Finitary proof systems for Kozen's μ

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Modal μ-calculus

Syntax: $p \mid \overline{p} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \Box\varphi \mid \mathsf{x} \mid \mu\mathsf{x}\,\varphi \mid \nu\mathsf{x}\,\varphi$

Semantics: For Kripke structure $K = (W, \rightarrow, \lambda)$ and valuation $V : \mathrm{Var} \rightarrow 2^W$

$$\|p\|_V^K = \{u \in W \mid p \in \lambda(u)\} \qquad \|\varphi \wedge \psi\|_V^K = \|\varphi\|_V^K \cap \|\psi\|_V^K$$

$$\|\Diamond\varphi\|_V^K = \{u \in W \mid \exists v(u \rightarrow v \wedge v \in \|\varphi\|_V^K)\} \qquad \|\mathsf{x}\|_V^K = V(\mathsf{x})$$

and similarly for $\overline{p}$, $\vee$ and $\Box$.

$$\begin{aligned}\|\mu\mathsf{x}\,\varphi\|_V^K &= \text{least fixpoint of the function } X \mapsto \|\varphi\|_{V[\mathsf{x}\mapsto X]}^K. \\ &= \bigcap \left\{X \subseteq W \mid \|\varphi\|_{V[\mathsf{x}\mapsto X]}^K \subseteq X\right\} \\ \|\nu\mathsf{x}\,\varphi\|_V^K &= \text{greateast fixpoint of the function } X \mapsto \|\varphi\|_{V[\mathsf{x}\mapsto X]}^K. \\ &= \bigcup \left\{X \subseteq W \mid X \subseteq \|\varphi\|_{V[\mathsf{x}\mapsto X]}^K\right\}\end{aligned}$$

Examples: $\mu\mathsf{x}(\Diamond\mathsf{x} \vee p)$; $\nu\mathsf{x}(\Diamond\mathsf{x} \wedge p)$; $\nu\mathsf{x}\mu\mathsf{y}(\Diamond\mathsf{y} \vee (p \wedge \Diamond\mathsf{x}))$.

Duality: Define $\overline{\varphi}$ as the De Morgan dual of φ:

$$\overline{\mu\mathsf{x}\varphi(\mathsf{x})} = \nu\mathsf{x}\overline{\varphi}(\mathsf{x}) \qquad \|\overline{\varphi}\|_V^K = W \setminus \|\varphi\|_{\mathsf{y}\mapsto\overline{V(\mathsf{y})}}^K$$

Validity and proofs

Let φ be a closed formula.
Define

- $K \models \varphi$ iff $\|\varphi\|^K = W$ where $K = (W, \rightarrow, \lambda)$,
- $\models \varphi$ (φ is valid) iff $K \models \varphi$ for every K.

Theorem (Kozen 1983; Walukiewicz 2000)

For every closed φ, $\models \varphi$ iff $\mathsf{Koz} \vdash \varphi$.

Soundness: Proved by Kozen (1983).
Completeness: Kozen: aconjuntive fragment; Walukiewicz: full μ-calculus:

1. Completeness of disjunctive fragment: tableaux.
2. Provable equivalence between disjunctive and μ-formulæ: tableaux, games/automata.

‘almost always’ implies ‘infinitely often’

Consider the formula $\psi = \mu x \nu y \varphi(x, y) \to \nu y \mu x \varphi(x, y)$.

- ψ is valid — easy semantic argument.
- $\mathsf{Koz} \vdash \psi$ — non-trivial.

Questions:

1. Is there a more direct/constructive proof of completeness?
2. Is cut necessary?
3. Are there other natural sound and complete finitary proof systems?

Tableaux proofs

A tableau is a Fix-tree in which every infinite path contains a ν-thread.

Theorem (Niwinski, Walukiewicz 1996; Studer 2008; Friedman 2013)

For every closed guarded~~guarded~~ formula φ, $\models \varphi$ iff there exists a tableau for φ.

$$\dfrac{\dfrac{\vdots}{(\dagger)\ Y(\nu x Y)\,,\, X(\nu y X)}}{\nu x Y\,,\, X(\nu y X)}\,\nu_x \qquad \begin{array}{c}\vdots\\ Y(\nu x Y)\,,\,\nu y X \quad (\ddagger)\end{array}$$

$$- -$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\overline{\varphi}(\nu x Y, Y(\nu x Y))\,,\,\varphi(X(\nu y X), \nu y X)}}{Y(\nu x Y)\,,\, X(\nu y X)\ (\dagger)}\,\mu_y + \mu_x}{Y(\nu x Y)\,,\,\nu y X\ (\ddagger)}\,\nu_y}{\nu x \mu y \overline{\varphi}\,,\,\nu y \mu x \varphi}\,\nu_x$$

where $Y(x) = \mu y \overline{\varphi}(x, y)$ and $X(y) = \mu x \varphi(x, y)$.

