Scaling quasistationary states in long-range interacting systems with dissipation

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1. Introduction

2. Self-Gravitating systems

3. Dissipative forces

4. Simulation

5. Generalized Self-gravitating models and Conclusion
Outline

1. Introduction
   - Additivity and extensivity
   - Ensemble inequivalence
   - Quasi-stationary states

2. Selfgravitating systems

3. Dissipative forces

4. Simulation

5. Generalized Self-gravitating models and Conclusion
Short-range interacting systems

- Extensity and additivity
- Ensemble equivalence
- Dynamics; Finite relaxation time toward equilibrium. Equilibrium independent of the initial states
Fully connected Ising model

Ising Hamiltonian

\[ H = -\frac{J}{N} \sum_{i,j} S_i S_j \]
Fully connected Ising model

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\[ H = - \frac{J}{N} \sum_{i,j}^{N} S_i S_j \]

\[ E_A = -N J \]
Fully connected Ising model

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\[ H = -\frac{J}{N} \sum_{i,j}^{N} S_i S_j \]

\[ E_B = -NJ \]
Fully connected Ising model

Ising Hamiltonian

$$H = -\frac{J}{N} \sum_{i,j} S_i S_j$$

$$E = 0 \neq E_A + E_B$$
Long range potential

\[ V(r) \sim -\frac{1}{r^\alpha} \]
Blume-Emery-Griffths model

\[ H = \Delta \sum_{i=1}^{N} S_i^2 - \frac{J}{2N} (\sum_{i} S_i)^2 \]

with \( S_i = 0, \pm 1 \)

Canonical ensemble
Hamiltonian mean-field

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{m} + \frac{J}{N} \sum_{i,j}^{N} (1 - \cos(\theta_i - \theta_j)) \]
Hamiltonian mean-field

HMF

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{m} + \frac{J}{N} \sum_{i,j}^{N} (1 - \cos(\theta_i - \theta_j)) \]

Canonical ensemble: Critical point at \( \beta = 2 \).
Microcanonical ensemble: Critical point at \( \epsilon = 3/4 \).
Hamiltonian Mean-Field

Time evolution of the magnetization

\[ \tau \sim N^{1.7} \]

\( M(t) \) vs. \( \log_{10} t \) for different values of \( N \):
- \( N = 10^2 \)
- \( N = 10^3 \)
- \( N = 2 \times 10^4 \)
Vlasov

Vlasov equation: (Liouville equation + Factorization approximation)

\[ \partial_t f(\theta, p, t) + p\partial_\theta f(\theta, p, t) + \bar{a}(\theta, t)\partial_v f(\theta, p, t) = 0 \]

where \( \bar{a}(\theta, t) = J \int \sin(\theta - \theta') f(\theta', p, t) d\theta' dp \)
Hamiltonian mean-field

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1. Stationary equation: infinite number of solutions (with equilibrium)
2. Indefinitely trapped in a stationary solution
3. Casimir invariants
Classical particles interacting with a gravitational potential $V(r)$
\[ \Delta V(r) = Gm \delta(r) \]
In three dimensions, $V(r) = - \frac{Gm^2}{r}$
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- Regularization at short distance
- Particles can escape to infinity
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In one dimension, $V(x) = Gm|x|$ (Sheet model)

- No regularization at short distance
- Particles are confined
Classical particles interacting with a gravitational potential $V(r)$

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Mean-field limit: the total energy $E$, total mass $M$ and system size $L$ fixed, with $N \to \infty$, and thus $m \sim N^{-1}$,
$V(r) = -\frac{gm^2}{nr^n}$ where $g$ is the coupling, $m$ the particle mass
Modèle 1D

\[ V(r) = -\frac{gm^2}{nr^n} \] where \( g \) is the coupling, \( m \) the particle mass

