The spectrum of $\kappa$-maximal cofinitary groups

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A subset $A \subseteq [\kappa]^{\kappa}$ is said to be $\kappa$-a.d. if for all distinct $a, b \in A$, $|a \cap b| < \kappa$. A $\kappa$-a.d. family of size $\geq \kappa$ is said to be maximal if it is maximal with respect to inclusion.

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Spectra

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Thus $\alpha(\kappa) = \min C_\kappa(\text{mad})$ and $\alpha_g(\kappa) = \min C_\kappa(\text{mcg})$. 
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A. Blass showed that if GCH holds and \( C \) is a closed set of cardinals such that \( \aleph_1 \in C \), \( \forall \nu \in C(\nu \geq \aleph_1), [\aleph_1, |C|] \subseteq C \) and \( \forall \lambda (\lambda \in C \land \text{cof}(\lambda) = \omega \rightarrow \lambda^+ \in C) \), then there is a ccc generic extension in which \( C_\omega(\text{mad}) = C \).
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Brendle, Spinas and Zhang obtained an analogous result regarding mcg’s: if $C$ is as above (and GCH holds), then there is a ccc generic extension in which $C_\omega(\text{mcg}) = C$. 
Theorem (V.F.)

(GCH) Let $\kappa$ be a regular infinite cardinal and let $C$ be a closed set of cardinals such that

1. $\kappa^+ \in C$, $\forall \nu \in C (\nu \geq \kappa^+)$,
2. $[\kappa^+, |C|] \subseteq C$,
3. $\forall \nu \in C (\text{cof}(\nu) \leq \kappa \rightarrow \nu^+ \in C)$.

Then there is a generic extension in which cofinalities have not been changed and such that $C = C_\kappa(mcg)$. 
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- adding a $\kappa$-m.c.g. of desired cardinality,
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- adding a $\kappa$-m.c.g. of desired cardinality,
- excluding certain cardinals as possible values for $\kappa$-mcg.
A mapping $\rho : B \to S(\kappa)$ induces a $\kappa$-cofinitary representation of $\mathbb{F}_B$ if the canonical extension of $\rho$ to a homomorphism $\hat{\rho} : \mathbb{F}_B \to S(\kappa)$ has the property that every non-identity element of $\text{im}(\hat{\rho})$ has $< \kappa$-many fixed points.
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Given a $\kappa$-cofinitary representation $\rho$ with domain $B$ and a set $A$ s.t. $A \cap B = \emptyset$, we will add generically a family of $\kappa$-permutations $\{g_a\}_{a \in A}$ such that the group $\mathcal{G}(\rho, A)$ generated by $\text{im}(\hat{\rho})$ and $\{g_a\}_{a \in A}$ is $\kappa$-cofinitary.
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We will work with approximations to the new generators of size $< \kappa$, i.e. with sets $s \in [A \times \kappa \times \kappa]^{< \kappa}$. For each $a \in A$ we interpret $s_a = \{((\gamma, \beta) : (a, \gamma, \beta))\}$ as a partial approximation to the generator $g_a$. 
To describe arbitrary members of $G(\rho, A)$, we work with the set of words $\hat{W}_{A \cup B}$ (referred to as good words) on the alphabet $A \cup B$ which start and end with a different letter, or a power of a single letter. Every word on $A \cup B$ is a conjugate of such a good word.
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Every approximation $s$ to the new generators gives an approximation to the permutations corresponding to arbitrary words $w \in \hat{W}_{A \cup B}$. This approximation is denoted $e_w[s, \rho]$ and is obtained by substituting every appearance of a letter $b$ from $B$ with $\rho(b)$ and every appearance of a letter $a \in A$ with $s_a$. We refer to $e_w[s, \rho]$ as the evaluation of $w$ given $s$ and $\rho$. 
Adding a $\kappa$-m.c.g. of desired cardinality

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Adding a $\kappa$-m.c.g. of desired cardinality

Let $A$ and $B$ be disjoint sets and let $\rho : B \to S(\kappa)$ be a function inducing a $\kappa$-cofinitary representation. The forcing notion $\mathbb{Q}_{A,\rho}^\kappa$ consists of all pairs $(s, F) \in [A \times \kappa \times \kappa]^{<\kappa} \times [\widehat{W}_{A \cup B}]^{<\kappa}$ such that $s_a$ is injective for every $a \in A$. The extension relation states that $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (t, E)$ if

- $s \supseteq t$, $F \supseteq E$ and
- for all $\alpha \in \kappa$ and $w \in E$, if $e_w[s, \rho](\alpha) = \alpha$ then already $e_w[t, \rho](\alpha)$ is defined and $e_w[t, \rho](\alpha) = \alpha$.

In case $B = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$. 
The above poset is clearly \(< \kappa\)-closed. In analogy with the Knaster property, we will say that a poset \(\mathbb{P}\) has the \(\kappa\)-Knaster property, if in every collection of \(\kappa\)-many conditions from \(\mathbb{P}\) there are \(\kappa\) many which are pairwise compatible.