Stirling's tableaux proofs with names

Fix a set of names for each variable: $N_x = \{x_0, x_1, \dots\}$ and $N = \bigcup_{x:\mathrm{Var}} N_x$.

An annotated sequent is an expression $a_0 \vdash \varphi_1^{a_1}, \dots, \varphi_n^{a_n}$ s.t. $a_0, \dots, a_n \in N^*$.

Definition

A Stirling proof is a finite tree built from rules $\mathsf{Fix}^{\mathsf{N}} + \mathsf{reset}_x + \nu_x + \mathsf{exp}$ s.t.

1. For every sequent $a \vdash \varphi_0^{a_0}, \dots, \varphi_k^{a_k}$ and every $i \leq k$, $a_i \sqsubset a$;
2. Every non-axiom leaf has the configuration

$$\left.\begin{array}{c} axb \vdash \Delta \\ \vdots \\ \dfrac{axb' \vdash \Pi}{axb' \vdash \Pi'}\,\mathsf{reset}_x \\ \vdots \\ axb \vdash \Delta \\ \vdots \end{array}\right\}$$

where:

2.1 x appears in every sequent between repetition

2.2 reset_x occurs between the two nodes.

We write $\mathsf{Stir} \vdash \Gamma$ if there exists a Stirling proof with root $\varepsilon \vdash \{\varphi^{\varepsilon} \mid \varphi \in \Gamma\}$.

Theorem (Stirling 2014)

Tableaux can be effectively transformed into Stirling proofs.

Stirling proofs: interpreting tableaux

$$
\dfrac{\dfrac{[xy \vdash Y(\nu\mathsf{x}Y)^{x}, X(\nu\mathsf{y}X)^{y}]^{\dagger}}{xyx' \vdash Y(\nu\mathsf{x}Y)^{xx'}, X(\nu\mathsf{y}X)^{y}}\ \mathsf{reset}_{x'}}{xy \vdash \nu\mathsf{x}Y^{x}, X(\nu\mathsf{y}X)^{y}}\ \nu_{x'}
\qquad
\dfrac{\dfrac{[xy \vdash Y(\nu\mathsf{x}Y)^{x}, X(\nu\mathsf{y}X)^{y}]^{\dagger}}{xyy' \vdash Y(\nu\mathsf{x}Y)^{x}, X(\nu\mathsf{y}X)^{yy'}}\ \mathsf{reset}_{y}}{xy \vdash Y(\nu\mathsf{x}Y)^{x}, \nu\mathsf{y}X^{y}}\ \nu_{y'}
$$

$$
- -
$$

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ xy \vdash \overline{\varphi}(\nu\mathsf{x}Y, Y(\nu\mathsf{x}Y))^{x}, \varphi(X(\nu\mathsf{y}X), \nu\mathsf{y}X)^{y}\end{array}}{xy \vdash Y(\nu\mathsf{x}Y)^{x}, X(\nu\mathsf{y}X)^{y}\ (\dagger)}\ \mu_{\mathsf{x}} + \mu_{\mathsf{y}}}{x \vdash Y(\nu\mathsf{x}Y)^{x}, \nu\mathsf{y}X^{\varepsilon}}\ \nu_{y}}{\varepsilon \vdash \nu\mathsf{x}Y^{\varepsilon}, \nu\mathsf{y}X^{\varepsilon}}\ \nu_{x}
$$

Pros:

1. Fintary proofs
2. Proofs constructed ‘semantically’

Cons:

1. Non-locality
2. Guessing resets

Circular proofs with ν-closure

Consider the rule

$$\dfrac{\begin{array}{c}[\vdash \Gamma, \nu \mathsf{x}\varphi(\mathsf{x})^{ax}]^{\dagger} \\ \vdots \\ \vdash \Gamma, \varphi(\nu \mathsf{x}\varphi)^{ax}\end{array}}{\vdash \Gamma, \nu \mathsf{x}\varphi^{a}}\,\mathsf{clo}_{\dagger}$$

where $a \leq x$, x is not in Γ or a.

