Stein’s method and the many-worlds interpretation of quantum mechanics

Ian W. McKeague

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Joint with Bruce Levin
Many-interacting-worlds quantum theory

- Physicists have long tried to formulate quantum mechanics without wave functions.
- Recent proposal: quantum effects arise from the interaction of finitely many classical “worlds”.
- Wave function recovered (as a secondary object) from observations of particles in these worlds, without knowing the world from which any particular observation originates.
- Claimed that the stationary ground-state particle configuration is approximately Gaussian, as with Schrödinger’s equation.
Quantum Phenomena Modeled by Interactions between Many Classical Worlds

Michael J. W. Hall,1 Dirk-André Deckert,2 and Howard M. Wiseman1,*

1Centre for Quantum Dynamics, Griffith University, Brisbane, QLD 4111, Australia
2Department of Mathematics, University of California Davis, One Shields Avenue,
Davis, California 95616, USA
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We investigate whether quantum theory can be understood as the continuum limit of a mechanical theory, in which there is a huge, but finite, number of classical “worlds,” and quantum effects arise solely from a universal interaction between these worlds, without reference to any wave function. Here, a “world” means an entire universe with well-defined properties, determined by the classical configuration of its particles and fields. In our approach, each world evolves deterministically, probabilities arise due to ignorance as to which world a given observer occupies, and we argue that in the limit of infinitely many worlds the wave function can be recovered (as a secondary object) from the motion of these worlds. We introduce a simple model of such a “many interacting worlds” approach and show that it can reproduce some generic quantum phenomena—such as Ehrenfest’s theorem, wave packet spreading, barrier tunneling, and zero-point energy—as a direct consequence of mutual repulsion between worlds. Finally, we perform numerical simulations using our approach. We demonstrate, first, that it can be used to calculate quantum ground states, and second, that it is capable of reproducing, at least qualitatively, the double-slit interference phenomenon.
FIG. 2. Oscillator ground state for $N = 11$ worlds. The steps of the stepped blue curve occur at the values $q = \xi_1, \xi_2, \ldots, \xi_{11}$, corresponding to the stationary world configurations $x_1, x_2, \ldots, x_{11}$ via Eq. (41). The height of the step between $\xi_n$ and $\xi_{n+1}$ is $P_N(\xi_n) := N^{-1}(\xi_{n+1} - \xi_n)^{-1}$, which from Eq. (21) is expected to approximate the quantum ground-state distribution for a one-dimensional oscillator, $P_{\psi_0}(\xi) = (2\pi)^{-1/2}e^{-\xi^2/2}$, for large $N$. The latter distribution is plotted for comparison (smooth magenta curve). All quantities are dimensionless.
Quantum harmonic oscillator

Particle in a parabolic trap $V(x) = \frac{1}{2}m\omega^2x^2$, where $m =$ mass, $\omega =$ angular frequency.

Wave function $\psi(x, t)$ satisfies Schrödinger’s equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

$\hbar =$ reduced Planck constant (about $10^{-34}$).

Born rule: $|\psi(\cdot, t)|^2$ is the pdf of particle location at time $t$

Stationary ground state solution: $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 = \hbar/(2m\omega)$. 
“About your cat, Mr. Schrödinger—I have good news and bad news.”
Many-interacting-worlds harmonic oscillator

Stationary ground-state configuration \( \mathbf{x} = (x_1, \ldots, x_N) \): particle locations in \( N \) interacting “worlds,” \( x_1 > x_2 > \ldots > x_N \) under a repulsive interaction.

Hall et al. (2014) derived the recursion relation

\[
x_{n+1} = x_n - \frac{1}{x_1 + \ldots + x_n}.
\]

Observers have access to draws from the empirical distribution

\[
\mathbb{P}_N(A) = \frac{\#\{n: x_n \in A\}}{N}
\]

but do not know the world from which any observation originates.

**Problem:** Find a bound on the Wasserstein distance to \( \mathcal{N} = \mathcal{N}(0, 1) \):

\[
d_W(\mathbb{P}_N, \mathcal{N}) = \inf \{ E|X - Z| : X \sim \mathbb{P}_N, Z \sim \mathcal{N} \}.
\]
Stationary ground state solutions

In the ground state, the Hamiltonian only depends on locations (each particle has zero momentum):

$$H(x) = V(x) + U(x)$$

where

$$V(x) = \sum_{n=1}^{N} x_n^2$$

is the classical potential for $N$ (non-interacting) particles of equal mass in a parabolic trap, and

$$U(x) = \sum_{n=1}^{N} \left( \frac{1}{x_{n+1} - x_n} - \frac{1}{x_n - x_{n-1}} \right)^2$$

is the hypothesized “interworld” potential.
Setting $x_0 = \infty$ and $x_{N+1} = -\infty$, gives

$$(N - 1)^2 = \left[ \sum_{n=1}^{N-1} \frac{x_{n+1} - x_n}{x_{n+1} - x_n} \right]^2$$

$$= \left[ \sum_{n=1}^{N} \left( \frac{1}{x_{n+1} - x_n} - \frac{1}{x_n - x_{n-1}} \right) (x_n - \bar{x}_N) \right]^2$$