The poset \(Q^\kappa_{A,\rho}\) is \(\kappa^+\)-Knaster.
Some Basic Properties

Let \((s, F) \in \mathbb{Q}_A^\kappa, a \in A\).  

1. **Domain Extension** Let \(\alpha \in \kappa \setminus \text{dom}(s_a)\). Then there is an index set \(l = l_{a,\alpha}\) such that \(|\kappa \setminus l| < \kappa\) and for all \(\beta \in l\) \((s \cup \{(a, \alpha, \beta)\}, F)\) extends \((s, F)\).

2. **Range Extension** Let \(\beta \in \kappa \setminus \text{ran}(s_a)\). Then there is an index set \(J = J_{a,\beta}\) such that \(|\kappa \setminus J| < \kappa\) and for all \(\alpha \in J\) \((s \cup \{(a, \alpha, \beta)\}, F)\) extends \((s, F)\).
The generic cofinitary representation

If $G$ is $\mathbb{Q}_{A,\rho}^\kappa$-generic, then the mapping $\rho_G : A \cup B \to S(\kappa)$, which is defined by

- $\rho_G{|}_B = \rho$ and
- $\rho_G(a) = \bigcup\{ s_a : \exists F(s,F) \in G \}$ for every $a \in A$,

induces a $\kappa$-cofinitary representation of $A \cup B$ which extends $\rho$. 
Complete Embeddings and Quotients

Let $A_0 \subseteq A$ and $A_1 = A \setminus A_0$. Then:

- $\mathbb{Q}_A^\kappa, \rho \prec \mathbb{Q}_A^\kappa, \rho$,
- $\mathbb{Q}_A, \rho = \mathbb{Q}_{A_0}, \rho \ast \mathbb{Q}_{A_1}, \rho \dot{\mathcal{G}}$, where $\dot{\mathcal{G}}$ is the canonical name for the $\mathbb{Q}_{A_0}, \rho$-generic filter.
Generic Hitting

Let $\rho : B \to S(\kappa)$ induce a $\kappa$-cofinitary representation and let $\sigma \in S(\kappa) \setminus \text{im}(\hat{\rho})$ be such that $\langle \text{im}(\hat{\rho}) \cup \{\sigma\} \rangle$ is $\kappa$-cofinitary. Let $a \notin B$. Then for every $\Omega \in \kappa$ the set of all $(s, F) \in Q^{\kappa\{a\}, \rho}$ such that for some $\alpha > \Omega$ we have $s_a(\alpha) = \sigma(\alpha)$ is dense in $Q^{\kappa\{a\}, \rho}$. As an immediate corollary we obtain that if $G$ is $Q^{\kappa\{a\}, \rho}$-generic, then there are $\kappa$-many $\alpha$ such that $\rho G(a)(\alpha) = \sigma(\alpha)$. Thus in particular, in $V_{Q^{\kappa\{a\}, \rho}}$ the group $\langle \text{im}(\hat{\rho} G) \cup \{\sigma\} \rangle$ is not $\kappa$-cofinitary.
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As an immediate corollary we obtain that if $G$ is $Q^\kappa_{\{a\}, \rho}$-generic, then there are $\kappa$-many $\alpha$ such that $\rho_G(a)(\alpha) = \sigma(\alpha)$. Thus in particular, in $V^{Q^\kappa_{\{a\}, \rho}}$ the group $\langle \text{im}(\hat{\rho}_G) \cup \{ \sigma \} \rangle$ is not $\kappa$-cofinitary.
Maximality

If $|A| > \kappa$ then $\mathbb{Q}_{A,\rho}^\kappa$ adds a maximal $\kappa$-cofinitary group.
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If $|A| > \kappa$ then $\mathbb{Q}^\kappa_{A,\rho}$ adds a maximal $\kappa$-cofinitary group.

Proof:
Let $G$ be $\mathbb{Q}^\kappa_{A,\rho}$-generic. Suppose that $\text{im}(\hat{\rho}_G)$ is not maximal. Then there is a $\sigma \notin \text{im}(\hat{\rho}_G)$ such that $\langle \text{im}(\hat{\rho}_G) \cup \{\sigma\} \rangle$ is cofinitary. By the $\kappa^+\text{-c.c.}$, there is $A_0 \subset A$, $|A_0| = \kappa$ such that $\sigma \in V[H]$ where $H = G \cap \mathbb{Q}_{A_0,\rho}$. Take any $a \in A \setminus A_0$. Then by the Generic Hitting Lemma in $V[G]$ we have that $\rho_G(a)(\alpha) = \sigma(\alpha)$ for $\kappa$-many $\alpha$, which is a contradiction.
Theorem (V.F.)

(GCH) Let $\kappa$ be a regular infinite cardinal and let $C$ be a closed set of cardinals such that

1. $\kappa^+ \in C$, $\forall \nu \in C (\nu \geq \kappa^+)$,
2. if $|C| \geq \kappa^+$ then $[\kappa^+, |C|] \subseteq C$,
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Then there is a generic extension in which cofinalities have not been changed and such that $C = C_\kappa(\text{mcg})$. 
Proof:
For each $\xi \in C$, let $I_\xi := \{ (\gamma, \xi) : \gamma < \xi \}$ and let $I = \bigcup_{\xi \in C} I_\xi$. Let $\mathbb{P} = \prod_{\xi \in C} \mathbb{Q}_{I_\xi}^\kappa$ with supports of size $< \kappa$.