Definition

$\mathsf{Clo} \vdash \Gamma$ iff there exists a finite tree built from rules $\mathsf{Fix}^{\mathsf{N}} + \nu_x + \mathsf{clo}$ satisfying

1. Every sequent has the form $\varepsilon \vdash \varphi_1^{a_1}, \ldots, \varphi_k^{a_k}$;
2. Every leaf is either an axiom or discharged by an application of clo:

Theorem

$\mathsf{Stir} \vdash \Gamma$ *implies* $\mathsf{Clo} \vdash \Gamma$.

Proof

1. Unravel Stirling proofs; 2. Delete resets; 3. Search for clo.

Example: Stirling to circular proofs (I)

$$
\dfrac{\dfrac{[xy \vdash Y(\nu \mathsf{x} Y)^{x}, X(\nu \mathsf{y} X)^{y}]^{\dagger}}{xyx' \vdash Y(\nu \mathsf{x} Y)^{xx'}, X(\nu \mathsf{y} X)^{y}}\ \mathsf{reset}_{x'}}{xy \vdash \nu \mathsf{x} Y^{x}, X(\nu \mathsf{y} X)^{y}}\ \nu_{x'}
\qquad
\dfrac{\dfrac{[xy \vdash Y(\nu \mathsf{x} Y)^{x}, X(\nu \mathsf{y} X)^{y}]^{\dagger}}{xyy' \vdash Y(\nu \mathsf{x} Y)^{x}, X(\nu \mathsf{y} X)^{yy'}}\ \mathsf{reset}_{y}}{xy \vdash Y(\nu \mathsf{x} Y)^{x}, \nu \mathsf{y} X^{y}}\ \nu_{y'}
$$

$$
\vdots
$$

$$
\dfrac{\dfrac{\dfrac{xy \vdash \overline{\varphi}(\nu \mathsf{x} Y, Y(\nu \mathsf{x} Y))^{x}, \varphi(X(\nu \mathsf{y} X), \nu \mathsf{y} X)^{y}}{xy \vdash Y(\nu \mathsf{x} Y)^{x}, X(\nu \mathsf{y} X)^{y}\ (\dagger)}\ \mu_{\mathsf{x}} + \mu_{\mathsf{y}}}{x \vdash Y(\nu \mathsf{x} Y)^{x}, \nu \mathsf{y} X^{\varepsilon}}\ \nu_{y}}{\varepsilon \vdash \nu \mathsf{x} Y^{\varepsilon}, \nu \mathsf{y} X^{\varepsilon}}\ \nu_{x}
$$

Example: Stirling to circular proofs (II)

$$\dfrac{\dfrac{\dfrac{\vdots}{Y(\nu\mathsf{x}Y)^x, X(\nu\mathsf{y}X)^y}}{Y(\nu\mathsf{x}Y)^{xx'}, X(\nu\mathsf{y}X)^y}\,\mathsf{reset}_{x'}}{\nu\mathsf{x}Y^x, X(\nu\mathsf{y}X)^y}\,\nu_{x'} \qquad \dfrac{\dfrac{\dfrac{\vdots}{Y(\nu\mathsf{x}Y)^x, X(\nu\mathsf{y}X)^y}}{Y(\nu\mathsf{x}Y)^{x}, X(\nu\mathsf{y}X)^{yy'}}\,\mathsf{reset}_{y}}{Y(\nu\mathsf{x}Y)^x, \nu\mathsf{y}X^y}\,\nu_{y'}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\overline{\varphi}(\nu\mathsf{x}Y, Y(\nu\mathsf{x}Y))^x, \varphi(X(\nu\mathsf{y}X), \nu\mathsf{y}X)^y}}{Y(\nu\mathsf{x}Y)^x, X(\nu\mathsf{y}X)^y}\,\mu^*}{Y(\nu\mathsf{x}Y)^{xx'}, X(\nu\mathsf{y}X)^y}\,\mathsf{reset}_{x'}}{\nu\mathsf{x}Y^x, X(\nu\mathsf{y}X)^y}\,\nu_{x'} \qquad \dfrac{\dfrac{\dfrac{\vdots}{Y(\nu\mathsf{x}Y)^x, X(\nu\mathsf{y}X)^y}}{Y(\nu\mathsf{x}Y)^{x}, X(\nu\mathsf{y}X)^{yy'}}\,\mathsf{reset}_{y}}{Y(\nu\mathsf{x}Y)^x, \nu\mathsf{y}X^y}\,\nu_{y'}$$

