Computability theory and uncountable structures

Noah Schweber

Sets and Computations, April 17 2015

Joint with Greg Igusa, Julia Knight and Antonio Montalbán
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Generically presentable structures</td>
</tr>
<tr>
<td>2</td>
<td>Computability in generic extensions</td>
</tr>
<tr>
<td>3</td>
<td>Versions of the reals</td>
</tr>
</tbody>
</table>
1. Generically presentable structures

2. Computability in generic extensions

3. Versions of the reals
Global behavior tends to be independent of the generic: if $\mathbb{P}$ is reasonably homogeneous, then the theory of $V[G]$ does not depend on $G$. On the other hand, individual sets in generic extensions must vary wildly:

**Theorem (Solovay)**

Suppose $G_0, G_1$ are two mutually generics. Then $V[G_0] \cap V[G_1] = V$.

**Proof.** Take $\nu[G_0] \in (V[G_0] \cap V[G_1]) - V$ of minimal rank. Then $\nu[G_0] \subset V$.

If $\nu[G_0] = \mu[G_1]$, then this is forced by some $(p, q) \in \mathbb{P}^2$.

The set $\{x \in V : \exists r \leq p (r \models x \in \nu)\}$ is in $V$, and must equal $\nu[G_0]$.

□
Generically presentable structures

Solovay: if a set is in every generic extension by some forcing, it exists already.

Definition

A generically presentable structure up to \( \cong \) is a pair \((\nu, P)\) such that

\[
\models_P "\nu \text{ is a structure with domain } \omega" \quad \text{and} \quad \models_{P_2} "\nu[G_0] \cong \nu[G_1]".
\]

A copy of \((\nu, P)\) is a \( \mathcal{A} \in V \) with \( \models_P "\mathcal{A} \cong \nu". \) (Maybe \( \text{dom}(\mathcal{A}) \neq \omega. \))

Recently and independently introduced by Kaplan and Shelah.

Question

If \((\nu, P)\) is a generically presentable structure, what hypotheses ensure that it has a copy in \( V \)?
Looking at the forcing: positive results

Theorem (Knight, Montalbán, S.)

1. If \( A \) is generically presentable by a forcing not making \( \omega^2 \) countable, then \( A \) has a copy in \( V \).

2. If \( A \) is generically presentable by a forcing not making \( \omega^1 \) countable, then that copy is countable.

Independently proved by Kaplan and Shelah.

Proof. For (1), the age of the Morleyization of \( A \) lives in \( V \); by Delhomme-Pouzet-Sagi-Sauer, Fraisse limits of ages of size \( \aleph_1 \) exist.

For (2), Scott sentence is in \( L_{V \omega^1} \omega \) since \( \omega_{V[\mathbb{G}]} = \omega \), and existence of countable models is absolute. □
Theorem (Knight, Montalbán, S.)

1. If $A$ is generically presentable by a forcing not making $\omega_2$ countable, then $A$ has a copy in $V$;

2. If $A$ is generically presentable by a forcing not making $\omega_1$ countable, then that copy is countable.
Looking at the forcing: positive results

Theorem (Knight, Montalbán, S.)

1. If $A$ is generically presentable by a forcing not making $\omega_2$ countable, then $A$ has a copy in $V$;
2. If $A$ is generically presentable by a forcing not making $\omega_1$ countable, then that copy is countable.

Independently proved by Kaplan and Shelah.

Proof. For (1), the age of the Morleyization of $A$ lives in $V$; by Delhomme-Pouzet-Sagi-Sauer, Fraïssé limits of ages of size $\aleph_1$ exist. For (2), Scott sentence is in $\mathcal{L}^{V}_{\omega_1 \omega}$ since $\omega_1^V = \omega_1^{V[G]}$, and existence of countable models is absolute. □
Counterexamples to Vaught’s conjecture

Corollary (Harrington)

Any counterexample to Vaught’s conjecture has models of size $\aleph_1$ with Scott rank arbitrarily high below $\omega_2$.

Independently by Larson, and by Baldwin/S. Friedman/Koerwien/Laskowski.

Proof. Given $\alpha < \omega_2$, collapse $\omega_1$ to $\omega$, get $B \models T$ with $sr(B) > \alpha$. If $B$ not generically presentable, can get perfect set of such models. So $B$ is generically presentable, hence has a copy in $V$ since $\omega_2^V$ is still uncountable. □

Remark

Hjorth showed that counterexamples need not have models of size $\aleph_2$. 
Looking at the structure: positive results

Theorem (Knight, Montalbán, S.)