Lemma A
$\mathbb{P}$ is $< \kappa$-closed and $\kappa^+$-Knaster.
Proof of Lemma A:
Let \( \{p_\alpha\}_{\alpha < \kappa^+} \) be given. Without loss of generality \( \{\text{supt}(p_\alpha)\}_{\alpha < \kappa^+} \) form a \( \Delta \)-system with root \( R_0 \), where \( |R_0| < \kappa \).

- For \( p \in \mathbb{P} \) and \( \xi \in \text{supt}(p) \) recall that \( p(\xi) \in \mathcal{Q}_{I_\xi}^\kappa \). That is \( p(\xi) = (s^\xi, F^\xi) \) where \( s^\xi \in [I_\xi \times \kappa \times \kappa]^{<\kappa} \) and \( F^\xi \in [\hat{W}_{I_\xi}]^{<\kappa} \).

By \( \text{oc}(p(\xi)) \) we denote the set of all letters of \( I_\xi \) which occur in \( (s^\xi, F^\xi) \). Thus in particular \( \text{oc}(p(\xi)) \in [I_\xi]^{<\kappa} \).

- Since \( \{\prod_{\xi \in R_0} \text{oc}_A(p_\alpha)(\xi)\}_{\alpha < \kappa^+} \) are \( \kappa^+ \)-many sets each of size \( < \kappa \), by the \( \Delta \)-system lemma we can assume that they form a \( \Delta \)-system with root \( \Delta \) where \( \Delta = \prod_{\xi \in R_0} \Delta_\xi \).
Proof cont’d:

- For every $\alpha$ let $p_\alpha(\xi) = (s^{\alpha,\xi}, F^{\alpha,\xi})$. Then the sets
  $\{\prod_{\xi \in R_0} s^{\alpha,\xi} \upharpoonright \Delta \times \kappa \times \kappa\}_{\alpha < \kappa^+}$ must coincide on a set of size $\kappa^+$, since $|\prod_{\xi \in R_0} (\Delta \times \kappa \times \kappa)| = \kappa$.

- Thus there is some $t = \prod_{\xi \in R_0} t^\xi$ such that for all $\xi \in R_0$ and (wlg) all $\alpha < \kappa^+$ we have that $s^{\alpha,\xi} \upharpoonright \Delta \times \kappa \times \kappa = t^\xi$.

- This implies that $\prod_{\xi \in R_0} (s^{\alpha,\xi} \cup s^{\beta,\xi}, F^{\alpha,\xi} \cup F^{\beta,\xi})$ is a common extension of $p_\alpha \upharpoonright R_0$ and $p_\beta \upharpoonright R_0$. Thus we can find a subset if size $\kappa^+$ of pairwise compatible conditions.
Lemma B

In $V^P$ there is a $\kappa$-mcg of size $\xi$ for all $\xi \in C$.

Proof

Let $\xi_0 \in C$ and let $G_{\xi_0}$ be the mcg added by $Q^K_{I_{\xi_0}}$. We will show that $G_{\xi_0}$ remains maximal in $V^P$. If not then there are a $p \in P$ and a $P$-name for a $\kappa$-cofinitary permutation $\tau$ such that $p \upharpoonright_P \langle \text{"im}(\hat{\rho}_{\xi_0}) \cup \{\check{\tau}\} \rangle$ is a $\kappa$-cofin. group”. Wlg $\tau$ is a nice name. Since $P$ is $\kappa^+$-cc there are $\kappa$-many antichains $\{B_\alpha\}_{\alpha \in \kappa}$ each of size $\kappa$, such that $\forall b \in B_\alpha \exists \beta_b \in \kappa$ with $b \upharpoonright_P \check{\tau}(\alpha) = \check{\beta}_b$. 
Proof of Lemma B, cnt.’d:

- For $b \in B_\alpha$ let $K_{\alpha,b}$ denote the support of $b$. Then the set $C' = [(\bigcup_{\alpha \in \kappa, b \in B_\alpha} K_{\alpha,b}) \cup \text{supt}(p)] \setminus \{\xi_0\}$ is of size at most $\kappa$.

- Let $A_{\xi_0} = [\bigcup_{\alpha \in \kappa, b \in B_\alpha} \text{oc}(b(\xi_0))] \cup \text{oc}(p(\xi_0))$. That is $A_{\xi_0}$ is the collection of all letters from $l_{\xi_0}$ occurring in $\tau$ and $p$.