Vlasov equation

\[ \partial_t f(x, v, t) + v \partial_x f(x, v, t) + \bar{a}(x, t) \partial_v f(x, v, t) = 0 \]

where \( \bar{a}(x) \) is the mean-field acceleration given by

\[ \bar{a}(x, t) = g \int sgn(x - x')|x - x'|^{-(n+1)} f(x', v', t) dx' dv' \]
Phase Portrait

\[ N = 1024 \]

100 realizations
\[ \tau = 0, 3.5, 10.5, 21.2, 31.8, 70 \]
\[ R0 = 0.01. \]
Virial ratio $R = \frac{2K}{nU}$
Correlation parameter $\phi_{11} = \frac{\langle |\mathbf{x}| \mathbf{v} \rangle}{\langle |\mathbf{x}| \rangle \langle |\mathbf{v}| \rangle} - 1$

Equilibrium $\phi_{11} = 0$. 
Outline

1 Introduction

2 Selfgravitating systems

3 Dissipative forces
   - Dissipative collisions
   - Friedmann–Lemaître–Robertson–Walker Metric
   - Modified Vlasov equation

4 Simulation

5 Generalized Self-gravitating models and Conclusion
Inelastic collisions

\[ (\vec{v}_2' - \vec{v}_1') = -c(\vec{v}_2 - \vec{v}_1) \]

where \( 0 < c \leq 1 \) is the coefficient of restitution
Inelastic collisions

\[ (\vec{v}'_2 - \vec{v}'_1) = -c(\vec{v}_2 - \vec{v}_1) \]

where \(0 < c \leq 1\) is the coefficient of restitution

\[ \Delta E = \frac{(1 - c^2)}{4} (\vec{v}_2 - \vec{v}_1)^2 \leq 0 \]
Granular gases in a nutshell

\[ N = 40000, \text{ MD 500 collisions/part.} \]

*Goldhirsch & Zanetti PRL (1993)*

- **Pseudo-Liouville equation**
  - Including dissipation
- **Boltzmann equation**
  - No equilibrium state
- **External heating:**
  - Velocity distribution is not a Gaussian
  - No equipartition
- **No heating:**
  - Homogeneous cooling state:
    - Haff law
    - Finite time before clustering
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  - Finite time before clustering
FLRW Metric

- Exact solution of the Einstein’s equation of general relativity
- Homogeneous, isotropic expanding or contracting universe

\[
\frac{d^2 r_i}{dt^2} + 2H \frac{dr_i}{dt} = - \frac{Gm}{a^3} \sum_j \frac{r_i - r_j}{|r_i - r_j|^3}
\]

where \( r_i \) is a comoving coordinate

\[
H(t) = \frac{1}{a} \frac{da}{dt},
\]

\( a(t) \) is the scale factor.
Change of variable $\tau = \int_0^t \frac{dt}{a^{3/2}}$

$$\frac{d^2 r_i}{d\tau^2} + \Gamma(\tau) \frac{d r_i}{d\tau} = -Gm \sum_j \frac{r_i - r_j}{|r_i - r_j|^3}$$

where $\Gamma(\tau)$ is a constant for a Einstein-de Sitter model and slightly dependent on time for a $\Lambda CDM$. 
Change of variable \( \tau = \int_0^t \frac{dt}{a^{3/2}} \)

\[
\frac{d^2 \mathbf{r}_i}{d\tau^2} + \Gamma(\tau) \frac{d\mathbf{r}_i}{d\tau} = -Gm \sum_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}
\]

where \( \Gamma(\tau) \) is a constant for a Einstein-de Sitter model and slightly dependent on time for a \( \Lambda CDM \).