If $\mathcal{A}$ is generically presentable and rigid (no nontrivial automorphisms), then $\mathcal{A}$ has a copy in $V$.

Independently by Paul Larson.
Proof uses amalgamation argument — unique way to amalgamate is even better than lots of ways to amalgamate. Given $p \in \mathbb{P}$ presenting $\mathcal{A}$, look at portion $\mathcal{A}_p$ of structure built by $p$; can glue these together in unique way, so inside $V$. □
Looking at the structure: positive results

Theorem (Knight, Montalbán, S.)

*If $A$ is generically presentable and rigid (no nontrivial automorphisms), then $A$ has a copy in $V$."

Independently by Paul Larson.

Proof uses amalgamation argument — unique way to amalgamate is even better than lots of ways to amalgamate. Given $p \in P$ presenting $A$, look at portion $A_p$ of structure built by $p$; can glue these together in unique way, so inside $V$. □

Theorem (Zapletal, unpublished)

*Generically presentable trees have copies.*

Kaplan-Shelah, following Zapletal: study when generically presentable linear orders, models of superstable theories, already exist.
Theorem (Knight, Montalbán, S.)

If forcing with $P$ makes $\omega^2$ countable, then there is a structure $A$, generically presentable by $P$, which has no copy in the ground model. Independently by Kaplan-Shelah.

Uses construction of Laskowski and Shelah, and later Hjorth: theory with predicate $U$ and no atomic models if $|U| = \aleph_2$, but countable atomic model in which $U$ is set of indiscernibles. In generic extension, we can attach $(\omega^2, <)$ to indiscernibles of this model; resulting structure has a copy after making $\omega^2$ countable but has no copy in ground model.
Looking at the forcing: negative results

**Theorem (Knight, Montalbán, S.)**

*If forcing with $\mathbb{P}$ makes $\omega_2$ countable, then there is a structure $\mathcal{A}$, generically presentable by $\mathbb{P}$, which has no copy in the ground model.*
Looking at the forcing: negative results

**Theorem (Knight, Montalbán, S.)**

*If forcing with \( \mathbb{P} \) makes \( \omega_2 \) countable, then there is a structure \( \mathcal{A} \), generically presentable by \( \mathbb{P} \), which has no copy in the ground model.*

Independently by Kaplan-Shelah

Uses construction of Laskowski and Shelah, and later Hjorth: theory with predicate \( U \) and no atomic models if \( |U| = \aleph_2 \), but countable atomic model in which \( U \) is set of indiscernibles. In generic extension, we can attach \((\omega_2, <)\) to indiscernibles of this model; resulting structure has a copy after making \( \omega_2 \) countable but has no copy in ground model.
Aside: generically presentable cardinalities

**Definition**

A generically presentable cardinality is a pair \((\nu, \mathbb{P})\) where \(\nu\) is a \(\mathbb{P}\)-name and \(\models_{\mathbb{P} \times \mathbb{P}} \nu[G_0] \equiv \nu[G_1].\) \((\nu, \mathbb{P})\) has no copy in \(V\) if for no \(A \in V\) do we have \(\models_{\mathbb{P}} A \equiv \nu.\)

**Question**

Is it consistent with ZF that there are generically presentable cardinalities with no copies?

Note that forcing over ZF-models can add new cardinalities (Ex: Truss ????)

**Question (Zapletal)**

Is it consistent with ZF that there is a generically presentable cardinality \((\nu, \mathbb{P})\) with no copy in \(V\), such that \(\nu\) is a name for a set of reals?
1. Generically presentable structures
2. Computability in generic extensions
3. Versions of the reals
Classical computable structure theory

We study the complexity of a structure by looking at its copies: for a countable structure $S$, a copy of $S$ is a structure $S'$ with domain $\omega$ isomorphic to $S$.

Throughout, structures have finite signature.

If $\mathcal{A}$ is a countable structure, a copy of $\mathcal{A}$ is a structure which is isomorphic to $\mathcal{A}$ and has domain $\omega$; we identify copies with the reals coding them.

**Definition (Muchnik reducibility)**

If $\mathcal{A}$, $\mathcal{B}$ are structures, $\mathcal{A}$ is *Muchnik reducible to* $\mathcal{B}$ if (nonuniformly) every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$; we write $\mathcal{A} \leq_w \mathcal{B}$. 

