Stein’s Method and Convex Hulls

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New Directions in Stein’s Method, Singapore, May 2015
convex hull of random points

extreme points = vertices

We assume points are i.i.d. uniform on a convex body $K$. 
Statistics of random polytopes

\[ K_n = \text{convex hull of } n \text{ i.i.d. points hosted by the convex set } K. \]

\[ f_0(K_n) = \text{number of vertices of } K_n \]
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\[ \text{Vol}(K_n) = \text{volume of } K_n. \]
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Investigation has focussed on behavior of $f_{\ell}(K_n)$, $\ell \in \{0, \ldots, d - 1\}$, as input size $n \to \infty$.

Behavior of $f_{\ell}(K_n)$ is sensitive to geometry of the boundary of $K$. 

Expectation asymptotics
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- Reitzner (2005) \( \partial K \) of class \( C^2 \), \( \kappa := \) Gaussian curvature

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\lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \mathbb{E} f_\ell(K_n) = c_{d, \ell} \int_{\partial K} \kappa(x) \frac{1}{d+1} \, dx;
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(flag is a maximal chain of faces, each a sub-face of the next in the chain)

- \(K_n\) is convex hull of \(n\) i.i.d. standard normal r.v. on \(\mathbb{R}^d\):

\[
\lim_{n \to \infty} (\log n)^{-\frac{d-1}{2}} \mathbb{E} f_\ell(K_n) = g_{d,\ell}.
\]

Auffentranger + Schneider ('92), Baryshnikov + Vitale ('94).
\[ \ell \in \{0, \ldots, d - 1\}, \ d \geq 2, \ \text{then} \]

\[
\sup_{t \in \mathbb{R}} \left| P \left[ \frac{f_\ell(K_n) - \mathbb{E} f_\ell(K_n)}{\sqrt{\text{Var} f_\ell(K_n)}} \leq t \right] - \Phi(t) \right| \leq c(K) \epsilon(n) = o(1).
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Central limit theorems via Stein’s method

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- Gaussian input: Bárány and Vu (2007)
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- \text{Gaussian input: Bárány and Vu (2007)}
- K \text{ convex: Bárány and Reitzner (2008)}
- CLT for Vol(K_n).
Chen and Shao (2004): use concentration inequalities in Stein’s method to give (uniform and non-uniform) rates of normal convergence for sums of locally dependent r.v.

Their results applicable to sums \( \sum_{x \in P} \lambda \xi(x, P) \) whenever \( \xi \) is exponentially stabilizing and satisfies moment condition; cf. Penrose and Y. (2005, Stein’s Method and Applications) and Barbour and Xia (2006).

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Normal approximation under local dependence

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In some cases local dependence gives optimal rates: Last, Pecatti, and Schulte (2014).
\( \mathcal{P}_\lambda \) intensity \( \lambda \) PPP; let \( K_\lambda \) be convex hull of \( \mathcal{P}_\lambda \cap K \).

Can one write \( f_\ell(K_\lambda) \) as a sum \( \sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda) \) of stabilizing scores yielding a CLT via Chen-Shao methods and also yielding:

- \( \lim_{n \to \infty} \text{Var} f_\ell(K_n), \ell \in \{0, ..., d - 1\} \),
- scaling limit of the boundary of convex hull, and
- functional CLT for defect volume?

We answer these questions positively when:

(i) the input is uniform on \( K \), where either \( \partial K \) is of class \( C^2 \), or \( K \) is convex polytope, or
(ii) input consists of i.i.d. standard normal r.v. on \( \mathbb{R}^d \), \( d \geq 2 \).
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(ii) input consists of i.i.d. standard normal r.v. on \( \mathbb{R}^d, d \geq 2 \).
Normal approximation in the re-scaled picture

\( \mathcal{X} \): locally finite set in upper half-space \( \mathbb{R}^{d-1} \times \mathbb{R}^+ \).

\[ \xi_{\text{Par}}(x, \mathcal{X}) := \begin{cases} 1 & \text{if } x \text{ is extreme in } \mathcal{X} \text{ wrt parabolas} \\ 0 & \text{otherwise.} \end{cases} \]
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Put \(\mathcal{P}^{(\lambda)} := T^{(\lambda)}(\mathcal{P}_\lambda)\). The total number of extreme points in the convex hull of \(\mathcal{P}_\lambda\) is

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\( \xi_{\text{Par}}(x, \mathcal{P}^{(\lambda)}) \) is locally defined: its value is determined by input on a cylinder centered at \( x \) with radius having an exponentially decaying tail (non-trivial).

Thus by the results of either (i) Penrose and Y. ('05) or (ii) Barbour and Xia ('06), one deduces rates of normal convergence for the total number of extreme points ... and the total number of \( \ell \) faces in \( K_\lambda \).
**Main results (w. Pierre Calka):** $K = \text{unit ball in } \mathbb{R}^d$

**Thm (CLT)** Let $K$ be the unit ball, $\ell \in \{0, \ldots, d - 1\}$, then

$$\sup_{t \in \mathbb{R}} \left| P \left[ \frac{f_\ell(K_\lambda) - \mathbb{E} f_\ell(K_\lambda)}{\lambda^{(d-1)/2(d+1)}} \leq t \right] - P[N(0, \sigma^2_\ell) \leq t] \right| \leq \epsilon(\lambda) = o(1),$$

where $\sigma^2_\ell = \lim_{\lambda \to \infty} \lambda^{-\frac{d-1}{d+1}} \text{Var} f_\ell(K_\lambda)$. 

