Domino tilings and other games

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1. Introduction: Domino tilings of an Aztec diamond

2. Domino tiling of a double Aztec diamond: $K_{-}$ and $L$-processes

3. Tacnode Process ($n \to \infty$)

4. The Edge-Tacnode ($n \to \infty$)

5. The Edge-Tacnode kernel and coupled GUE-matrices
1. Introduction: Domino tilings of an Aztec diamond:

4 types of coverings by domino’s!

Kasteleyn 1961, Elkies, Kuperberg, Larsen, Propp '92
Cohn, Elkies, Propp '96, Jockusch, Propp, Shor '98,
Johansson ’00, ’03, ’05
Johansson-Nordenstam ’06
A weight on domino’s and a probability on domino tilings:

- put the weight $0 < a \leq 1$ on vertical dominoes
- put the weight 1 on horizontal dominoes,

Define:

$$P(\text{domino tiling } T) = \frac{a \#\text{vertical domino’s in } T}{\sum_{\text{all possible tilings } T} a \#\text{vertical domino’s in } T}$$
In the large size limit for $0 < a \leq 1$:

(horizontal domino’s are more likely than vertical ones for $0 < a < 1$)
Arctic circle for $n = 50, \quad a = 1.$  
(courtesy Sunil Chhita)
Double Aztec diamond of type $n = 7$, with overlap $\rho = 3$
Coordinate system: $(\xi, \eta)$
Figure 4. Height function on domino’s and level-lines.
Random cover with domino’s: two groups of non-intersecting paths

level lines corresponding to height \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, n - \frac{1}{2}, \frac{n + 1}{2}, \ldots, 2n - \frac{1}{2} \)

A–level lines

B–level lines

A- level lines for this random surface

B- level lines for this random surface

Figure 4. Height function on domino’s and level-lines.
Two processes:

\[ K \text{-process} \]

\[
\text{dots = (lines}\{ z = 2k \}\cap \text{level curves})
\]

\[
\# \text{ dots per line = } n
\]

\[ L \text{-process} \]

\[
\text{dots = (lines}\{ \xi = 2k \}\cap \text{level curves})
\]

\[
\# \text{ dots per line varies from } n \text{ to } \rho
\]
For $n = 50$, overlap $\rho = 3$ and $a = 0.95$:

- \( K \)-process: red and green domino's
- \( L \)-process: blue and green domino's

level lines
For the $L$-process: Which probability are we computing?

\[
P\{\text{The interval } [k, \ell] \subset \{\xi = 2s\} \text{ contains no dot-particles}\}
\]

\[
= P\{\text{The random surface is flat along the interval } [k, \ell] \subset \{\xi = 2s\}\}
\]

\[
= P\{\text{Dominoes covering interval } [k, \ell] \subset \{\xi = 2s\} \text{ are red or yellow }\}
\]
Determinantal point processes $\mathbb{K}$ and $\mathbb{L}$

$$(-1)^{x-y}\mathbb{K}_{n,\nu}(z, x; z', y) = \tilde{\mathbb{K}}_{n,\nu}^{\text{oneAzt}}(z, x; z', y)$$

$$- \left\langle ((I - K_n)^{-1}A_{-y, z'}) (k), B_{-x, z} (k) \right\rangle \geq n - \rho + 1.$$ 

$$\frac{1}{1 + a^2} \mathbb{L}_{n, \rho}(\xi_1, \eta_1; \xi_2, \eta_2) = \tilde{\mathbb{L}}_{n, \rho}^{\text{oneAzt}}(\xi_1, \eta_1; \xi_2, \eta_2)$$

$$- \left\langle ((I - K_n)^{-1}A_{\xi_1, \eta_1}) (k), B_{\xi_2, \eta_2} (k) \right\rangle \geq n - \rho + 1.$$
Reminder:

$K_n^{\text{OneAzt}}(2r, x; 2s, y) = \frac{(-1)^{x-y}}{(2\pi i)^2} \oint_{\Gamma_0} du \oint_{\Gamma_{0,u,a}} dv \frac{v^{-x} (1 + au)^{n-s}(1 - a)^s}{v - u u^{1-y} (1 + av)^{n-r}(1 - a)^r}$

$- \mathbb{1}_{s > r} \oint_{\Gamma_{0,a}} dz \frac{z^{x-y}}{2\pi i z} \left( \frac{1 + a z}{1 - a} \right)^{s-r}$,
3. Tacnode Process \((n \to \infty)\).
Disordered region