Noah Schweber
Computability theory and uncountable structures
Classical computable structure theory

We study the complexity of a structure by looking at its copies: for a countable structure $S$, a copy of $S$ is a structure $S'$ with domain $\omega$ isomorphic to $S$.

Throughout, structures have finite signature.

If $\mathcal{A}$ is a countable structure, a copy of $\mathcal{A}$ is a structure which is isomorphic to $\mathcal{A}$ and has domain $\omega$; we identify copies with the reals coding them.

**Definition (Muchnik reducibility)**

If $\mathcal{A}$, $\mathcal{B}$ are structures, $\mathcal{A}$ is *Muchnik reducible to* $\mathcal{B}$ if (nonuniformly) every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$; we write $\mathcal{A} \leq_w \mathcal{B}$.

- $\mathcal{A}$ is computably presentable $\implies \mathcal{A} \leq_w \mathcal{B}$
- For $X \subseteq \mathcal{A}$ finite, the substructure generated by $X$ is $\leq_w \mathcal{A}$
- $\mathcal{L}, \mathcal{L}_0, \mathcal{L}_1$ linear orders $\implies \mathcal{L} \leq_w \mathcal{L}_0 + 1 + \mathcal{L} + 1 + \mathcal{L}_1$
- If $\mathcal{L} \prec \hat{\mathcal{L}}$ are linear orders, need not have $\mathcal{L} \leq_w \hat{\mathcal{L}}$ (Harrison order)
Uncountable computable structure theory

For $\mathcal{A}$ uncountable, $\mathcal{A}$ has no copies whatsoever, so $\leq_w$ is not useful. There are many ways one might generalize computability structure theory to uncountable settings. Today: want notion which agrees with $\leq_w$ on countable structures, and is generally not contingent on set-theoretic axioms.
Uncountable computable structure theory

For $\mathcal{A}$ uncountable, $\mathcal{A}$ has no copies whatsoever, so $\leq_w$ is not useful. There are many ways one might generalize computability structure theory to uncountable settings.

Today: want notion which agrees with $\leq_w$ on countable structures, and is generally not contingent on set-theoretic axioms.

We ask, “what would the complexity of $\mathcal{A}$ be if $\mathcal{A}$ were countable?”

**Definition (Generic Muchnik reducibility (S.))**

For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, we write $\mathcal{A} \leq^*_w \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in every generic extension of the universe in which both are countable.

Can similarly study other computability-theoretic reductions between uncountable structures.
Absoluteness

Definition (Generic Muchnik reducibility (S.))

For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, $\mathcal{A} \leq^* \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in every generic extension in which both are countable.
Absoluteness

Definition (Generic Muchnik reducibility (S.))
For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, $\mathcal{A} \leq^* \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in every generic extension in which both are countable.

Theorem (Shoenfield Absoluteness)
If $\varphi$ is $\Pi^1_2$ with parameters from $\mathbb{R}$, we have:

$$V[G] \models \varphi \iff V \models \varphi.$$  

Corollary
- We can replace “every generic extension” by “some generic extension” in definition of generic Muchnik reducibility.
- For $\mathcal{A}, \mathcal{B}$ countable, $\mathcal{A} \leq^* \mathcal{B} \iff \mathcal{A} \leq_w \mathcal{B}$. 
Back to generic presentability

Theorem (Knight, Montalbán, S.)

If $\mathcal{A}$ is generically presentable, and is generically Muchnik reducible to a structure $\mathcal{B} \in V$ with $|\mathcal{B}| \leq \aleph_1$, then $\mathcal{A}$ has a copy in $V$.

Proof.
In $V[G]$ with $\omega = |\omega_1^V| < |\omega_2^V|$, let $B \cong \mathcal{B}$ with domain $\omega$.
Let $V[G][H]$ be further extension in which $\mathcal{A}$ has a copy.
$\exists e$ such that $V[G][H] \models \Phi^B_e \cong \mathcal{A}$.
In $V[G]$, $\Phi^B_e$ satisfies Scott sentence of $\mathcal{A}$.
Existence of countable models of $\mathcal{L}_{\omega_1 \omega}$-sentences is absolute.
So $\mathcal{A}$ has a copy in $V[G]$, and hence in $V$.

Proposition (Knight, Montalbán, S.)

Counterexample to “Shoenfield for structures” is $\leq^*_w (\omega_2, <)$.

