A determinantal structure for the O’Connell-Yor polymer model

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(Based on a collaboration with T. Imamura)

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0. The model and result

2001 O’Connell Yor

Semi-discrete directed polymer in random media

$B_i, 1 \leq i \leq N$: independent Brownian motions

Energy of the polymer $\pi$

$$E[\pi] = B_1(s_1) + B_2(s_1, s_2) + \cdots + B_N(s_{N-1}, t)$$

with $B_j(s, t) = B_j(t) - B_j(s), \ j = 2, \cdots, N$ for $s < t$

Partition function

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta E[\pi]} ds_1 \cdots ds_{N-1}$$

$\beta = 1/k_B T$: inverse temperature
Zero-temperature limit

In the $T \to 0$ (or $\beta \to \infty$) limit

$$f_N(t) := \lim_{\beta \to \infty} F_N(t) = \max_{0<s_1<\cdots<s_{N-1}<t} E[\pi]$$

2001 Baryshnikov Connection to random matrix theory

$$\text{Prob} (f_N(1) \leq s) = \int_{(-\infty,s]^N} \prod_{j=1}^N dx_j \cdot P_{\text{GUE}}(x_1, \cdots, x_N),$$

$$P_{\text{GUE}}(x_1, \cdots, x_N) = \prod_{j=1}^N \frac{e^{-x_j^2/2}}{j! \sqrt{2\pi}} \cdot \prod_{1 \leq j < k \leq N} (x_k - x_j)^2$$

where $P_{\text{GUE}}(x_1, \cdots, x_N)$ is the probability density function of the eigenvalues in the Gaussian Unitary Ensemble (GUE)
A generalization to finite $\beta$

$$\mathbb{E} \left( e^{-\frac{e^{-\beta u Z_N(t)}}{\beta^2(N-1)}} \right) = \int_{\mathbb{R}^N} \prod_{j=1}^N dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t)$$

$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^N \frac{1}{j!} \prod_{1 \leq j < k \leq N} (x_k - x_j) \cdot \det (\psi_{k-1}(x_j; t))^{N}_{j,k=1}$$

where $f_F(x) = 1/(e^{\beta x} + 1)$ is the Fermi distribution function and

$$\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - \frac{w^2}{2}t} \frac{(iw)^k}{\Gamma(1 + iw/\beta)^N}$$

Proof by generalizing Warren’s process on the Gelfand-Tsetlin cone.
1. Universal distribution in surface growth

- Paper combustion, bacteria colony, crystal growth, etc
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Connections to integrable systems, representation theory, etc
Simulation models

Ex: ballistic deposition

Height fluctuation

$O(t^\beta), \beta = 1/3$

Universality: exponent and height distribution
**Totally ASEP (q = 0)**

**ASEP (asymmetric simple exclusion process)**

\[
\begin{array}{c|ccc|c|ccc|c}
q & p & q & p & q & p & q \\
\hline
\cdots & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \cdots \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

Mapping to a surface growth model (single step model)
TASEP with step i.c.

For the step (wedge for surface) initial condition for TASEP 2000 Johansson

$$\lim_{t \to \infty} \mathbb{P} \left[ \frac{h(0, t) - t/4}{-2^{-4/3} t^{1/3}} \leq s \right] = F_2(s)$$

where $F_2(s)$ is the GUE Tracy-Widom distribution

$$F_2(s) = \det(1 - P_s K_{Ai} P_s)_{L^2(\mathbb{R})}$$

where $P_s$: projection onto the interval $[s, \infty)$ and $K_{Ai}$ is the Airy kernel

$$K_{Ai}(x, y) = \int_{0}^{\infty} d\lambda \text{Ai}(x + \lambda) \text{Ai}(y + \lambda)$$
KPZ equation

\( h(x, t) \): height at position \( x \in \mathbb{R} \) and at time \( t \geq 0 \)

1986 Kardar Parisi Zhang

\[
\partial_t h(x, t) = \frac{1}{2} \lambda (\partial_x h(x, t))^2 + \nu \partial^2_x h(x, t) + \sqrt{D} \eta(x, t)
\]

where \( \eta \) is the Gaussian noise with mean 0 and covariance

\[
\langle \eta(x, t) \eta(x', t') \rangle = \delta(x - x') \delta(t - t')
\]

By a simple scaling we can and will do set \( \nu = \frac{1}{2}, \lambda = D = 1 \).