- Let $\bar{P} = \prod_{\xi \in C'} Q^K_{\xi}$ with supports of size $< \kappa$ and $\bar{Q} = Q^K_{A_{\xi_0}}$. Then $Q^K_{A_{\xi_0}} \prec Q^K_{l_{\xi_0}}$. Also $p$ is in $\bar{P} \times \bar{Q}$ and $\tau$ is a $\bar{P} \times \bar{Q}$-name for a $\kappa$-cofinitary permutation. Furthermore

$$p \models_{\bar{P} \times \bar{Q}} \langle \text{im}(\hat{\rho}_{\xi_0}) \cup \{\hat{\tau}\} \rangle \text{ is a } \kappa\text{-cofin. group}.$$
Proof, cnt.'d:

- Since $A_{\xi_0}$ is of size at most $\kappa$ and $I_{\xi_0}$ is of size $\xi_0 > \kappa$, there is some $a \in I_{\xi_0} \setminus A_{\xi_0}$. Let $G$ be $\bar{P} \times \bar{Q}$ generic and $p \in G$. Then in $V[G]$ by the Generic Hitting Lemma we have that $\Vdash_{\bar{Q}^\kappa}_{I_{\xi_0} \setminus A_{\xi_0} \setminus \rho_{A_{\xi_0}}} \forall \Omega < \kappa \exists \beta > \Omega(\rho_{I_{\xi_0} \setminus A_{\xi_0}}(a)(\beta) = \tau(\beta))$.

- However, $(\bar{P} \times \bar{Q}^\kappa_{A_{\xi_0}}) \ast \bar{Q}^\kappa_{I_{\xi_0} \setminus A_{\xi_0} \setminus \rho_{A_{\xi_0}}} = \bar{P} \times (\bar{Q}^\kappa_{A_{\xi_0}} \ast \bar{Q}^\kappa_{I_{\xi_0} \setminus A_{\xi_0} \setminus \rho_{A_{\xi_0}}}) = \bar{P} \times \bar{Q}^\kappa_{I_{\xi_0}}$. Therefore $p \Vdash_{\bar{P} \times \bar{Q}^\kappa_{I_{\xi_0}}} \forall \Omega < \kappa \exists \beta > \Omega(\rho_{I_{\xi_0}}(a)(\beta) = \check{\tau}(\beta))$.

Since $\bar{P} \times \bar{Q}^\kappa_{I_{\xi_0}} \prec \bar{P}$, we reach a contradiction.
Lemma C

In $V^P$ for every $\lambda \not\in C$ there are no $\kappa$-maximal cofinitary groups of size $\lambda$.

The proof follows very closely to Blass’s proof regarding the spectrum of maximal almost disjoint families on $\omega$ and relies on homogeneity properties shared by the two constructions.
Proof of Lemma C:
We will show that for every $\lambda \notin C$, $\lambda$ is not $OD(\kappa \kappa)$ definable. That is we will show that in $V[G]$ if $\langle X_\alpha \rangle_{\alpha \in \lambda}$ is a sequence of $OD(\kappa \kappa)$ definable sets which covers $\kappa \kappa$, then there is a proper subsequence which also covers $\kappa \kappa$. Fix such a sequence and for each $\alpha$ an ordinal $\Theta_\alpha$ and a function $u_\alpha \in \kappa \kappa$ such that in $V[G]$, $X_\alpha$ is the $\Theta_\alpha$-th set definable from $u_\alpha$.

Let $\mu$ be the largest element of $C$ below $\lambda$. Then $cof(\mu) \geq \kappa^+$. By $GCH$ (in $V$) we have $\mu^\kappa = \mu$. Recursively we will define a sequence $\langle M_\gamma \rangle_{\gamma \in \kappa^+}$, where $|M_\gamma| \leq \mu$ for all $\gamma$, such that the $X_\alpha$'s with indexes in $\bigcup M_\gamma$ cover $\kappa \kappa$. Let $M_0 := \emptyset$ and for $\gamma$ limit, let $M_\gamma := \bigcup_{\delta < \gamma} M_\delta$. 

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For each \( \alpha \in \lambda \) choose \( J_\alpha \subseteq I = \bigcup_{\xi \in C} I_\xi \) of size \( \kappa \) such that for every \( p \) which is involved either in \( \dot{u}_\alpha \) or in \( \dot{\Theta}_\alpha \) and each \( \xi \) in the support of \( p \) we have that \( \text{oc}(p(\xi)) \subseteq J_\alpha \). Let \( S \) be the union of \( \{ I_\gamma : \gamma \in \mu \cap C \} \) and \( \{ J_\alpha : \alpha \in \lambda \} \). Then \( |S| = \lambda \).

**K-support**

Let \( K \subseteq S \) be of size \( \mu \) such that \( \bigcup_{\gamma \in \mu \cap C} I_\gamma \subseteq K \). A subset \( J \) of \( I \) such that \( |J| = \kappa \) is called a **K-support** for the name \( \dot{x} \) of a function in \( \kappa \kappa \) if

- for every \( p \) involved in \( \dot{x} \) and every \( \xi \) in the support of \( p \) we have that \( \text{oc}(p(\xi)) \subseteq J \) and
- if \( J \cap I_\gamma \setminus K \neq \emptyset \) then \( |J \cap I_\gamma \setminus K| = \kappa \).
Since every \( \gamma \in C \setminus (\mu \cup \{ \mu \}) \), \( \gamma > \lambda \), we have \( |I_\gamma \setminus S| = |I_\gamma \setminus K| = \gamma \). Thus whenever we are given a \( K \) as above and a name for a function in \( \kappa \kappa \), we can assume that it has a \( K \)-support.
Let $G$ be the group of those permutations of $I$ that map each $I_\gamma$ into itself and that fixes all members of $K$. Then $G$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each $p$ to a condition $g(p)$ naturally defined from $g$ and $p$. 
Let $G$ be the group of those permutations of $I$ that map each $I_\gamma$ into itself and that fixes all members of $K$. Then $G$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each $p$ to a condition $g(p)$ naturally defined from $g$ and $p$.