- - -

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\overline{\varphi}(\nu\mathsf{x}Y, Y(\nu\mathsf{x}Y))^x, \varphi(X(\nu\mathsf{y}X), \nu\mathsf{y}X)^y}}{Y(\nu\mathsf{x}Y)^x, X(\nu\mathsf{y}X)^y}\,\mu_\mathsf{x} + \mu_\mathsf{y}}{Y(\nu\mathsf{x}Y)^x, \nu\mathsf{y}X^\varepsilon}\,\nu_y}{\nu\mathsf{x}Y^\varepsilon, \nu\mathsf{y}X^\varepsilon}\,\nu_x$$

Example: Stirling to circular proofs (III)

$$
\dfrac{\dfrac{\vdots}{\dfrac{Y(\nu x Y)^{x x' x''}, X(\nu y X)^{y}}{Y(\nu x Y)^{x x' x''}, X(\nu y X)^{y}}\,\mathsf{reset}_{x''}}}{\nu x Y^{x x'}, X(\nu y X)^{y}}\,\mathsf{clo}_{x''}
\qquad
\dfrac{\dfrac{\vdots}{\dfrac{Y(\nu x Y)^{x x'}, X(\nu y X)^{y y'}}{Y(\nu x Y)^{x x'}, X(\nu y X)^{y y'}}\,\mathsf{reset}_{y}}}{Y(\nu x Y)^{x x'}, \nu y X^{y}}\,\mathsf{clo}_{y'}
$$

- -

$$
\dfrac{\dfrac{\vdots}{\overline{\varphi}(\nu x Y, Y(\nu x Y))^{x x'}, \varphi(X(\nu y X), \nu y X)^{y}}\,\mu^{*}}{\dfrac{\dfrac{Y(\nu x Y)^{x x'}, X(\nu y X)^{y}}{Y(\nu x Y)^{x x'}, X(\nu y X)^{y}}\,\mathsf{reset}_{x'}}{\nu x Y^{x}, X(\nu y X)^{y}}\,\mathsf{clo}_{x'}}
\qquad
\dfrac{\dfrac{\vdots}{\dfrac{Y(\nu x Y)^{x}, X(\nu y X)^{y y'}}{Y(\nu x Y)^{x}, X(\nu y X)^{y y'}}\,\mathsf{reset}_{y}}}{Y(\nu x Y)^{x}, \nu y X^{y}}\,\mathsf{clo}_{y'}
$$

- -

$$
\dfrac{\dfrac{\vdots}{\overline{\varphi}(\nu x Y, Y(\nu x Y))^{x}, \varphi(X(\nu y X), \nu y X)^{y}}\,\mu^{*}}{\dfrac{\dfrac{Y(\nu x Y)^{x}, X(\nu y X)^{y}}{Y(\nu x Y)^{x}, \nu y X^{\varepsilon}}\,\mathsf{clo}_{y}}{\nu x Y^{\varepsilon}, \nu y X^{\varepsilon}}\,\mathsf{clo}_{x}}
$$

Example: Stirling to circular proofs (IV)

$$
\dfrac{\dfrac{\cancel{Y(\nu x Y)^{x}, X(\nu y X)^{y}}}{\cancel{Y(\nu x Y)^{x x'}, X(\nu y X)^{y}}}\,\cancel{\mathsf{reset}_{x'}}}{[\nu x Y^{x x'}, X(\nu y X)^{y}]^{\ddagger}}\,\cancel{\mathsf{clo}_{x'}}
\qquad
\dfrac{\dfrac{\cancel{Y(\nu x Y)^{x x'}, X(\nu y X)^{y}}}{\cancel{Y(\nu x Y)^{x x'}, X(\nu y X)^{y y'}}}\,\cancel{\mathsf{reset}_{y}}}{[Y(\nu x Y)^{x x'}, \nu y X^{y}]^{\dagger}}\,\cancel{\mathsf{clo}_{y'}}
$$

$$\vdots$$

$$
\dfrac{\varphi(\nu x Y, Y(\nu x Y))^{x x'}, \overline{\varphi}(X(\nu y X), \nu y X)^{y}}{\dfrac{Y(\nu x Y)^{x x'}, X(\nu y X)^{y}}{\dfrac{Y(\nu x Y)^{x x'}, X(\nu y X)^{y}}{\nu x Y^{x}, X(\nu y X)^{y}}\,\mathsf{clo}_{x'}\,\ddagger}\,\cancel{\mathsf{reset}_{x'}}}\,\mu^{*}
\qquad
\dfrac{\dfrac{\begin{matrix}\vdots\\ \cancel{Y(\nu x Y)^{x}, X(\nu y X)^{y}}\end{matrix}}{\cancel{Y(\nu x Y)^{x}, X(\nu y X)^{y y'}}}\,\cancel{\mathsf{reset}_{y}}}{[Y(\nu x Y)^{x}, \nu y X^{y}]^{\dagger}}\,\cancel{\mathsf{clo}_{y'}}
$$