Proof. Theory of Laskowski-Shelah has computable atomic model. □
Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

\[ \mathcal{C} = (\mathbb{C}; +, \times), \quad \mathcal{W} = (\omega, \mathcal{P}(\omega); \text{Succ}, \in), \quad \mathcal{R} = (\mathbb{R}; +, \times) \]
Consider the following uncountable structures:

\[ C = (\mathbb{C}; +, \times) \], \quad \mathcal{W} = (\omega, \mathcal{P}(\omega); \text{Succ}, \in), \quad \mathcal{R} = (\mathbb{R}; +, \times) \]

**Observation**

\( C \) is “generically computably presentable:” \( C \) has a computable copy in every generic extension in which it is countable.
Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

\[ \mathbb{C} = (\mathbb{C}; +, \times), \quad \mathbb{W} = (\omega, \mathcal{P}(\omega); \text{Succ}, \in), \quad \mathbb{R} = (\mathbb{R}; +, \times) \]

**Observation**

\( \mathbb{C} \) is “generically computably presentable:” \( \mathbb{C} \) has a computable copy in every generic extension in which it is countable.

**Observation**

Every countable structure is generically Muchnik reducible to \( \mathbb{W} \) and to \( \mathbb{R} \).
Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

\[ \mathcal{C} = (\mathbb{C}; +, \times), \quad \mathcal{W} = (\omega, \mathcal{P}(\omega); \text{Succ}, \in), \quad \mathcal{R} = (\mathbb{R}; +, \times) \]

Observation

\[ \mathcal{C} \text{ is \textit{"generically computably presentable:}" } \mathcal{C} \text{ has a computable copy in every generic extension in which it is countable.} \]

Observation

\[ \text{Every countable structure is generically Muchnik reducible to } \mathcal{W} \text{ and to } \mathcal{R}. \]

Theorem (Igusa, Knight)

\[ \mathcal{W} \text{ is strictly less complicated than } \mathcal{R}: \mathcal{W} <^* \mathcal{R}. \]

Some examples, II/III: $\omega_1$
Some examples, II/III: $\omega_1$

**Proposition**

For $A$ countable:

$$A \preceq^* (\omega_1, <) \iff \exists \text{ countable ordinal } \alpha \text{ with } A \preceq^* (\alpha, <).$$

**Proof.** Suppose $A \preceq^* (\omega_1, <)$. Let $V[G]$ be generic extension in which $\omega_1$ is countable. Then we have

$$V[G] \models "A \preceq_w (\alpha, <) \text{ for some countable ordinal } \alpha."$$

This is a $\Sigma^1_2$ sentence with a real parameter (since $A$ is countable), so already true in $V$. □

**Question**

What families of countable structures are captured by some single uncountable structure?
Some examples, III/III: $\omega_1$ and $\mathbb{R}$

**Proposition (Richter)**

*If a real $X$ is computable in every copy of a linear order $\mathcal{L}$, then $X$ is computable.*

**Corollary**

$(\omega_1, <) \not\preceq^*_{w} \mathcal{W}$.

**Proposition (Ash, Knight)**

*If $X'$ computes a copy of a linear order $\mathcal{L}$, then $X$ computes a copy of $\omega \cdot \mathcal{L}$.*

**Corollary**

$(\omega_1, <) \prec^*_{w} \mathcal{W}$. 
1 Generically presentable structures

2 Computability in generic extensions

3 Versions of the reals

Noah Schweber
Computability theory and uncountable structures
Versions of the reals

\[ \mathcal{W} = (\omega \sqcup 2^\omega; \text{Succ}, \in), \quad \mathcal{B} = (\omega \sqcup \omega^\omega; \text{Succ}, \circ) \]

\[ \mathcal{R} = (\mathbb{R}; +, \times) \]

\[ \mathcal{R}^* = \omega_1\text{-saturated real closed field realizing all types in } \mathcal{V} \]

\[ \mathcal{R}_f = (\mathbb{R}; +, \times, f), \quad \mathcal{R}^+ = (\mathbb{R}; +) \]
Versions of the reals

\[ \mathcal{W} = (\omega \sqcup 2^\omega; \text{Succ}, \in), \quad \mathcal{B} = (\omega \sqcup \omega^\omega; \text{Succ}, \circ) \]

\[ \mathcal{R} = (\mathbb{R}; +, \times) \]

\[ \mathcal{R}^* = \omega_1\text{-saturated real closed field realizing all types in } V \]

\[ \mathcal{R}_f = (\mathbb{R}; +, \times, f), \quad \mathcal{R}^+ = (\mathbb{R}; +) \]