More precisely: let $p \in \mathbb{P}$, $\xi \in \text{supt}(p)$ and $p(\xi) = (s^\xi, F^\xi)$ where $s^\xi \in [I_\xi \times \kappa \times \kappa]^{<\kappa}$, $F^\xi \in [W_{I_\xi}]^{<\kappa}$. Then let $\text{supt}(g(p)) := \text{supt}(p)$. For $\xi \in \text{supt}(p)$, let $g(p(\xi)) := (g(s^\xi), g(F^\xi))$ where $\text{oc}(g(s^\xi)) = g(\text{oc}(s^\xi))$ and for every $(\alpha, \xi) \in \text{oc}(g(s^\xi)) = g(\text{oc}(s^\xi))$ if $(\alpha_0, \xi) \mapsto (\alpha, \xi)$ then $[g(s^\xi)]_{(\alpha, \xi)} := s^\xi_{(\alpha_0, \xi)}$. Furthermore for a word $w \in F^\xi$ define $g(w)$ to be the word obtained by substituting every appearance of a letter $a = (\alpha, \xi)$ in $w$ with $g(\alpha, \xi)$. Then let $g(F^\xi)$ be the set of all $g(w)$ for $w \in F^\xi$. 
Let $G$ be the group of those permutations of $I$ that map each $I_\gamma$ into itself and that fixes all members of $K$. Then $G$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each $p$ to a condition $g(p)$ naturally defined from $g$ and $p$.

- Note that each such automorphism $g$ preserves not only maximal antichains, but also the forcing relation. In particular, if $J$ is a support of a name $\dot{x}$, then $g(J)$ is a support of the name $g(\dot{x})$. If in addition $g$ fixes all members of $J$, then it also fixes the name $\dot{x}$.

- If $J$ is a support then its $G$-orbit is determined by $J \cap K$ and $\bar{J} = \{ \gamma \in C : J \cap I_\gamma - K \neq \emptyset \}$. That is, if $J'$ is another support with $J' \cap K = J \cap K$ and $J' = \bar{J}$, then there is $g \in G$ with $g(J) = J'$. 

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The spectrum of $\kappa$-maximal cofinitary groups
This implies that there are only $\mu$ many orbits of supports. Indeed:

- Since $J \cap K$ is of size $\leq \kappa$ and $|K| = \mu = \mu^\kappa$, there are only $\mu$ possibilities for $J \cap K$.
- If $[\kappa^+, |C|] \neq \emptyset$, then $[\kappa^+, |C|] \subseteq C$. Thus in this case $|C| \leq \mu$.
- If $[\kappa^+, |C|] = \emptyset$, i.e. $|C| \leq \kappa$, then since $\mu \geq \kappa^+$ we have again $|C| \leq \mu$. Thus there are no more than $\mu^\kappa = \mu$ many possibilities for $\bar{J} \in [C]^{\leq \kappa}$. 
For each $G$-orbit of supports, fix a member $J$ such that $J \cap S = J \cap K$. Such orbits are referred to as standard supports. For each fixed support $J$ there are only $\kappa^\kappa = \kappa^+$ (by GCH in $V$) many names. Since $\mu \geq \kappa^+$, there are only $\mu$-many names that have standard supports.

For each name $\dot{x}$ with a standard support, fix a set $A = A(\dot{x}) \in [\lambda]^{\leq \kappa} \cap V$ such that $P$ forces $"(\exists \alpha \in \mathcal{A}) \dot{x} \in \dot{X}_\alpha"$. Let $B = \bigcup\{A(\dot{x}) : \dot{x} \text{ has a standard support}\}$. Then $|B| \leq \mu$. 
We will proceed with the successor step in the inductive definition of $\langle M_\sigma \rangle_{\sigma < \kappa^+}$. Let

$$K_\sigma = \bigcup_{\alpha \in M_\sigma} J_\alpha \cup \bigcup_{\gamma \leq \mu \cap C} I_\gamma.$$ 

Then $|K_\sigma| = \mu$. Let $M_{\sigma + 1}$ be obtained from $K_\sigma$ in the same way that $B$ was obtained from $K$ above. Then $|M_{\sigma + 1}| \leq \mu$. Define $M = \bigcup_{\sigma \in \kappa^+} M_\sigma$ and $K = \bigcup_{\sigma \in \kappa^+} K_\sigma$.

