Brownian loops and conformal fields

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Conformal field theories are special field theories which enjoy a particular type of symmetry called conformal symmetry.

Particle physics (Standard Model)

Field theories are very useful in physics

Condensed matter physics (e.g., large scale properties of critical and off-critical systems)

Example from statistical mechanics: 2D critical Ising model (on square lattice, \( \mathbb{Z}^2 \))

Continuum scaling limit: replace \( \mathbb{Z}^2 \) by \( \mathbb{R} \times \mathbb{R} \) and let \( \epsilon \to 0 \).

\[ \{ S_x \}_{x \in \mathbb{R} \times \mathbb{R}} \]

Local magnetization (sum of all spins in a "mesoscopic" region)

\[ \sum_x S_x \]

\[ \frac{1}{15/8} \sum_{x \in \mathbb{R} \times \mathbb{R}} S_x S_x \to \delta \text{ Dirac delta at } x \]

"Conformal field" (random distribution) (Camia, Garban, Newman; 2012)
Rest of talk: family of conformal fields built from Brownian motion.

Why are they interesting?

- Natural construction involving Brownian motion.
- New fields with unusual properties.
- Connections with Conformal Loop Ensembles (CLEs) and Schramm–Loewner Evolution (SLE).
- Amenable to rigorous mathematical analysis.

**Random Walk Loop Soup**

Consider all loops $f$ (walks in $\mathbb{Z}^2$ starting and ending at same vertex) that stay inside $D$.

Unrooted loop $Y$: equivalence class of loops obtained by forgetting the starting point.

Measure $\mu_Y$ on unrooted loops: $\mu_Y(Y_f) = e_y \frac{1}{|Y_f|} \left( \frac{1}{4} \right)^{|Y_f|}$

$|Y_f|$: length of loop (number of steps)

$e_y$: number of loops in $Y_f$
\textbf{RWLS in }D \text{ with intensity } \lambda:\n
Take a collection \( \{ N_y \} \) of random variables, one for each unrooted loop \( y \) in \( D \).

\( N_y \) has Poisson distribution with intensity \( \lambda \nu_D(y) \):

\[ P(N_y = m) = \frac{1}{m!} e^{-\lambda \nu_D(y)} (\lambda \nu_D(y))^m. \]

\( N_y \) represents the multiplicity of \( y \) in the Random Walk Loop Soup.

\textbf{Brownian Loop Soup in }D \text{ with intensity } \lambda:\n
Take scaling limit of \( \text{RWLS} \) (Brownian scaling):

\[ \text{RWLS in } D \text{ of } \mathbb{Z}^2 \text{ with int. } \lambda \rightarrow_{s \to 0} \text{BLS in } D \text{ with int. } \lambda \]

Let \( \Gamma \) denote the collection of loops of a BLS in \( D \) with intensity \( \lambda \).

\textbf{Goal - I want to study the "field" } e^{iB N_\omega(z)} \text{, where:}

\( z \in D \), \( \beta \in (0, 2\pi) \) and \( N_\omega(z) = \sum \Theta_\chi(z) \).

\( S > 0 \) cutoff \rightarrow \text{diam}(\Gamma) > \delta\)

\textbf{Scaling limit} \( (s \to 0) \)?
Theorem (C., Gandolfi, Kleban)

If \( n \in \mathbb{N} \), \( D \subseteq \mathbb{C} \) is bounded and \( \beta_1, \ldots, \beta_n \in (0, 2\pi) \), then the limit

\[
\lim_{s \to 0} \prod_{j=1}^{n} \frac{\ell^s i^n N_{\beta_j}^s (z_j)}{\ell, N_{\beta_j}^s (z_j)} = \phi_D (z_1, \ldots, z_n; \beta_1, \ldots, \beta_n)
\]

exists and is finite and nontrivial iff \( \Delta (\beta_j) = \frac{\lambda \beta_j (2\pi - \beta_j)}{8\pi^2} \).

Moreover, if \( D' \) is another bounded domain and \( f : D \to D' \) is a conformal map, then

\[
\phi_D (f(z_1), \ldots, f(z_n); \beta_1, \ldots, \beta_n) = \prod_{j=1}^{n} \frac{f'(z_j) e^{-2\Delta (\beta_j)}}{f(z_j)} \phi_D (z_1, \ldots, z_n; \beta_1, \ldots, \beta_n). \]

Theorem (C., Lies)

Let \( V_{\psi}^s (z) = s - 2\Delta (\beta) e^{-i\beta N_{\psi}^s (z)} \), with \( \Delta (\beta) = \frac{\lambda \beta (2\pi - \beta)}{8\pi^2} \), and \( \langle V_{\psi}^s, \varphi \rangle = \int_D V_{\psi}^s (z) \varphi (z) \, d\, z \). If \( D \) is bounded, \( \exists \, D \) is smooth, \( \Delta < \frac{1}{2} \), then \( \forall \alpha > 1 \), as \( s \to 0 \), \( V_{\psi}^s \) converges in second mean in the Sobolev space \( H^{2-\alpha} (D) \) to a random distribution \( \varphi_{\psi} \in H^{-\alpha} (D) \).