Frozen North

Frozen South

Frozen West

Frozen East

\[
\sqrt{2} \quad \sqrt{2(1-r)}
\]

\[
(p - \frac{r}{2}, q + \frac{r}{2})
\]

\[
(p + \frac{r}{2}, q - \frac{r}{2})
\]

\[
(1 + \frac{r}{2}, 1 - \frac{r}{2})
\]

\[
\frac{r}{2}, 1 + \frac{r}{2}
\]

\[
\frac{r}{2}, 1 - \frac{r}{2}
\]

\[
(\frac{r}{2}, \frac{r}{2})
\]

\[
(\frac{r}{2}, -\frac{r}{2})
\]

\[
(-\frac{r}{2}, \frac{r}{2})
\]

\[
(-1 - \frac{r}{2}, \frac{r}{2})
\]

\[
(0, 0)
\]
New statistics near the point of tangency of the two ellipses, when \( n \to \infty \)

- Overlap of the two diamonds = \( \rho = n(1 - \frac{2}{a+a^{-1}}) \)

- Scaling \( z \sim \tau n^{-1/3} \) in the tangential direction, giving new time \( \tau \)
- Scaling \( x \sim \xi n^{-2/3} \) in the oblique direction, giving new space \( \xi \)

\[
K^{\text{tac}}(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{q_\sigma(\tau_1, \xi_1)}{q_\sigma(\tau_2, \xi_2)} K^{\text{AiryProcess}}(\tau_2, \sigma - \xi_2 + \tau_2^2; \tau_1, \sigma - \xi_1 + \tau_1^2) \\
+ 2^{1/3} \int_{\tilde{\sigma}}^{\infty} ((1 - K_{\text{Ai}})^{-1} A^{\tau_1}_{\xi_1-\sigma}) (\lambda) A^{-\tau_2}_{\xi_2-\sigma} (\lambda) d\lambda.
\]

(Adler-Johansson-PvM 2011)

$n=100$, $a=1/2$ and overlap=20
4. **The Edge Tacnode (a limit of the $\mathbb{L}$-process!)**: Two Aztec diamonds overlapping.

Letting the size $n \to \infty$ and keeping the overlap fixed.

![Diagram of Double Aztec diamond of size $n = 7$ and overlap $\rho = 3$.](image)

**Figure 1.** Double Aztec diamond of size $n = 7$ and overlap $\rho = 3$. 
Each time a line $\xi = 2k$ intersects an

1. $A$-level line: put a black dot in that square
2. $B$-level line: put a red dot in that square

\} interlacing!
Diamond size $n = 100$ and overlap $\rho = 4$ with weight $a = 1$

(Courtesy Sunil Chhita)
• (Discrete) interlacing:

Any such set of (discrete) blue and red dots, respecting the interlacing corresponds to a tiling with domino's!
• (Discrete) interlacing:

\[
\begin{align*}
\mathcal{L} & \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad \cdots \quad x_{\rho+\delta} \\
\mathcal{L}' & \quad y_1 \quad y_2 \quad y_3 \quad \cdots \quad \cdots \quad y_{\rho+\delta}
\end{align*}
\]

\[
\begin{array}{c}
\rho = 4 \\
\delta = 2
\end{array}
\]

Any such set of (discrete) blue and red dots, respecting the interlacing corresponds to a tiling with domino’s!
Given the position of the dots along the lines \( \mathcal{L} \) and \( \mathcal{L}' \), the blue and red dots in between are uniformly distributed, respecting the numbers and the interlacing.
• **Edge-Tacnode kernel, as a scaling limit of the \( L \)-kernel**

Take scaling limit:

\[
\begin{aligned}
&\text{size } n \to \infty, \\
\rho &= \text{fixed overlap of two diamonds}, \\
a &= 1 - \beta \sqrt{\frac{2}{n}} = \text{weight of vertical domino's, with } \beta \geq 0.
\end{aligned}
\]

Scaling of the coordinates \((\xi, \eta)\), \(\implies\) coordinates \((u, y) \in \mathbb{Z} \times \mathbb{R} :\)

\[
\begin{aligned}
\xi &= 2n - 2u, \\
\eta &= n + [y \sqrt{2n}] - 1.
\end{aligned}
\]

**LIMITING KERNEL: Edge-Tacnode Kernel**

\[
\lim_{n \to \infty} (-a)^{1/2}(\eta_1 - \eta_2)(-\sqrt{n/2})^{1/2}(\xi_1^2 - \xi_2^2)\mathbb{I}_{n,\rho}(\xi_1, \eta_1; \xi_2, \eta_2)\frac{\Delta \eta_2}{2} = K_{\beta, \rho}^{\text{EdgeTac}}(u_1, y_1; u_2, y_2) dy_2.
\]
Edge-Tacnode kernel, two parameters: \[
\begin{aligned}
\beta &= \text{speed at which } a \to 1 \\
\rho &= \text{overlap of two diamonds}
\end{aligned}
\]

\[
\frac{1}{2} K_{\beta,\rho}^{\text{Edge-Tac}}(u_1, y_1; u_2, y_2) = \frac{1}{2} K_{\text{minor}}(u_1, \beta - y_1; u_2, \beta - y_2)
\]