Let $\dot{x}$ be a $P$-name for a function in $\kappa$. We will show that $P$ forces that “$(\exists \alpha \in M) \dot{x} \in \dot{X}_\alpha$”.
Let $J \subset I$ of size $\kappa$ such that for every $p$ involved in $\dot{x}$ and every $\xi$ in the support of $p$ we have $\text{oc}(p(\xi)) \subseteq J$. Fix $\sigma < \kappa^+$ such that $J \cap K \subseteq K_\sigma$. For each $\gamma \in C$ such that $J \cap I_\gamma - K_\sigma \neq \emptyset$, we have that $\gamma > \lambda(> \mu)$. Then $|I_\gamma - K| = \lambda$. Thus enlarging $J$ is necessary we can assume that it is a $K_\sigma$-support and $J \cap K \subseteq K_\sigma$.

Consider the group of all permutations of $I$ which fix $K_\sigma$ and map each $I_\gamma$ to itself. There is $g \in G$ such that $g(J)$ is a $K_\sigma$-standard support. Then neither $J$ nor $g(J)$ meets $K_{\sigma+1} - K_\sigma$ and so there is a permutation $h$ which agrees with $g$ on $J$ and with the identity map on $K_{\sigma+1} - K_\sigma$. In particular $h(J) = g(J)$ is standard and $h$ leaves $K_{\sigma+1}$ pointwise fixed.
Since $h(\dot{x})$ has standard support $h(J)$, it is one of the $\mu$ names for which we chose a set $A = A(h(\dot{x}))$ to include in $M_{\sigma+1}$. Thus

$$\Vdash \exists \alpha \in \check{A} h(\dot{x}) \in \dot{X}_\alpha,$$

which implies that

$$\Vdash \exists \alpha \in \check{A} [h(\dot{x}) \text{ is in the } \dot{\Theta}_\alpha \text{th set ordinal-definable from } \dot{u}_\alpha].$$

However $A \subseteq M_{\sigma+1}$ and $\forall \alpha \in A(J_\alpha \subseteq K_{\sigma+1})$. Thus $h$ fixes $J_\alpha$ pointwise, and so $h$ fixes $\dot{\Theta}_\alpha$ and $\dot{u}_\alpha$. Therefore

$$\Vdash \exists \alpha \in \check{A} [h(\dot{x}) \text{ is in the } h(\dot{\Theta}_\alpha) \text{th set ordinal-definable from } h(\dot{u}_\alpha)].$$
Now since $h$ preserves the forcing relation, we have

$$\Vdash_{P} \exists \alpha \in \dot{A}[\dot{x} \text{ is in the } \dot{\Theta}_{\alpha} \text{th set ordinal-definable from } \dot{u}_{\alpha}]$$.

Now since $M_{\sigma+1} \subseteq M$ we obtain that

$$\Vdash_{P} \exists \alpha \in \dot{M}(\dot{x} \in \dot{X}_{\alpha})$$,

which completes the proof that $\lambda$ is not $OD(\kappa\kappa)$-definable.
Following standard notation, $F_{n_{<\kappa}}(\kappa, \kappa)$ denotes the $\kappa$-Cohen poset, e.g. the poset of all partial functions from $\kappa$ to $\kappa$ of cardinality $< \kappa$ with extension relation superset.

**Theorem (V.F.)**

(GCH) There is a $\kappa$-Cohen indestructible $\kappa$-maximal cofinitary group.
Proof:
Let \( \{ \langle p_\xi, \tau_\xi \rangle : \kappa \leq \xi < \kappa^+, \xi \in \text{Succ}(\kappa^+) \} \) enumerate all pairs \( \langle p, \tau \rangle \) where \( p \in \text{Fn}_{<\kappa}(\kappa, \kappa) \) and \( \tau \) is a name for a \( \kappa \)-cofinitary permutation. Recursively we will construct a family \( \{ \rho_\xi \}_{\kappa \leq \xi < \kappa^+} \) of \( \kappa \)-cofinitary representations such that

1. for all \( \xi \), \( \rho_\xi : \xi \to S(\kappa) \),
2. for all \( \eta < \xi \), \( \rho_\eta = \rho_\xi \upharpoonright \eta \), and
3. \( \bigcup_{\kappa \leq \xi < \kappa^+} \rho_\xi : \kappa^+ \to S(\kappa) \) induces a cofinitary representation \( \hat{\rho} \) such that \( \text{im}(\hat{\rho}) \) is a \( \kappa \)-maximal cofinitary group, which is \( \text{Fn}_{<\kappa}(\kappa, \kappa) \)-indestructible.
Proof cnt.’d: Let $\rho_\kappa$ be a cofinitary representation of $\kappa$ given by $\mathbb{Q}_\kappa^\kappa$. Suppose for all $\xi : \kappa \leq \xi < \eta$, $\rho_\xi$ has been defined and $\eta = \xi + 1$ for some $\xi$. Consider the pair $\langle p_\xi, \dot{\tau}_\xi \rangle$. If

$\vdash p_\xi \Vdash \text{Fn}_{<\kappa}(\kappa, \kappa) \quad \langle \tau_\xi \notin \text{im}(\hat{\rho}_\xi) \rangle$, and