The KPZ equation now looks like

\[
\partial_t h(x, t) = \frac{1}{2} (\partial_x h(x, t))^2 + \frac{1}{2} \partial^2_x h(x, t) + \eta(x, t)
\]
If we set

\[ Z(x, t) = \exp (h(x, t)) \]

this quantity (formally) satisfies

\[
\frac{\partial}{\partial t} Z(x, t) = \frac{1}{2} \frac{\partial^2 Z(x, t)}{\partial x^2} + \eta(x, t) Z(x, t)
\]

This can be interpreted as a (random) partition function for a directed polymer in random environment \( \eta \).

The polymer from the origin: \( Z(x, 0) = \delta(x) = \lim_{\delta \to 0} c_\delta e^{-|x|/\delta} \) corresponds to narrow wedge for KPZ.
**KPZ equation for sharp wedge i.c.**

For the initial condition \( Z(x, 0) = \delta(x) \) (narrow wedge for KPZ)

\[
\lim_{t \to \infty} \mathbb{P} \left[ \frac{h(0, t) + \frac{t}{24}}{(t/2)^{1/3}} \leq s \right] = F_2(s)
\]

- The Tracy-Widom distribution appears universally in various surface growth models in the KPZ class.

- Experiment

- Technically there is a big difference between TASEP and KPZ equation. The structure for TASEP is well-understood but for KPZ equation, not really yet.
2. "Determinantal"s

Random matrix theory

GUE (Gaussian unitary ensemble): For a matrix $H$: $N \times N$ hermitian matrix

$$P(H)dH \propto e^{-\text{Tr}H^2}dH$$

Each independent matrix element is independent Gaussian.

Joint eigenvalue density

$$\frac{1}{Z} \prod_{i<j} (x_j - x_i)^2 \prod_i e^{-x_i^2}$$

This is written in the form of a product of two determinants using

$$\prod_{i<j} (x_j - x_i) = \det(x_i^{j-1})_{i,j=1}^N$$
From this follows

- All $m$ point correlation functions can be written as determinants using the "correlation kernel" $K(x, y)$.

- The largest eigenvalue distribution

$$
P_{x_{\text{max}} \leq s} = \frac{1}{Z} \int_{[-\infty, s]^N} \prod_{i<j} (x_j - x_i)^2 \prod_i e^{-x_i^2} \prod_i dx_i$$

can be written as a Fredholm determinant using the same kernel $K(x, y)$. 
In the limit of large matrix dimension, we get

$$\lim_{N \to \infty} \mathbb{P} \left[ \frac{x_{\max} - \sqrt{2N}}{2^{-1/2} N^{-1/6}} \leq s \right] = F_2(s) = \det (1 - P_s K_2 P_s)_{L^2(\mathbb{R})}$$

where $P_s$: projection onto $[s, \infty)$ and $K_2$ is the Airy kernel

$$K_2(x, y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda) \text{Ai}(y + \lambda)$$

$F_2(s)$ is known as the **GUE Tracy-Widom distribution**
Determinantal process

- The point process whose correlation functions are written in the form of determinants are called a determinantal process.

- Eigenvalues of the GUE is determinantal.

- This is based on the fact that the joint eigenvalue density can be written as a product of two determinants. The Fredholm determinant expression for the largest eigenvalue comes also from this.

- Once we have a measure in the form of a product of two determinants, there is an associated determinantal process and the Fredholm determinant appears naturally.
"TASEP is determinantal": Schur measure

- Finite $t$ formula

$$
\mathbb{P} \left[ \frac{h(0, t) - t/4}{-2^{-4/3}t^{1/3}} \leq s \right] = \frac{1}{Z} \int_{[0,s]^N} \prod_{i<j} (x_j - x_i)^2 \prod_i e^{-x_i} \prod_i dx_i
$$

As $t \to \infty$ we get $F_2(s)$.