$\leq \sum_{n=1}^{N} \left( \frac{1}{x_{n+1} - x_n} - \frac{1}{x_n - x_{n-1}} \right)^2 \sum_{n=1}^{N} (x_n - \bar{x}_N)^2$

$\leq U(x) V(x)$

where the first inequality is Cauchy–Schwarz.
So $U \geq (N - 1)^2/V$, leading to

$$H = U + V \geq (N - 1)^2/V + V \geq 2(N - 1)$$

with the last inequality being equality for $V = N - 1$.

We conclude that $x$ is a ground state solution if and only if

$$x_1 + \ldots + x_N = 0, \quad x_1^2 + \ldots + x_N^2 = N - 1$$

and

$$x_n = \frac{\alpha}{x_{n+1} - x_n} - \frac{\alpha}{x_n - x_{n-1}}$$

for some constant $\alpha$. The sum of the rhs telescopes, so

$$x_{n+1} = x_n - \frac{1}{x_1 + \ldots + x_n}$$

by rearranging and setting $\alpha = 1$. 
Lemma (Existence)

Every zero-median solution \( x_1, \ldots, x_N \) of the recursion relation

\[
x_{n+1} = x_n - \frac{1}{x_1 + \ldots + x_n}
\]

defines the following properties:

(P1) Zero-mean: \( x_1 + \ldots + x_N = 0 \).

(P2) Variance-bound: \( x_1^2 + \ldots + x_N^2 = N - 1 \).

(P3) Symmetry: \( x_n = -x_{N+1-n} \) for \( n = 1, \ldots, N \).

Further, there exists a unique zero-median solution \( x_1, \ldots, x_N \) that maximizes \( x_1 \), in the sense that if \( \tilde{x}_1, \ldots, \tilde{x}_N \) is any other zero-median solution then \( \tilde{x}_1 < x_1 \), and this solution satisfies

(P4) Strictly decreasing: \( x_1 > \ldots > x_N \).
Example: $x_n$ starting from $x_1 = 3.1$
Theorem (IM & BL, 2015)

If \( \{x_n, n = 1, \ldots, N\} \) is a monotonic zero-median solution to the recursion relation

\[
x_{n+1} = x_n - \frac{1}{x_1 + \ldots + x_n}
\]

for each \( N \geq 1 \), then

\[
d_W(\mathbb{P}_N, \mathcal{N}) = O \left( \frac{\log \log N}{\log N} \right).
\]

A remarkably slow rate of convergence compared to usual CLTs! Caused by the range of \( \mathbb{P}_N \) expanding extremely slowly.
Proof based on zero bias approach

For every mean-zero unit-variance r.v. $X$, there exists a r.v. $X^*$ such that

$$E[f'(X^*)] = E[Xf(X)] \text{ for all } f \in \mathcal{F}$$

$X^*$ has pdf $p^*(x) = E[X1(X > x)]$.

Stein’s equation: $\mathcal{N}$ is the unique fixed point of the zero-bias transformation, so if $X$ is close to $X^*$, it should be close to $\mathcal{N}$.

Theorem (Goldstein, 2004)

Let $X \sim F$ be a mean-zero unit-variance r.v. and let $X^*$ have the $X$-zero-bias distribution, both defined on the same probability space. Then

$$d_W(F, \mathcal{N}) \leq 2E|X^* - X|.$$
Sketch of proof

First we show \( x_1 \to \infty \) as \( N \to \infty \) (where wlog \( x_1 > \ldots > x_N \)).

Argue by contradiction. Suppose there is a constant \( C \) such that \( x_1 < C \) for infinitely many \( N \). For such an \( N \), and denoting \( S_n = x_1 + \ldots + x_n \),

\[
x_n - x_{n+1} = \frac{1}{S_n} > \frac{1}{(nC)}
\]

\( n = 1, \ldots, m - 1 \), where \( m = (N + 1)/2 \). The median is \( x_m = 0 \) if \( N \) is odd, so

\[
x_1 = \sum_{n=1}^{m-1} (x_n - x_{n+1}) > \frac{1}{C} \sum_{n=1}^{m-1} \frac{1}{n} \sim \frac{\log N}{C}
\]

which gives the contradiction.
Sketch of proof (cont’d)

The same argument with $C$ replaced by $C \log N / \log \log N$ gives

$$\delta_N \equiv \max_{n=1,\ldots,N-1} |x_n - x_{n+1}| \leq 1/x_1 = O\left(\frac{\log \log N}{\log N}\right).$$

$P_N$-zero-bias pdf:

$$p_N^*(x) = S_n/(N - 1), \quad \text{for } x_{n+1} \leq x < x_n.$$  

By the recursion relation we can write it as

$$p_N^*(x) = \left[(N - 1)(x_n - x_{n+1})\right]^{-1}, \quad \text{for } x_{n+1} \leq x < x_n$$

so $X_N \sim P_N$ can be coupled to $X_N^* \sim p_N^*$ with $|X_N^* - X_N| \leq \delta_N \to 0$.