$\vdash p_\xi \Vdash \text{Fn}_{<\kappa}(\kappa, \kappa) \quad \langle \text{im}(\hat{\rho}_\xi) \cup \{ \dot{\tau}_\xi \} \rangle$ is a $\kappa$-cofin. group

then proceed as follows:
Let $q \leq \rho_{\xi}$. Then

$$q \Vdash Fn_{<\kappa}(\kappa, \kappa) "\langle \text{im}(\hat{\rho}_{\xi}) \cup \{ \hat{\tau}_{\xi} \} \rangle \text{ is a cofin. group".}$$

The Generic Hitting implies that if $G$ is $Fn_{<\kappa}(\kappa, \kappa)$-generic and $q \in G$, then in $V[G]$ for all $\Omega \in \kappa$ the set

$$D_{\hat{\tau}_{\xi}[G],\Omega} = \{(s, F) \in \mathbb{Q}\{\xi\},\rho_{\xi} : \exists \alpha > \Omega(s(\alpha) = \tau_{\xi}[G](\alpha))\}$$

is dense. Thus for every $\Omega \in \kappa$ and every $(s, F) \in \mathbb{Q}\{\xi\},\rho_{\xi}$ there are $q' \leq Fn_{<\kappa}(\kappa, \kappa) q$, $\alpha > \Omega$ and $(s', F') \leq (s, F)$ such that

$$q' \Vdash Fn_{<\kappa}(\kappa, \kappa) \check{s}'(\alpha) = \hat{\tau}_{\xi}(\alpha).$$
Proof cnt.’d: Therefore the set

\[ D^q_\Omega = \{(s,F) \in \mathbb{Q}\{\xi\},\rho_\xi : \exists \alpha > \Omega \exists q' \leq q(q' \Vdash s(\alpha) = \dot{\iota}_\xi(\alpha))\} \]

is dense in \( \mathbb{Q}\{\xi\},\rho_\xi \). Now let \( G \subseteq \mathbb{Q}\{\xi\},\rho_\xi \) be a filter meeting the dense sets

- \( D^\text{domain}_\alpha = \{(s,F) \in \mathbb{Q}\{\xi\},\rho_\xi : \alpha \in \text{dom}(s)\} \),
- \( D^\text{range}_\alpha = \{(s,F) \in \mathbb{Q}\{\xi\},\rho_\xi : \alpha \in \text{range}(s)\} \),
- \( D_w = \{(s,F) \in \mathbb{Q}\{\xi\},\rho_\xi : w \in F\} \) and \( D^q_\Omega \),

where \( \alpha, \Omega \in \kappa, q \leq \text{Fn}_{\kappa}(\kappa,\kappa) \rho_\xi \) and \( w \in \hat{W}_{\{\xi\} \cup \xi} \).
Since these are only $\kappa$ many dense sets and the forcing notion $\mathbb{Q}_\kappa^{\{\xi\}, \rho_\xi}$ is $<\kappa$-closed such a filter $G$ exists. Then the mapping $\rho_{\xi+1} : \xi + 1 \to S(\kappa)$ where

\[
\begin{align*}
\rho_{\xi+1} \upharpoonright \xi &= \rho_\xi, \\
\rho_{\xi+1}(\xi) &= \bigcup \{ s : \exists F(s, F) \in G \}
\end{align*}
\]

induces a $\kappa$-cofinitary representation extending $\rho_\xi$. 
Claim

\( p_\xi \models \mathcal{F}n_{<\kappa}(\kappa,\kappa) \) "\( \forall \Omega \in \kappa \exists \alpha > \Omega (\tau_\xi(\alpha) = \rho_{\xi+1}(\xi)(\alpha)) \)".

Proof:

Suppose not. Then there are \( q \leq p_\xi \) and \( \Omega \in \kappa \) such that

\[ q \models \mathcal{F}n_{<\kappa}(\kappa,\kappa) \{ \alpha : \dot{\tau}_\xi(\alpha) = \rho_{\xi+1}(\xi)(\alpha) \} \subseteq \check{\Omega}. \]

Then let \((s, F) \in G \cap D^\Omega_q \). Then there are \( \alpha > \Omega \) and \( q' \leq \mathcal{F}n_{<\kappa}(\kappa,\kappa) \) \( q \) such that \( q' \models \mathcal{F}n_{<\kappa}(\kappa,\kappa) \dot{\tau}_\xi(\alpha) = s(\alpha) \). It remains to observe that \( \rho_{\xi+1}(\xi)(\alpha) = s(\alpha) \) and so we have reached a contradiction.
If $\xi$ is a limit, then define $\rho_\xi := \bigcup_{\eta < \xi} \rho_\eta$ and note that $\rho_\xi : \xi \rightarrow S(\kappa)$ induces a cofinitary representation.

Indeed, let $w \in F_\xi$. Then there is a good word $w' \in \hat{W}_\xi$ such that for some $u \in W_\xi$ we have $w = u^{-1}w'u$. However in each of those words there are only finitely many letters involved and so there is $\eta < \kappa^+$ such that $w, u, w'$ are in fact elements in $W_\eta$. Then $e_{w'}[\rho_\xi] = e_{w'}[\rho_\eta]$ and since by Inductive Hypothesis $\rho_\eta$ induces a $\kappa$-cofinitary representation we have that the set of all fixed points of $e_{w'}[\rho_\xi]$ is of cardinality smaller than $\kappa$. However $|\text{fix}(e_w[\rho_\xi])| = |\text{fix}(e_{w'}[\rho_\xi])|$, which completes our argument.
With this the inductive construction of the sequence $\langle \rho_\xi \rangle_{\kappa \leq \xi < \kappa^+}$ is complete. Let $\rho := \bigcup_{\kappa \leq \xi < \kappa^+} \rho_\xi$.