In 1D,

\[
\frac{d^2 x_i}{d\tau^2} + \Gamma(\tau) \frac{dx_i}{d\tau} = -Gm \sum_j \text{sgn}(x_i - x_j)
\]
Modified Vlasov equation: \( V(r) = -\frac{gm^2}{nr^n} \)

\[
\partial_t f(x, v, t) + v \partial_x f(x, v, t) + \bar{a}(x, t) \partial_v f(x, v, t) = J_d[f]
\]

with

\[
\bar{a}(x, t) = g \int sgn(x - x') |x - x'|^{-(n+1)} f(x', v', t) dx' dv'
\]
Modified Vlasov equation:

\[
V(r) = -\frac{gm^2}{nr^n}
\]

\[
\partial_t f(x, v, t) + v \partial_x f(x, v, t) + \bar{a}(x, t) \partial_v f(x, v, t) = J_d[f]
\]

with

\[
\bar{a}(x, t) = g \int \text{sgn}(x - x')|x - x'|^{-(n+1)} f(x', v', t) dx' dv'
\]

Viscous damping forces:

\[
f = -m\eta \|v\|^{\alpha-1}v
\]

→ Damping operator added to the Vlasov equation

\[
J_{d,1} = \eta \partial_v (v^\alpha f(x, v, t))
\]
Inelastic collisions: $c$ coefficient of restitution

\[ J_{d,2} = \frac{N}{M} \int dv_1 |v - v_1| \left[ \frac{f(x, v^{**}, t) f(x, v_1^{**}, t)}{c^2} \right. \\
- \left. f(x, v, t) f(x, v_1, t) \right] \]
Inelastic collisions: $c$ coefficient of restitution

$$J_{d,2} = \frac{N}{M} \int dv_1 |v - v_1| \left[ \frac{f(x, v^{**}, t)f(x, v_1^{**}, t)}{c^2} - f(x, v, t)f(x, v_1, t) \right]$$

Quasi-elastic limit: $c \to 1$

$$\gamma = \frac{(1 - c)N}{2}$$

$$J_{d,2} = -\partial_v (a_1(x, v, t)f(x, v, t))$$

$$a_1(x, v, t) = \frac{\gamma}{M} \int du (u - v)|u - v|f(x, u, t)$$
Scaling solution

\[ f(x, v, t) = \frac{M}{\bar{x}(t)\bar{v}(t)} F \left( \frac{x}{\bar{x}(t)}, \frac{v}{\bar{v}(t)} \right) \]
Scaling solution

\[ f(x, v, t) = \frac{M}{\bar{x}(t)\bar{v}(t)} F \left( \frac{x}{\bar{x}(t)}, \frac{v}{\bar{v}(t)} \right) \]

Results

- Virial ratio \( R = 1 \)

\[
\bar{E}(t) = E_0 \begin{cases} 
(1 + \text{sgn}(\beta) \frac{t}{t_c})^{-\frac{2}{\beta}} & \beta \neq 0 \\
\exp\left(-\frac{2t}{t_c}\right) & \beta = 0
\end{cases}
\]

with \( \beta = \alpha - 1 \) for the VDM, \( \beta = (n + 2)/n \) for the ICM and \( t_c \) as the inverse of the strength of the dissipation.
Outline

1. Introduction
2. Selfgravitating systems
3. Dissipative forces
4. Simulation
   - Simulation Method and models
   - Energy decay
   - Virial ratio
   - Distribution function
   - Correlation parameter
   - Interrupted dissipation
5. Generalized Self-gravitating models and Conclusion

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Simulation method and models

- Event-driven method
- Heap Algorithm
- Round-off errors

Expanding Universe: viscous damping model (VDM)
Inelastic collisions: ICM

Pascal Viot (LPTMC)
4 May 2015 29 / 44
Simulation method and models

- Event-driven method
- Heap Algorithm
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Models
- Expanding Universe: viscous damping model (VDM)
- Inelastic collisions: ICM
Initial configurations: rectangular waterbag $N = 1024$
100 realizations $\tau = 0, 3.5, 10.5, 21.2, 31.8, 70$
$R_0 = 1. \text{ (initial virial ratio)}$
ICM: \( E = E_0(1 - \frac{t}{t_c})^\delta \) with \( \delta = 2.00 \pm 0.01; \)

Left: ICM, \( R_0 = 0.01 \). Inset: zoom on the shorter time evolution.
VDM: $E = E_0 \exp\left(-\frac{t}{t_c}\right)$

Right: VDM, semi-log plot of $\frac{E(t)}{E(0)}$ versus $\frac{t}{\tau_{mf}}$ $R_0 = 0.01$. 