- The proof is based on Robinson-Schensted-Knuth (RSK) correspondence. For a discrete TASEP with parameters $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_M)$ associated with the Schur measure for a partition $\lambda$

$$
\frac{1}{Z} s_\lambda(a)s_\lambda(b)
$$

The schur function $s_\lambda$ can be written as a single determinant (Jacobi-Trudi identity).
Dyson’s Brownian motion

In GUE, one can replace the Gaussian random variables by Brownian motions. The eigenvalues are now stochastic process, satisfying SDE

\[ dX_i = dB_i + \sum_{j \neq i} \frac{dt}{X_i - X_j} \]

known as the Dyson’s Brownian motion.
Warren’s Brownian motion in Gelfand-Tsetlin cone

Let \( Y(t) \) be the Dyson’s BM with \( m \) particles starting from the origin and let \( X(t) \) be a process with \((m + 1)\) components which are interlaced with those of \( Y \), i.e.,

\[
X_1(t) \leq Y_1(t) \leq X_2(t) \leq \ldots \leq Y_m(t) \leq X_{m+1}(t)
\]

and satisfies

\[
X_i(t) = x_i + \gamma_i(t) + \{L_i^-(t) - L_i^+(t)\}.
\]

Here \( \gamma_i \), \( 1 \leq i \leq m \) are indep. BM and \( L_i^\pm \) are local times.

Warren showed that the process \( X \) is distributed as a Dyson’s BM with \((m + 1)\) particles.
\(m = 3\) Dyson BM

\(m = 3, 4\) Dyson BM
Warren’s Brownian motion in Gelfand-Tsetlin cone

• Repeating the same procedure for $m = 1, 2, \ldots, n - 1$, one can construct a process $X^j_i$, $1 \leq j \leq n$, $1 \leq i \leq j$ in Gelfand-Tsetlin cone

• The marginal $X^i_i$, $1 \leq i \leq n$ is the diffusion limit of TASEP (reflective BMs). One can understand how the random matrix expression for TASEP appears.
The formula for KPZ equation

**Thm** *(2010 TS Spohn, Amir Corwin Quastel)*

For the initial condition \( Z(x, 0) = \delta(x) \) (narrow wedge for KPZ)

\[
\mathbb{E} \left[ e^{-e^{h(0,t)} + \frac{t}{24} - \gamma_t s}} \right] = \det(1 - K_{s,t})_{L^2(\mathbb{R}_+)}
\]

where \( \gamma_t = (t/2)^{1/3} \) and \( K_{s,t} \) is

\[
K_{s,t}(x, y) = \int_{-\infty}^{\infty} \text{d}\lambda \frac{\text{Ai}(x + \lambda)\text{Ai}(y + \lambda)}{e^{\gamma_t(s - \lambda)} + 1}
\]

The final result is written as a Fredholm determinant, but this was obtained without using a measure in the form of a product of two determinants (Bethe ansatz, Macdonald measure, replica, \( \delta \)-Bose gas).
3 O’Connell-Yor polymer

2001 O’Connell Yor

Semi-discrete directed polymer in random media

$B_i, 1 \leq i \leq N$: independent Brownian motions

Energy of the polymer $\pi$

$$E[\pi] = B_1(s_1) + B_2(s_1, s_2) + \cdots + B_N(s_{N-1}, t)$$

Partition function

$$Z_N(t) = \int_{0 < s_1 < \cdots < s_{N-1} < t} e^{\beta E[\pi]} ds_1 \cdots ds_{N-1}$$

$\beta = 1/k_B T$: inverse temperature

In a limit, this becomes the polymer related to KPZ equation.
**Whittaker measure: non-determinantal**

O’Connell discovered that the OY polymer is related to the quantum version of the Toda lattice, with Hamiltonian

\[ H = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N-1} e^{x_i - x_{i-1}} \]

and as a generalization of Schur measure appears a measure written as a product of the two Whittaker functions (which is the eigenfunction of the Toda Hamiltonian):

\[ \frac{1}{Z} \Psi_0(\beta x_1, \cdots, \beta x_N) \Psi_\mu(\beta x_1, \cdots, \beta x_N) \]