There seem to be two levels of complexity:

\[ \mathcal{W} \equiv^*_w \mathcal{B} <^*_w \mathcal{R}^+ \equiv^*_w \mathcal{R} \equiv^*_w \mathcal{R}_f \quad (f \text{ analytic}) \]
Simple reductions

**Proposition (Igusa, Knight)**

\[ \mathcal{R}^* \succeq^w B \succeq^w \mathcal{W}. \]

**Proposition (Igusa, Knight)**

\[ \mathcal{W} \equiv^w \mathcal{R}^*. \]
Simple reductions

Proposition (Igusa, Knight)

\[ \mathcal{R}^* \succeq_w \mathcal{B} \succeq_w \mathcal{W}. \]

Proposition (Igusa, Knight)

\[ \mathcal{W} \equiv_w \mathcal{R}. \]

Theorem (Macintyre-Marker)

*If* \( S \) *is a Scott set and* \( T \in S \) *is a consistent theory, any enumeration of* \( S \) *computes the complete diagram of a recursively saturated model of* \( T \) *realizing exactly the types in* \( S \).

*Proof of Prop.* After collapse, \( \mathcal{W} \) is still a Scott set, and each ground real — including \( \text{Th}(\mathcal{R}) \) — appears in \( \mathcal{W} \); apply Macintyre-Marker. □
\[ \mathbb{R}^* \prec_w \mathbb{R}, \text{I/II} \]

**Definition**

If \( K \) is a real closed field:

- \( K \) is **Archimedean** if every element of \( K \) is below some \( q \in \mathbb{Q} \).
- The **residue field** \( \text{Res}(K) \) of \( K \) be the quotient of the finite elements by the infinitesimal elements.
- A **residue field section** of \( K \) is a subfield of \( K \) isomorphic to \( \text{Res}(K) \).
- \( FT(K) \) ("finite transcendental") is the set of finite elements not infinitesimally close to an algebraic element.
Definition

If $K$ is a real closed field:

- $K$ is *Archimedean* if every element of $K$ is below some $q \in \mathbb{Q}$.
- The *residue field* $\text{Res}(K)$ of $K$ be the quotient of the finite elements by the infinitesimal elements.
- A *residue field section* of $K$ is a subfield of $K$ isomorphic to $\text{Res}(K)$.
- $\text{FT}(K)$ ("finite transcendental") is the set of finite elements not infinitesimally close to an algebraic element.

Lemma

If $K$ is a real closed field with domain $\omega$, then:

- If $\text{Res}(K)$ has a $\Sigma^0_2(K)$ copy, then $\text{FT}(K)$ is $\Sigma^0_2(K)$.
- . . . And so $K$ has a residue field section which is $\Sigma^0_2(K)$. 
Lemma

If $K$ is a real closed field with domain $\omega$, then:

- If $\text{Res}(K)$ has a $\Sigma_2^0(K)$ copy, then $\text{FT}(K)$ is $\Sigma_2^0(K)$. . . .
- . . . And so $K$ has a residue field section which is $\Sigma_2^0(K)$. 
Lemma

If $K$ is a real closed field with domain $\omega$, then:

- If $\text{Res}(K)$ has a $\Sigma^0_2(K)$ copy, then $\text{FT}(K)$ is $\Sigma^0_2(K)$.
- . . . And so $K$ has a residue field section which is $\Sigma^0_2(K)$.

Theorem (Igusa, Knight)

- (Reduction) If $\text{Res}(K) \leq^*_w K$, then $\text{FT}(K)$ is $\Sigma^c_2$-definable in $K$.
- (Undefinability) If $K$ is a recursively saturated real closed field, the set $\text{FT}(K)$ is not $\Sigma^c_2$-definable in $K$.

Since “recursively saturated” is absolute, this gives:

Theorem (Igusa, Knight.)

$\mathcal{R}^* <^*_w \mathcal{R}$.
Expansions of \( \mathcal{R}, \text{I/II} \)

What happens if we add expressive power to \( \mathbb{R} \)?

**Definition**

A function \( f : \mathbb{R} \to \mathbb{R} \) is *trivial* if, in any \( V[G] \) where \( \mathbb{R} \) is countable, we have: Any copy \( \mathcal{A} \) of \( \mathcal{R} = (\mathbb{R}^V; +, \times) \) with domain \( \omega \) computes a copy \( \mathcal{B} \) of \( \mathcal{R}_f = (\mathbb{R}^V; +, \times, f) \) with \( \mathcal{B} \upharpoonright \{+, \times\} = \mathcal{A} \).