$$\vdots$$

$$
\dfrac{\varphi(\nu x Y, Y(\nu x Y))^{x}, \overline{\varphi}(X(\nu y X), \nu y X)^{y}}{\dfrac{Y(\nu x Y)^{x}, X(\nu y X)^{y}}{\dfrac{Y(\nu x Y)^{x}, \nu y X^{\varepsilon}}{\nu x Y^{\varepsilon}, \nu y X^{\varepsilon}}\,\mathsf{clo}_{x}}\,\mathsf{clo}_{y}\,\dagger}\,\mu^{*}
$$

Example: Stirling to circular proofs (V)

A proof in Clo:

$$[\nu \mathsf{x} Y^{xx'}, X(\nu \mathsf{y} X)^{y}]^{\ddagger} \qquad \dfrac{[Y(\nu \mathsf{x} Y)^{x}, X(\nu \mathsf{y} X)^{y}]^{\dagger}}{Y(\nu \mathsf{x} Y)^{xx'}, \nu \mathsf{y} X^{y}}\ \mathsf{exp}$$

$$\vdots$$

$$\dfrac{\dfrac{\overline{\varphi}(\nu \mathsf{x} Y, Y(\nu \mathsf{x} Y))^{xx'}, \varphi(X(\nu \mathsf{y} X), \nu \mathsf{y} X)^{y}}{Y(\nu \mathsf{x} Y)^{xx'}, X(\nu \mathsf{y} X)^{y}}\ \mu^{*}}{\nu \mathsf{x} Y^{x}, X(\nu \mathsf{y} X)^{y}}\ \mathsf{clo}_{\ddagger} \qquad [Y(\nu \mathsf{x} Y)^{x}, \nu \mathsf{y} X^{y}]^{\dagger}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\overline{\varphi}(\nu \mathsf{x} Y, Y(\nu \mathsf{x} Y))^{x}, \varphi(X(\nu \mathsf{y} X), \nu \mathsf{y} X)^{y}}{Y(\nu \mathsf{x} Y)^{x}, X(\nu \mathsf{y} X)^{y}}\ \mu^{*}}{Y(\nu \mathsf{x} Y)^{x}, \nu \mathsf{y} X^{\varepsilon}}\ \mathsf{clo}_{\dagger}}{\nu \mathsf{x} Y^{\varepsilon}, \nu \mathsf{y} X^{\varepsilon}}\ \mathsf{clo}$$

Taking stock

We have shown

$$\varphi \text{ valid} \Rightarrow \varphi \text{ has a tableau} \Rightarrow \mathsf{Stir} \vdash \varphi \Rightarrow \mathsf{Clo} \vdash \varphi$$

Corollary

Circular proofs with ν-closure are complete for the modal μ-calculus.

Can we get closer to a sequent calculus?

- no discharge rules.
- no annotations.

Eliminating assumptions

Let $Z = \nu z \psi(z)$.

$$(\nu z\psi)^{az_1\cdots z_i} := \nu z(\overline{\Gamma_i} \vee \cdots \vee \overline{\Gamma_1} \vee \psi^a)$$