A determinant formula for \( \Psi \) is not known.
From this connection one can find a formula

$$\text{Prob} ( F_N(t) \leq s ) = \int_{(-\infty, s]^N} \prod_{j=1}^{N} dx_j \cdot m_t(x_1, \cdots, x_N)$$

where $m_t(x_1, \cdots, x_N) \prod_{j=1}^{N} dx_j$ is given by

$$m_t(x_1, \cdots, x_N) = \Psi_0(\beta x_1, \cdots, \beta x_N)$$

$$\times \int_{(i\mathbb{R})^N} d\lambda \cdot \Psi_{-\lambda}(\beta x_1, \cdots, \beta x_N) e^{\sum_{j=1}^{N} \chi_j^2 t/2} s_N(\lambda)$$

where $s_N(\lambda)$ is the Sklyanin measure

$$s_N(\lambda) = \frac{1}{(2\pi i)^N N!} \prod_{i<j} \Gamma(\lambda_i - \lambda_j)$$

Doing asymptotics using this expression has not been possible.
Macdonald measure and Fredholm determinant formula

Borodin, Corwin (2011) introduced the Macdonald measure

\[
\frac{1}{Z} P_\lambda(a) Q_\lambda(b)
\]

Here \( P_\lambda(a) \), \( Q_\lambda(b) \) are the Macdonald polynomials, which are also not known to be a determinant.

By using this, they found a formula for OY polymer

\[
\mathbb{E}[e^{-\beta u Z_N(t)}] = \det (1 + L)_{L^2(C_0)}
\]

where the kernel \( L(v, v'; t) \) is written as

\[
\frac{1}{2\pi i} \int_{i\mathbb{R}+\delta} dw \frac{\pi/\beta}{\sin(v' - w)/\beta} \frac{w^N e^{w(t^2/2-u)}}{v'^N e^{v'(t^2/2-u)}} \frac{1}{w - v} \frac{\Gamma(1 + v'/\beta)^N}{\Gamma(1 + w/\beta)^N}
\]

By using this expression, one can study asymptotics.
Our formula for finite $\beta$

$$
\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}}{\beta^2(N-1)}\right) = \int_{\mathbb{R}^N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t)
$$

$$
W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \prod_{1 \leq j < k \leq N} (x_k - x_j) \cdot \det(\psi_{k-1}(x_j; t))
$$

where $f_F(x) = 1/(e^{\beta x} + 1)$ is Fermi distribution function and

$$
\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx-w^2t/2} \frac{(iw)^k}{\Gamma(1+iw/\beta)^N}
$$

A formula in terms of a determinantal measure $W$ for finite temperature polymer.

From this one gets the Fredholm determinant by using standard techniques of random matrix theory and does asymptotics.
Proof of the formula

We start from a formula by O’Connell

\[
E \left( e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^2(N-1)}} \right) = \int_{(i\mathbb{R}-\epsilon)^N} \prod_{j=1}^{N} \frac{d\lambda_j}{\beta} e^{-u\lambda_j + \lambda_j^2t/2} \Gamma \left( -\frac{\lambda_j}{\beta} \right)^N s_N \left( \frac{\lambda}{\beta} \right)
\]

where \( \epsilon > 0 \).

This is a formula which is obtained by using Whittaker measure.

In this sense, we have not really found a determinant structure for the OY polymer itself.
An intermediate formula

\[
E \left( e^{-\beta u Z_N(t)} \right) = \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} dx_{\ell} f_F(x_{\ell} - u) \cdot \det (F_{jk}(x_j; t))_{j,k=1}^{N}
\]

with \((0 < \epsilon < \beta)\)

\[
F_{jk}(x; t) = \int_{i\mathbb{R} - \epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x + \lambda^2 t/2}}{\Gamma \left( \frac{\lambda}{\beta} + 1 \right)^N} \left( \frac{\pi}{\beta \cot \frac{\pi \lambda}{\beta}} \right)^{j-1} \lambda^{k-1}
\]

Now it is sufficient to prove the relation

\[
\int_{\mathbb{R}^N} \prod_{\ell=1}^{N} dt_{\ell} f_F(t_{\ell} - u) \cdot \det (F_{jk}(t_j; t))_{j,k=1}^{N}
\]

\[
= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t).
\]
A determinantal measure on $\mathbb{R}^{N(N+1)/2}$

