Metastability for the Ising model on a random graph

Joint project with
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Outline

- A little about metastability
- Some key questions
- A little about the Ising model on a multi-graph, and the Configuration model random graph
- Significance of low temperature limit
- The communication height and some of its properties
We observe (Markovian) dynamics on some state space $\Omega$ with energy function $\mathcal{H}$ that has a global minimum $s$ and or more local minima $m$. Dynamics started at $m$ may appear to be in equilibrium, prevented from moving far due to a
Concept of metastability

We observe (Markovian) dynamics on some state space $\Omega$ with energy function $\mathcal{H}$ that has a global minimum $s$ and or more local minima $m$. Dynamics started at $m$ may appear to be in equilibrium, prevented from moving far due to a potential barrier.
m and s

- s is a **stable** state (trivially unique in our model).
\( \mathbf{m} \) and \( \mathbf{s} \)

- \( \mathbf{s} \) is a *stable* state (trivially unique in our model).
- \( \mathbf{m} \) is a *metastable* state if no other state has a greater potential barrier between itself and configurations of lower energy (along best path).

\[
V(\mathbf{m}) \geq V(\mathbf{q})
\]
Main questions

What can we say about the \textit{crossover} time

\[ m \rightarrow s ? \]

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What can we say about the *crossover* time

\[ m \to s \]

In particular,

- what is the average crossover time \( E_m [\tau_s] \)?
- how does it vary with \(|\Omega|\)?
- what kind of configurations does the dynamics see "near the top"?

We investigate these in the low temperature setting \( \beta \to \infty \) (this is a parameter of the model).
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Ising model on a multi-graph

Let $G = (V, E)$ be a multi-graph.

- **Configuration (state) space**: $\Omega = \{+1, -1\}^{|V|}$ is the set of $+1/-1$ configurations on $G$.
- **Hamiltonian** on configuration space: for $\omega \in \Omega$,

$$H(\omega) = -\frac{\hat{J}}{2} \sum_{\langle u, v \rangle \in E} \omega(v) \omega(u) - \frac{\hat{h}}{2} \sum_{v \in V} \omega(v)$$

with $\hat{J} > 0$ and $\hat{h} > 0$. 

$$\mu_\beta(\omega) = \frac{1}{Z_\beta} \exp\{-\beta H(\omega)\}$$

Assigns exponential preference to configurations with low energy.
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- **Gibbs** measure on $\Omega$

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Configuration model random graph

A family of random graphs that are of particular interest to us, are based on the *Configuration* model. This begins with a given set of vertices \( \{v_1, ..., v_n\} \), and a prescribed degree sequence \( \mathbf{d} = (d_1, ..., d_n) \).
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Glauber Dynamics, significance of $\beta \to \infty$

Dynamics on configuration space: continuous-time Glauber dynamics with transition rates

$$c_\beta(\omega, \omega') = \begin{cases} \exp \left( -\beta \left[ \mathcal{H}(\omega') - \mathcal{H}(\omega) \right]_+ \right) & \text{if } \omega \sim \omega' \\ 0 & \text{otherwise} \end{cases}$$

where $\omega \sim \omega'$ iff $\omega$ and $\omega'$ differ at exactly one vertex.
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- Note that 'up' moves become exponentially expensive.
- Bounds on $R_{\text{eff}}$ (effective resistance) show that w.h.p. dynamics avoid ” unnecessarily expensive” configurations when going from $m \to s$, and will take an optimal path (see for example *Metastability* - A. Bovier, F. den Hollander).
Communication Height

The communication height between \( m \) and \( s \) on \( G \) is

\[
\Gamma^* := \Phi(m, s) - \mathcal{H}(m) = \min_\gamma \max_{\sigma \in \gamma} \mathcal{H}(\sigma) - \mathcal{H}(m)
\]

- represents the "lowest barrier" between \( m \) and \( s \) which the dynamics must overcome.
- paths \( \gamma \) that correspond to the minimum are called optimal paths.
Communication Height

For example, for the previous graph we have the following optimal path for suitable values of $\tilde{J}$ and $\tilde{h}$ (note: easy to show here that $m = \square$ and $s = \square$)
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Significance of $\Gamma^*$

How is $\Gamma^*$ related to the crossover time $m \rightarrow s$? It follows from standard universal metastability theorems that

$$\lim_{\beta \rightarrow \infty} \exp \left( -\beta \Gamma^* \right) \mathbb{E}_{\square} [\tau_{\square}] = K$$

Furthermore, in $\mathbb{T}^2 \subset \mathbb{Z}^2$ the pre-factor $K$ is related to $|C^*|$, where $C^* := \text{Configurations on optimal paths where height } \Gamma^* \text{ is first seen.}$

We expect that a similar relation holds for our model.
Some properties of $\Gamma^*$

When $\min_i d_i \geq 3$, the resulting Configuration model is an expander graph. Therefore $\Gamma^* = \Theta(n)$, with $n = |V|$ (under some assumptions on the distribution of $d$).
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**Upper bound on $\Gamma^*$:** by a coupling argument, we can show that

$$\Gamma^* \leq \ell_n/4$$

where $\ell_n$ is the total degree of the graph. We would like to make this sharper.
Some properties of $\Gamma^*$

We also expect that

$$\frac{\Gamma^*_n}{n} \to C$$

for some constant $C$ depending on the parameters of the model, when $d$ is regular, and possibly also for $d_i \sim i.i.d.$ under some constraints.
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To this avail, we can show that if $A \subset V$ is a random set/configuration of total degree $\ell_A$, then

$$\frac{|\partial A|}{|d|} \rightarrow_{L^2} \ell_A \left(1 - \frac{\ell_A - 1}{|d| - 1}\right)$$
Some properties of $\Gamma^*$

\[ \frac{|\partial A|}{|d|} \rightarrow_{L_2} \ell_A \left( 1 - \frac{\ell_A - 1}{|d| - 1} \right) \]

We need to show that this holds not only for random sets, but also for extremal.
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We need to show that this holds not only for random sets, but also for extremal.
But what are \( m \) and \( s \)?
It is easy to see that \( s = +1 \), i.e. the configuration with +1 assigned at every vertex.
It is also easy to see that \( -1 \) is a local minimum (for non-degenerate values of \( J \) and \( h \)).
Showing that \( m = -1 \) and nothing else (in the case when degree sequence satisfies \( d_i \geq 3 \)) is difficult.
Conclusion

• There are many unforeseen challenges when assigning the Ising model to a random graph.
• Global properties of the random graph model play a key role.
• Precise descriptions of $\Gamma^*$, $C^*$ exist in lattice setting, but may be too difficult for random graphs. But good estimates should be attainable.