$$\begin{array}{c}
[\Gamma_1, Z^{az_1}] \quad \ldots \quad [\Gamma_k, Z^{az_1\cdots z_k}] \\
\vdots \qquad \vdots \qquad \vdots \\
\text{- - - - - - - - - - - - - - -} \\
\vdots \\
\dfrac{\Gamma_k, \psi(Z)^{az_1\cdots z_k}}{\Gamma_k, Z^{az_1\cdots z_{k-1}}}\ \mathsf{clo}_{z_k} \\
\vdots \\
\dfrac{\Gamma_i, \psi(Z)^{az_1\cdots z_i}}{\Gamma_i, Z^{az_1\cdots z_{i-1}}}\ \mathsf{clo}_{z_i} \\
\vdots \\
\dfrac{\Gamma_1, \psi(Z)^{az_1}}{\Gamma_1, Z^{a}}\ \mathsf{clo}_{z_1} \\
\vdots
\end{array}
\quad \mapsto \quad
\begin{array}{c}
\vdots \\
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_i, \psi^a(\nu z.\ \overline{\Gamma_i} \vee \cdots \vee \overline{\Gamma_1} \vee \psi^a)}{\Gamma_i, \psi^a(\overline{\Gamma_i} \vee \nu z.\ \overline{\Gamma_i} \vee \cdots \vee \overline{\Gamma_1} \vee \psi^a)}\ \vee^d}{\Gamma_i, \nu z.\ \overline{\Gamma_{i-1}} \vee \cdots \vee \overline{\Gamma_1} \vee \psi^a(\overline{\Gamma_i} \vee z))}\ \vee + \nu_z}{\Gamma_i, \nu y \nu z.\ \overline{\Gamma_{i-1}} \vee \cdots \vee \overline{\Gamma_1} \vee \psi^a(y \vee z))}\ \mathsf{ind}}{\Gamma_i, \nu z.\ \overline{\Gamma_{i-1}} \vee \cdots \vee \overline{\Gamma_1} \vee \psi^a}\ \mathsf{con} \\
\vdots
\end{array}$$

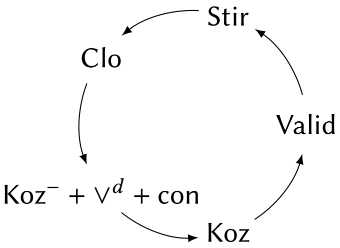
Theorem

$\mathsf{Clo} \vdash \Gamma \Rightarrow \mathsf{Koz}^- + \mathsf{con} + \vee^d \vdash \Gamma$.

Theorem

$\mathsf{Koz}^- + \mathsf{con} + \vee^d \vdash \Gamma$ *implies* $\mathsf{Koz} \vdash \Gamma$.

Summary



1. We introduce two sound and complete cut-free proof systems;
2. Doing so we provide a new proof of completeness for Koz yielding a procedure for obtaining proofs from tableaux;
3. Koz^{-} is complete iff $\vee^{d}$ and con are admissible.

Proving ‘almost always’ implies ‘infinitely often’

$$\nu\mathsf{x}Y', X(\nu\mathsf{y}X') \qquad \dfrac{Y(\nu\mathsf{x}Y), \nu\mathsf{y}X'}{Y(\nu\mathsf{x}Y'), \nu\mathsf{y}X'}\,\vee^d$$

$$\text{- -}$$

$$\vdots$$

$$\dfrac{\overline{\varphi}(\nu\mathsf{x}Y', Y(\nu\mathsf{x}Y')), \varphi(X(\nu\mathsf{y}X'), \nu\mathsf{y}X')}{Y(\nu\mathsf{x}Y'), X(\nu\mathsf{y}X')}\,\mu_\mathsf{x} + \mu_\mathsf{y}$$

$$(\nu_\mathsf{x} + \vee^d + \mathsf{ind} + \mathsf{con})\ \dfrac{Y(\nu\mathsf{x}Y'), X(\nu\mathsf{y}X')}{\nu\mathsf{x}Y, X(\nu\mathsf{y}X')} \qquad Y(\nu\mathsf{x}Y), \nu\mathsf{y}X'$$

$$\text{- -}$$

$$\vdots$$

$$\dfrac{\overline{\varphi}(\nu\mathsf{x}Y, Y(\nu\mathsf{x}Y)), \varphi(X(\nu\mathsf{y}X'), \nu\mathsf{y}X')}{Y(\nu\mathsf{x}Y), X(\nu\mathsf{y}X')}\,\mu_\mathsf{x} + \mu_\mathsf{y}$$

$$(\nu_\mathsf{y} + \vee^d + \mathsf{ind} + \mathsf{con})\ \dfrac{Y(\nu\mathsf{x}Y), X(\nu\mathsf{y}X')}{\dfrac{Y(\nu\mathsf{x}Y), \nu\mathsf{y}X}{\nu\mathsf{x}\mu\mathsf{y}\overline{\varphi}, \nu\mathsf{y}\mu\mathsf{x}\varphi}\,\nu_\mathsf{x}}$$

where

$$Y(\mathsf{x}) = \mu\mathsf{y}\overline{\varphi}(\mathsf{x}, \mathsf{y}) \qquad Y' = \overline{X(\mu\mathsf{y}X')} \vee Y$$

$$X(\mathsf{y}) = \mu\mathsf{x}\varphi(\mathsf{x}, \mathsf{y}) \qquad X' = \overline{Y(\mu\mathsf{x}Y)} \vee X$$