$VDM\ \eta_{\tau_{mf}} = 0.037$

$VDM\ \eta_{\tau_{mf}} = 0.012$
Virial ratio $R = \frac{2K}{U}$
for the ICM with \( \gamma = 0.01 \) at different times \( t/\tau_{mf} = 10, 31, 53, 74, 95 \) and \( R_0 = 1 \) (upper panels) and for the VDM with \( \eta \tau_{mf} = 0.037 \) and \( R_0 = 0.01 \) (lower panels) at \( t/\tau_{mf} = 10, 20, \ldots, 100 \).
\begin{align*}
\phi_{11} & \text{ versus } \gamma t / \tau_{mf} \text{ for } \gamma = 0.01, 0.005, 0.001. \\
R_0 = 0.01 \text{ (3 upper curves), } R_0 = 0.1 \text{ (3 middle curves) and } R_0 = 1 \text{ (3 lower curves).}
\end{align*}
For the ICM model, the dissipation is interrupted after a finite time larger than $T_{mf}$.

The system is in a highly-correlated QSS!
(\vec{v}_2' - \vec{v}_1') = -c(\vec{v}_2 - \vec{v}_1)

where $0 < c \leq 1$ is the coefficient of restitution
Stochastic collision model

\[
\vec{v}_1 \rightarrow \vec{v}'_2 \rightarrow \vec{v}_2
\]

\[
\vec{v}'_1 \rightarrow \vec{v}'_1 \rightarrow \vec{v}'_2
\]

\[
(\vec{v}'_2 - \vec{v}'_1) = -c(\vec{v}_2 - \vec{v}_1)
\]

where \(0 < c \leq 1\) is the coefficient of restitution

\(c\) is now a discrete random variable with two states \(c_R\) and

\[
\tilde{c}_R = \sqrt{2 - c_R^2} \geq 1
\]
The stochastic collision model describes the interaction between two particles. Let $c$ be a discrete random variable with two states $c_R$ and $\tilde{c}_R = \sqrt{2 - c_R^2} \geq 1$.

The equation for the change in kinetic energy is:

$$\Delta K = \frac{(1 - c^2)}{4} (\vec{v}_2 - \vec{v}_1)^2$$

where $0 < c \leq 1$ is the coefficient of restitution.
Stochastic collision model

\[
\vec{v}_1 \rightarrow \vec{v}_2 \quad \text{and} \quad \vec{v}_1' \rightarrow \vec{v}_2'
\]

\[
(\vec{v}_2' - \vec{v}_1') = -c(\vec{v}_2 - \vec{v}_1)
\]

where \(0 < c \leq 1\) is the coefficient of restitution

\[
c \text{ is now a discrete random variable with two states } c_R \text{ and } \tilde{c}_R = \sqrt{2 - c_R^2} \geq 1
\]

\[
\Delta K = \frac{(1 - c^2)}{4}(\vec{v}_2 - \vec{v}_1)^2
\]

\[
\delta K_{superel} = -\delta K_{inel} \geq 0
\]
The volume of the phase space is conserved on average: After a collision the Jacobian is multiplied by $c$.

**Granular gas model**

- Steady states with large fluctuations
- Long tails of the velocity distribution
- Uniform states (no clustering!)
Stochastic collision model

Self-gravitating model

\[ \gamma_{SCM} = \frac{(1 - c_R)\sqrt{N}}{2} \]
Stochastic collision model

Scaling QSS in LRI with dissipation

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The steady states are QSS!
Conclusion

- QSS robustness: virialized states with strong correlations
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- Beyond the scaling solution?
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References

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