For $x_k := (x_i^{(j)}, 1 \leq i \leq j \leq k) \in \mathbb{R}^{k(k+1)/2}$, we define a measure $R_u(x_N; t)dx_N$ with $R_u$ given by

$$\prod_{\ell=1}^{N} \frac{1}{\ell!} \det \left( f_i(x_j^{(\ell)} - x_{i-1}^{(\ell-1)}) \right)_{i,j=1}^{\ell} \cdot \det \left( F_{1i}(x_j^{(N)}; t) \right)_{i,j=1}^{N}$$

where $x_0^{(\ell-1)} = u, x_N = \prod_{j=1}^{N} \prod_{i=1}^{j} dx_i^{(j)}$,

$$f_i(x) = \begin{cases} f_F(x) := 1/(e^{\beta x} + 1) & i = 1, \\ f_B(x) := 1/(e^{\beta x} - 1) & i \geq 2. \end{cases}$$

and $F_{1i}(x; t)$ is given by $F_{ji}(x; t)$ with $j = 1$ in the previous slide.
Two ways of integrations

\[ \int_{\mathbb{R}^{N(N+1)/2}} dx_N R_u(x_N; t) \]
\[ = \int_{\mathbb{R}^N} \prod_{j=1}^N dx_1^{(j)} f_F \left( x_1^{(j)} - u \right) \cdot \det \left( F_{jk} \left( x_1^{(N-j+1)}; t \right) \right)_{j,k=1}^N \]

\[ = \int_{\mathbb{R}^N} \prod_{j=1}^N dx_j^{(N)} f_F \left( x_j^{(N)} - u \right) \cdot W \left( x_1^{(N)}, \cdots, x_N^{(N)}; t \right) \]
Lemma

1. For $\beta > 0$ and $a \in \mathbb{C}$ with $-\beta < \text{Re } a < 0$, we have
   \[
   \int_{-\infty}^{\infty} e^{-ax} f_B(x) dx = \frac{\pi}{\beta} \cot \frac{\pi}{\beta} a.
   \]

2. Let $G_0(x) = f_F(x)$ and
   \[
   G_j(x) = \int_{-\infty}^{\infty} dy f_B(x - y) G_{j-1}(y), \ j = 1, 2, \ldots.
   \]
   Then we have for $m = 0, 1, 2, \ldots$
   \[
   G_m(x) = f_F(x) \left( \frac{x^m}{m!} + p_{m-1}(x) \right),
   \]
   where $p_{-1}(x) = 0$ and $p_k(x) (k = 0, 1, 2, \ldots)$ is some $k$th order polynomial.
Dynamics of $X^N_i$

The density for the positions of $X^N_i, 1 \leq i \leq N$ satisfies

$$\frac{\partial}{\partial t} W(x_1, \cdots, x_N; t)$$

$$= \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} W(x_1, \cdots, x_N; t)$$

$$- \sum_{i=1}^{N} \left( \sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial}{\partial x_i} W(x_1, \cdots, x_N; t)$$

which is the equation for the Dyson’s Brownian motion.
Dynamics of $X^i_i$'s

The transition density of $X^i_i$'s

$$R(x_1, \cdots , x_N; t) = \text{det} \left( F_{jk}(x_k; t) \right)_{j,k=1}^{N}$$

satisfy

$$\frac{\partial}{\partial t} R(x_1, \cdots , x_N; t) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \cdot R(x_1, \cdots , x_N; t)$$

$$- \frac{\beta^2}{\pi^2} \int_{-\infty}^{\infty} dx_{j+1} \frac{e^{-\beta (x_{j+1}-x_j)}}{e^{\beta (x_{j+1}-x_j)} - 1} R(x_1, \cdots , x_N; t) = 0$$

As $\beta \to \infty$, the latter becomes

$$\partial_{x_i} R(x_1, \cdots , x_N; t) |_{x_{i+1}=x_i+0} = 0$$

which represents reflective interaction like TASEP.
Summary

- A determinantal formula for finite temperature O’Connell-Yor polymer
- Techniques from random matrix theory can readily be applied. Asymptotics possible.
- We started from a formula which is obtained from Whittaker measure. In this sense we have not found a determinantal structure for the OY polymer model itself.
- The proof is by generalizing Warren’s process on Gelfand-Tsetlin cone. There are interesting generalizations of Dyson’s Brownian motion and reflective Brownian motions.