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**Thm (scaling limit for boundary)** For all $L \in (0, \infty)$, the interface $T^{(\lambda)}(\partial K_\lambda)$ converges in law as $\lambda \to \infty$ to parabolic festoon on $\mathcal{H}$ in $C([-L, L])$ equipped with the sup norm.
Main results: $K = $ unit ball in $\mathbb{R}^d$

$\mathcal{H}$: rate one PPP in upper half-space.

$$\xi(x, \mathcal{H}) := \xi_{\text{Par}}(x, \mathcal{H}) := \begin{cases} 1 & \text{if } x \text{ is extreme in } \mathcal{H} \text{ wrt parabolas} \\ 0 & \text{otherwise.} \end{cases}$$
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For all $w_1, w_2 \in \mathbb{R}^d$ put

$$c^\xi(w_1, w_2) := \mathbb{E} \xi(w_1, \mathcal{H} \cup \{w_2\})\xi(w_2, \mathcal{H} \cup \{w_1\}) - \mathbb{E} \xi(w_1, \mathcal{H})\mathbb{E} \xi(w_2, \mathcal{H})$$

and

$$V_{0,d} := \int_{-\infty}^{\infty} \mathbb{E} \xi((0, h), \mathcal{H})dh$$

$$+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^\xi((0, h), (v, h'))dh'dvdh.$$
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**Thm** (variance asymptotics when $K$ is unit ball) We have

$$\lim_{\lambda \to \infty} \lambda^{-\frac{d-1}{d+1}} \text{Var} f_0(K_\lambda) = c(d)V_{0,d}.$$
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**Thm** (functional CLT, $d = 2$) Define integrated defect volume

$$W_\lambda(\rho) := \int_0^\rho (1 - \partial K_\lambda(\theta))d\theta, \ \rho \in [0, 2\pi].$$

Then after centering and scaling, as $\lambda \to \infty$, $W_\lambda(\rho)$ converges in law to a Brownian motion in the space $C([0, 2\pi])$. 
Main results: $K$ a simple polytope

**Thm** (scaling limit for boundary) Put

$$d\mathcal{P} = e^{dh} dv dh.$$ 

For all $L \in (0, \infty)$ the interface $T^{(\lambda)}(\partial K_\lambda)$ converges in law as $\lambda \to \infty$ to hyperbolic festoon on $\mathcal{P}$ in $C([-L, L])$ equipped with the sup norm.
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**Thm** (CLT) Let $K$ be a simple polytope, $\ell \in \{0, ..., d-1\}$. Then

$$\sup_{t \in \mathbb{R}} \left| P \left[ \frac{f_\ell(K_\lambda) - \mathbb{E} f_\ell(K_\lambda)}{(\log \lambda)^{(d-1)/2}} \leq t \right] - P[N(0, \sigma_\ell^2) \leq t] \right| \leq \epsilon(\lambda) = o(1),$$

where $\sigma_\ell^2 = \lim_{\lambda \to \infty} (\log \lambda)^{-(d-1)} \text{Var} f_\ell(K_\lambda)$. 
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**Thm** (variance asymptotics when $K$ is simple polytope) We have

$$\lim_{\lambda \to \infty} (\log \lambda)^{-(d-1)} \text{Var} f_0(K_\lambda) = c(d) \text{ (number of vertices of } K) \ V_{0,d},$$

where $c(d)$ is explicit constant depending on $d$. 
Advantages to studying re-scaled picture

(i) spatial dependencies are easy to localize in re-scaled picture
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(ii) geometry of paraboloids and hyperboloids is actually easier to work with. Whether a point \((v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}^+\) is extreme depends only on the paraboloid (resp. hyperboloid) geometry inside a space-time cylinder (with axis through \(v\)) having a random radius \(R\), where \(R\) has exponentially decaying tails.

(iii) the space correlations decay exponentially fast wrt spatial distance. This leads to asymptotic independence and CLTs for e.g. the number of extreme points.

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Scaling limits for Gaussian polytopes

\[ R_n := \sqrt{2 \log n - \log(2 \cdot (2\pi)^d \cdot \log n)}. \]

Define scaling transform \( T^{(n)} : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \times \mathbb{R} \)

\[ T^{(n)}(x) := \left( R_n \exp^{-1} \frac{x}{|x|}, \ R_n^2 \left( 1 - \frac{|x|}{R_n} \right) \right), \quad x \in \mathbb{R}^d. \]
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$\mathcal{P} : \text{ Poisson pt process on } \mathbb{R}^{d-1} \times \mathbb{R} \text{ with intensity } d\mathcal{P}((v, h)) = e^h dh dv.$
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**Thm** (scaling limit for boundary of \( K_n \)) For all \( L \in (0, \infty) \), the interface \( T^{(n)}(\partial K_n) \) converges in law as \( n \to \infty \) to parabolic festoon on \( \mathcal{P} \) in \( \mathcal{C}(B_{d-1}(-L, L)) \) equipped with the sup norm.
THANK YOU