**Claim**

$\text{im}(\hat{\rho})$ is a $\kappa$-mcg which is $\kappa$-Cohen indestructible.
Proof:
Let $G$ be $Fn_{<\kappa}(\kappa, \kappa)$-generic filter. Suppose

$$V[G] \models (\text{im}(\hat{\rho}) \text{ is not a } \kappa \text{ maximal cof. group}).$$

Then

$$V[G] \models \exists \tau (\tau \notin \text{im}(\hat{\rho}) \land \langle \text{im}(\hat{\rho}) \cup \{\tau\} \rangle \text{ is a } \kappa \text{ cofin. group}).$$
Proof cnt’d.: 
Therefore there is \( p \in G \) and a \( \text{Fn}_{<\kappa}(\kappa, \kappa) \)-name for a cofinitary permutation \( \dot{\tau} \) such that 

\[
 p \models \text{Fn}_{<\kappa}(\kappa, \kappa) \left( \tau \notin \text{im}(\dot{\rho}) \land \langle \text{im}(\dot{\rho}) \cup \{ \dot{\tau} \} \rangle \text{ is a } \kappa\text{-cofin. group} \right).
\]

There is \( \xi : \kappa \leq \xi < \kappa^+ \), successor such that \( \langle p, \tau \rangle = \langle p_\xi, \tau_\xi \rangle \). Then by construction

\[
 p \models \forall \Omega \exists \alpha > \Omega(\rho(\xi + 1)(\alpha) = \dot{\tau}(\alpha)),
\]

which is a contradiction. \( \square \)
Theorem (V.F.)

(GCH) Let $\kappa^{++} \leq \lambda$ be regular uncountable cardinals and let $\mathbb{P} = Fn_{<\kappa}(\lambda \times \kappa, \kappa)$. Then in $V^\mathbb{P}$ every $\kappa$-maximal cofinitary group is either of size $\kappa^+$ or of size $2^\kappa = \lambda$.

An isomorphism of names argument shows that in the generic extension there are no $\kappa$-maximal cofinitary groups of size $\mu$, where $\kappa^+ < \mu < 2^\kappa$. 
Let $C$ denote either of the following sets: set of all $\kappa$-maximal cofinitary groups, the set of $\kappa$-maximal almost disjoint families, the set of $\kappa$-almost disjoint permutations on $\kappa^\kappa$, the set on $\kappa$-almost disjoint functions on $\kappa^\kappa$. Then:

**Theorem (V.F.)**

*(GCH)* Let $\kappa$ be a regular uncountable cardinal and let $C$ be a closed set of cardinals such that

1. $\kappa^+ \in C$, $\forall \nu \in C (\nu \geq \kappa^+)$,
2. $[\kappa^+, |C|] \subseteq C$ and
3. $\forall \nu \in C (cof(\nu) \leq \kappa \rightarrow \nu^+ \in C)$.

Then there is a generic extension in which cofinalities have not been changed and such that $C = \{|G| : G \in C\}$. 

Vera Fischer
Theorem (V.F., S.D. Friedman)

Assume GCH. Let $E$ be an Easton index function and let $P = P(E)$ be the Easton product. Then in $V^P$ for every $\kappa \in \text{dom}(E)$ we have that $a(\kappa) = a_g(\kappa) < d(\kappa)$.
Theorem (V.F., S.D. Friedman)

Assume GCH. Let $E$ be an Easton index function and let $\mathbb{P} = \mathbb{P}(E)$ be the Easton product. Then in $V^\mathbb{P}$ for every $\kappa \in \text{dom}(E)$ we have that $a(\kappa) = a_g(\kappa) < d(\kappa)$.

Can we control the spectra of $\kappa$-mad families (resp. $\kappa$-m.c.g.) globally?
The question of obtaining an optimal set of conditions on the potential spectrum of m.c.g. and also m.a.d. families, even on $\omega$ is still open.
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The question of obtaining an optimal set of conditions on the potential spectrum of m.c.g. and also m.a.d. families, even on $\omega$ is still open. In a recent paper S. Shelah and O. Spinas that the requirements $\aleph_1 \in C$ and $\forall \lambda \in C(\text{cof}(\lambda) = \omega \rightarrow \lambda^+ \in C)$ in Blass’s theorem are not necessary.

An analogous weakening on the requirements which we impose on the spectrum of $\kappa$-maximal cofinitary groups is of interest.

There are still many open questions regarding the possible sizes of $\kappa$-mad families and $\kappa$-maximal cofinitary groups. For example, it is not known if consistently $\text{cof}(a(\kappa)) = \kappa$, neither if consistently $\text{cof}(a_g(\kappa)) = \kappa$. 
Thank you!