $G$ on $X$, do we have that $x \in X$ is uniformly computably categorical on a cone $\iff$ its orbit is $G^\delta$.
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Lemma $x \in X$ is uniformly computably categorical $\iff$ the map $g \mapsto g \cdot x : G \to X$ is effectively open.

Theorem [Effross 65] For a point $x \in X$, TFAE:

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Corollary (1) For a structure $A$, TFAE:

1. $A$ has a $\Pi^0_2$ Scott sentence.
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Uniformly computable categoricity

**Definition:** A point \( x \in \mathcal{X} \) is *uniformly computably categorical* if there is a computable operator \( \Phi \) that, given a fast Cauchy sequence for \( y \equiv x \), outputs \( g \in \mathcal{G} \) with \( g \cdot x = y \).

**Lemma** \( x \in \mathcal{X} \) is uniformly computably categorical \( \iff \) the map \( g \mapsto g \cdot x : \mathcal{G} \to \mathcal{X} \) is effectively open.

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**Corollary** (1) For a structure \( \mathcal{A} \), TFAE:

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Recall: [Lopez-Escobar 65]: $\mathcal{A}$ has a $\Sigma^0_3$ Scott sentence $\iff \{ \mathcal{B} \in \text{Mod}(L) : \mathcal{B} \cong \mathcal{A} \}$ is $F_{\sigma\delta}$.
Back to Theorem 2

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**Theorem [Melnikov, M.]** For a point $x \in X$, TFAE:

1. $x$ is computably categorical on a cone.
2. The $G$-orbit of $x$ is $G_{\delta\sigma}$.
Theorem 3 – Knight’s group 90’s

Recall: Given a structure $\mathcal{A}$:

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Theorem Each of the following is a degree spectra of some structure:
- upper cones: $\{ Z \in 2^\omega : Z \geq_T C \}$ for some $C \in 2^\omega$ [Van der Waerden 30]
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Theorem [Knight et al. 90’s] The degree spectrum of a structure is never a non-trivial union of countably many upper cones.
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- upper cones: $\{ Z \in 2^\omega : Z \geq_T C \}$ for some $C \in 2^\omega$ [Van der Waerden 30]
- non-zero degrees: $\{ Z \in 2^\omega : Z \not\equiv_T \emptyset \}$ [Slaman 98; Wehner 98]
- non-$\Delta^0_2$ degrees: $\{ Z \in 2^\omega : Z \not\leq_T 0' \}$ [Kalimullin 08]
- the hyperimmune degrees [Csima, Kalimullin 10]
- non-hyp-degrees: $\{ Z \in 2^\omega : Z \not\in hypoth \}$ [Greenberg, Montalbán, Slaman 12]
- ...

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Recall: Given a structure $A$:

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In the general setting of Polish group actions:

**Theorem** [Melnikov, M.] The degree spectrum of a point

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**Theorem** Any structure in the following classes has computable dimension either 1 or \( \omega \):

- Boolean Algebras [Goncharov 73]
- Linear Ordering [Remmel 81][Goncharov and Dzgoev 80]
- Real algebraically closed fields [Nurtazin [1974]]
- Archimedean ordered group [Goncharov, Lempp and Solomon 2000]
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Proof: Show that if a structure has finite dimension on a cone, its orbits is $\Sigma^0_3$. 
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*If a computable structure has two computable copies which are $\Delta^0_2$-isomorphic but not computably isomorphic, then the structure has infinite computable dimension.*
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Theorem ([Melnikov, M.])

If in the orbit of a point there are two computable points
which are NH-equivalent but not computably equivalent,
then the point has infinite computable dimension.