This is stronger than \( \mathcal{R} \equiv^*_w \mathcal{R}_f \).
What happens if we add expressive power to $\mathbb{R}$?

**Definition**

A function $f : \mathbb{R} \to \mathbb{R}$ is *trivial* if, in any $V[G]$ where $\mathbb{R}$ is countable, we have: Any copy $\mathcal{A}$ of $\mathcal{R} = (\mathbb{R}^V; +, \times)$ with domain $\omega$ computes a copy $\mathcal{B}$ of $\mathcal{R}_f = (\mathbb{R}^V; +, \times, f)$ with $\mathcal{B} \upharpoonright \{+, \times\} = \mathcal{A}$.

This is stronger than $\mathcal{R} \equiv^*_w \mathcal{R}_f$.

**Proposition**

A function $f$ is trivial iff it is piecewise algebraic.

*Proof.* Right-to-left is immediate. For left-to-right, build (in $V[G]$) a sufficiently generic copy of $\mathcal{R} = (\mathbb{R}; +, \times)$ by forcing with $\mathcal{R}^{\prec \omega}$. Can diagonalize against $\Phi_e$ unless “$f(x) = y$” determined by finitely many $\{+, \times\}$-atomic formulas. □
Interlude: $O$-minimality and bases

**Definition**
An ordered structure $\mathcal{A}$ is $o$-minimal if every definable subset of $\mathcal{A}$ is a union of finitely many intervals.

**Theorem (Macintyre)**

*The structure $\mathcal{R}_{exp} = (\mathbb{R}; +, \times, \exp)$ is $o$-minimal*

**Definition**

A *basis* for $\mathcal{R}_{exp}$ is a set $U \subset \mathbb{R}$ such that
- Every real is definable over some tuple from $U$.
- No element of $U$ is definable over any disjoint tuple from $U$.

A tuple is *independent* if it can be extended to a basis.
Expansions of $\mathcal{R}$, II/II

### Definition
We let $\text{IND}_n(\mathcal{R}_{\text{exp}})$ be the set of independent $n$-tuples of $\mathbb{R}$.

### Lemma
The sets $\text{IND}_n(\mathcal{R}_{\text{exp}})$ of independent $n$-tuples are $\Delta^c_2, \text{Th}(\mathcal{R}_{\text{exp}})$ in any copy of $\mathcal{R}$.

**Proof.** $\vec{a}$ is independent iff there is an assignment of open boxes around $\vec{a}$ to formulas such that the formula holds of $\vec{a}$ iff it holds in whole box. □

### Theorem (Igusa, Knight, S.)
$\mathcal{R}_{\text{exp}}$ is generically Muchnik equivalent to $\mathcal{R}$.

**Proof.** We use lemma to get $\Delta^0_2$-approximation to a basis for $\mathcal{R}_f$, and build the “term” model generated by this basis. □
Since $|\mathbb{R}^\mathbb{R}| > \mathbb{R}$, there are functions in $V$ which add information.

**Question**

Is there a “reasonably definable” $f$ which adds information?

**Question**

Is there a continuous function $f$ adds information?

**Conjecture (Igusa, Knight, S.)**

Martin-Lof Brownian motion adds information.
Further versions of the reals

**Theorem (Igusa, Knight, S.)**

\[ R_f \equiv^*_w R \text{ for } f \text{ analytic.} \]

Uses Wilkie: \( R \) adjoined with analytic functions on compact intervals is \( o \)-minimal.

**Theorem (Igusa, Knight, S.)**

*The field \((\mathbb{R}; +, \times)\) is generically Muchnik reducible to the group \((\mathbb{R}; +)\).*

**Theorem (Igusa, Knight, S.)**

*For \( \langle a_i : i \in \omega \rangle \in \mathbb{R}^V \), the expansion \((\mathbb{R}; +, \times, a_0, a_1, ...)\) is generically Muchnik reducible to \( R \).*

**Proof.** In each case, we show that the independence relation over the larger language is \( \Sigma^c_2 \) in the smaller language.

Noah Schweber

Computability theory and uncountable structures
Thanks!

- Baldwin, S.-D. Friedman, Koerwien, Laskowski. “Three red herrings around Vaught’s conjecture.” submitted, on Baldwin’s webpage
- Knight, Igusa, S. In preparation.
- Kaplan, Shelah. “Forcing a countable structure to belong to the ground model.” on arXiv
- Knight, Montalbán, S. “Computable structures in generic extensions.” submitted, on arXiv