\[
+ \left\langle (1 - K^\beta(\lambda, \kappa))^{-1} A_{u_1}^{\beta; y_1 - \beta}(\kappa), B_{u_2}^{\beta; y_2 - \beta}(\lambda) \right\rangle_{\geq -\rho}
\]

where

\[
\frac{1}{2} \text{GUE-minor kernel} = \frac{1}{2} K_{\text{minor}}(n, x; n', x') := -\mathbb{1}_{n > n'} \frac{(2(x - x'))^{n-n'-1}}{(n-n'-1)!} \mathbb{1}_{x \geq x'}
\]

\[
+ \oint_{\Gamma} \frac{dz}{(2\pi i)^2} \oint_L \frac{dw}{w - z} e^{-z^2 + 2zx} w^{n'}
\]

where, given a GUE-matrix \(A\),

\[
\det(K_{\text{minor}}^{\beta}(n_i, x_i; n_j, x_j))_{1 \leq i, j \leq 2} dx_1 dx_2
\]

\[
= \mathbb{P} \left( \begin{array}{l}
\text{an eigenvalue of the } n_1\text{th minor } A^{(n_1)}(n_1) \in dx_1 \\
\text{an eigenvalue of the } n_2\text{th minor } A^{(n_2)}(n_2) \in dx_2
\end{array} \right)
\]
5. The Edge-Tacnode kernel and coupled GUE-matrices

The main message is the following equivalence:

\[
\begin{align*}
\left\{ \text{The statistics of the eigenvalues of} \right. & \left. \text{the consecutive minors } A^{(i)} \text{ and } B^{(i)} \right. \\
& \text{of the coupled GUE-matrices } A \text{ and } B \right\} \iff \left\{ \text{The statistics of the dot-particles in the rescaled double-Aztec diamonds} \right. \\
& \text{when its size } n \to \infty \right. \\
& \text{(Edge-tacnode Process)} \right\}
\end{align*}
\]

What coupling of GUE-matrices do we refer to? (Cond. probability)

\[
P(E) := \mathbb{P}^\text{GUE} \left( E \left| \bigcap_{1}^{\rho} \{ \max \text{ spec}(B^{(i)}) \leq \min \text{ spec}(A^{(\rho-i+1)}) , \text{ for } 1 \leq i \leq \rho \} \right. \right)
\]

Example: Two GUE-matrices \( A \) and \( B \) of size 6, conditioned by

\[
\rho = 4 \text{ conditions} \begin{cases} 
\text{spectrum (} A^{(1)} \text{)} < \text{spectrum (} B^{(4)} \text{)} \\
\text{spectrum (} A^{(2)} \text{)} < \text{spectrum (} B^{(3)} \text{)} \\
\text{spectrum (} A^{(3)} \text{)} < \text{spectrum (} B^{(2)} \text{)} \\
\text{spectrum (} A^{(4)} \text{)} < \text{spectrum (} B^{(1)} \text{)} 
\end{cases}
\]
• Extension of Baryshnikov result:

\[
\mathbb{P}\left( \bigcap_{1}^{n-1} \{ x^{(k)} \in dx^{(k)}, \ y^{(k)} \in dy^{(k)} \} \bigg| x^{(n)} = x, \ y^{(n)} = y \right) = \frac{d\mu_{x,y}}{\text{Vol}(C_{xy}^{(n)})},
\]
In the limit $n \to \infty$, for $u = -\delta$ (at a distance $\delta$ outside the overlap)

$$
\mathbb{P}(\text{particles in } dy_1, \ldots, dy_{\rho + \delta})
$$

$$
= C_{\rho, \delta} \Delta_{\rho + \delta}(y) \tilde{\Delta}_{\rho + \delta}^\beta(y) \prod_{1}^{\rho + \delta} \frac{e^{-(y_i - \beta)^2}}{\sqrt{\pi}} dy_1 \cdots dy_{\rho + \delta}
$$

where

$$
\tilde{\Delta}_{\rho + \delta}^\beta(y) := 
\begin{cases}
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
y_1 & y_2 & \cdots & y_{\rho + \delta} \\
\vdots & \vdots & \cdots & \vdots \\
y_{\delta - 1} & y_{\delta - 1} & \cdots & y_{\rho + \delta}
\end{vmatrix}
\end{cases}
$$

$$
\delta = \left\{ \begin{array}{l}
\text{distance to overlap} \\
\rho = \text{overlap}
\end{array} \right.
$$

where

$$
\Phi_{\delta}(\beta - y_k) \\
\Phi_{\delta + \rho - 1}(\beta - y_k)
$$

where

$$
\Phi_n(\eta) := \frac{2^n}{\sqrt{\pi n!}} \int_0^\infty \xi^n e^{-(\xi - \eta)^2} d\xi, \quad n \geq 0,
$$
Thank you!