Aside: Slight rescaling needed since $X_N$ has variance $(N - 1)/N$, not 1.
The pdf $p_N^*$ compared with $\mathcal{N}(0, 1)$
Alternative approach

\[
p_N^*(x_{n+1}) - p_N^*(x_n) \quad \frac{x_{n+1}/(N - 1)}{x_{n+1} - x_n} = \frac{x_n - S_n^{-1}}{-S_n^{-1}(N - 1)} = \frac{1}{N - 1} - x_n p_N^*(x_n).
\]

Helly selection theorem used to show that \( p_N^*(x) \to p(x) \) uniformly in \( x \), where \( p(x) \) is differentiable and satisfies the linear first-order ODE

\[
p'(x) + xp(x) = 0.
\]

General solution is \( p(x) = c \varphi(x) \), where \( c \) is a constant and \( \varphi(x) \) is the standard normal density.

With a bit more work we can show \( c = 1 \).
Time-dependent solutions to Schrödinger’s equation

Nelson’s stochastic mechanics

Time-dependent ground state solution of the quantum harmonic oscillator represented in terms an OU process.

The complete family of solutions results from adding this OU process to solutions of the classical harmonic oscillator.

We showed $\mathbb{P}_N \xrightarrow{d} \mathcal{N}$, stationary distribution of the OU process.

Next we show that it is possible to construct a sequential bootstrap that gives the OU process as a continuum limit.
Sequential bootstrap

At each time-step a bootstrap sample of size $m$ from $\mathbb{P}_N$ is updated by replacing a randomly selected observations by a new draw from $\mathbb{P}_N$. Let $Y_k$ be the sample mean at step $k = 0, 1, \ldots$.

**Theorem (IM & BL, 2015)**

If $m = m_N \to \infty$ and $m = O(\sqrt{\log N})$, the rescaled process

$$\bar{X}_t = \sqrt{m} Y_{[mt]}, \quad t \geq 0$$

converges in distribution to the OU process

$$dX_t = -X_t \, dt + \sqrt{2} \, dW_t, \quad t \geq 0,$$

where $W_t$ is a standard Wiener process and $X_0 \sim \mathcal{N}$.
Sketch of proof

First suppose the bootstrap draws come from $N$ instead of $P_N$.

Then the sum $Z_k$ of the observations at step $k$ is an autoregressive time series satisfying

$$Z_k = (1 - \lambda)Z_{k-1} + \epsilon_k, \quad k = 1, 2, \ldots$$

where $Z_0 \sim N(0, m)$, $\lambda = \lambda_m = 1/m$, and the $\epsilon_k \sim N(0, 2 - \lambda)$ are iid.

Using a result of Phillips (1987) concerning first-order autoregressions with a root near unity, we can show that

$$\bar{Z}_t = Z[mt]/\sqrt{m} \xrightarrow{d} X_t \quad \text{in} \quad D[0, \infty)$$

as $m \to \infty$. 
As we have seen, the zero bias transformation of $X_N$ is coupled so that $|X_N^* - X_N| \leq \delta_N$, which implies that $X_N$ can be coupled with $Z \sim N$:

$$E|X_N - Z| \leq 2\delta_N = O\left(\frac{\log \log N}{\log N}\right).$$

Thus we can couple $\overline{X}_t$ and $\overline{Z}_t$ over $t \in [0, T]$ using $m([mT] + 1)$ iid copies of coupled $(X_N, Z)$:

$$E\left\{ \sup_{t \in [0, T]} |\overline{X}_t - \overline{Z}_t| \right\} \leq m([mT] + 1)E|X_N - Z|/\sqrt{m} = m^{3/2}O\left(\frac{\log \log N}{\log N}\right)$$

which tends to zero given $m = O(\sqrt{\log N})$. \qed
We have used Stein’s method to establish a claim of Hall et al. (2014) on the convergence of an empirical distribution in many-interacting-worlds quantum theory, as well as giving a rate

Sequential bootstrap on many-interacting-worlds converges to the time-dependent ground state solution of Schrödinger’s equation (for the harmonic oscillator)

The sequential bootstrap convergence rate might be obtained via Stein’s method for exchangeable pairs (cf. its use in urn problems)?

Open problems abound: higher-energy states? more complex potentials? chaotic solutions?
References


Thank you!