How to decide qualitative properties for higher-order program with Nature?

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The General Context

Nature + Program + Environment

System Specification: Every (valid) job gets eventually printed except in very unlikely executions? Proposing tractable and robust definitions to capture negligible set of executions is at the heart of this work.
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Games With Nature: Basic Definitions

Three players
- Éloïse: Standing for the program ➔ controllable.
- Abélard: Standing for the environment ➔ uncontrollable + possibly hostile.
- Nature: Standing for nature ➔ uncontrollable + unpredictable.
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- An **arena** \(A\): A directed graph \(G = (V, E)\) whose vertices have been partitioned among the players.
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- An **initial vertex** $v_0 \in V$.
- A pebble is put on $v_0$ and is moved along edges by the players.
- A **winning condition**: A (say Borel) subset $\Omega$ of $V^\omega$.
- A play is an infinite path $\lambda \in V^\omega$: it is won by Éloïse if $\lambda \in \Omega$ and otherwise it is won by Abélard.
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**Popular winning conditions:**

- Reachability condition: Éloïse wins if a final state is eventually reached, i.e. $\Omega = V^* F V^\omega$.
- Büchi condition: Éloïse wins if final states are infinitely often visited, i.e. $\Omega = \cap_{i \geq 0} V^i V^* F V^\omega$.
- Parity condition: Every vertex gets a colour from a finite set; Éloïse wins if the smallest colour infinitely often visited is even.
Handling Higher-Order Programs

The general roadmap:

1. Abstract the system as a scheme.
2. Build an equivalent collapsible pushdown automaton.
3. Reflect who does what using the control states of the CPDA (a state is controlled by Éloïse, Abélard or Nature).

Special case:

1. Closed system (i.e. no environment): no Abélard.

Possible extra features: imperfect information, e.g. when having public/private variables.
**Games With Nature: Strategies**

**Strategies**

- A strategy for Éloïse is a map $\varphi_E : V^* V_E \rightarrow V$.
- A strategy for Abélard is a map $\varphi_A : V^* V_A \rightarrow V$.
- Nature is fully non-deterministic hence, does not use strategies.

\[ \begin{array}{c}
\text{Given } (\varphi_E; \varphi_A) \text{ one gets a set } \text{Outcomes } \varphi_E;\varphi_A v_0 \\
\text{of all plays where } \text{Éloïse/Abélard respects her/his strategy.}
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**Remark.** One can think of Outcomes $\varphi_E;\varphi_A v_0$ as the set of branches of the tree of partial plays where the players respect their strategies.
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Given $(\varphi_E, \varphi_A)$ one gets a set $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$ of all plays where Éloïse/Abélard respects her/his strategy.

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Games Without Nature (aka 2-Player Games)

Things work fine:

- For Borel winning conditions $\Omega$ the games are *determined*: Either Éloïse or Abélard has a winning strategy:

$$\exists \varphi_E \forall \varphi_A \text{ Outcomes}_{v_0}^{\varphi_E, \varphi_A} \subseteq \Omega \quad \text{or} \quad \exists \varphi_A \forall \varphi_E \text{ Outcomes}_{v_0}^{\varphi_E, \varphi_A} \cap \Omega = \emptyset$$

- For parity winning conditions, winning strategies can be chosen positional (i.e. as map from $V \to V$).

- For $\omega$-regular winning conditions one can decide who has a winning strategy and compute winning strategies for a large class of graphs: finite ones, pushdown graphs, collapsible pushdown graphs.

Consequence for program verification:

- One can do model-checking (and even program synthesis) for higher-order programs against $\omega$-regular specifications
How good is strategy $\varphi_E$ against strategy $\varphi_A$?

$\Rightarrow$ Look how “small” is the set of loosing plays for Eloise.

How good is strategy $\varphi_E$?

$\Rightarrow$ Look how good is $\varphi_E$ in the worse case (i.e. against the best counter-strategy $\varphi_A$ of Abélard).
The Probabilistic Setting (Stochastic Games)

- Nature is coming with a probabilistic semantics.

- A pair of strategies \((\varphi_E, \varphi_A)\) induces a probability measure 
  \(\mu_{v_0}^{\varphi_E, \varphi_A}\) on Outcomes \(\varphi_E, \varphi_A\) on Outcomes.

- \(\text{Val}_G(\varphi_E) = \inf\{\mu_{v_0}^{\varphi_E, \varphi_A}(\Omega) \mid \varphi_A \text{ Abélard strategy}\}\).

- \(\text{Val}(G) = \sup\{\text{Val}_G(\varphi_E) \mid \varphi_E \text{ Éloïse strategy}\}\).

- Typical (qualitative) questions: does \(\text{Val}(G) = 1\)? If so is there an optimal strategy, \(\text{i.e.} \ a \varphi_E \text{ such that } \text{Val}_G(\varphi_E) = 1\)?
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→ Well-understood (and decidable) when arena is finite + \(\Omega\) is \(\omega\)-regular + perfect information. In other situations: quickly becomes undecidable.
(Un)Decidability landscape:

- When Nature is not involved parity games on CPDA graphs are decidable [HMOS08] + finite effective presentation of the winning region [BCOS10] + effective winning strategies [CS12].

- When Nature is involved and even if Abélard is not: undecidable even for reachability (except under strong restrictions) [EY05].
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End of the story?
The Probabilistic Setting for Programs With (Higher-Order)-Recursion

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Not really... One should just fine alternative ways to define negligible sets of executions!
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- \( \text{CardLeak}(\varphi_E) = \sup \{ \text{Card}(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \setminus \Omega) \mid \varphi_A \text{ Abélard’s strat} \} \).
- \( \text{LeakVal}(G) = \inf \{ \text{CardLeak}(\varphi_E) \mid \varphi_E \text{ strategy of Éloïse} \} \).
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- \( \text{LeakVal}(G) = \inf \{ \text{CardLeak}(\varphi_E) \mid \varphi_E \text{ strategy of Éloïse} \} \).

**Proposition.** \( \text{LeakVal}(G) \in \mathbb{N} \cup \{ \mathbb{N}_0, 2^{\mathbb{N}_0} \} \).

**Proposition.** Optimal strategies for Éloïse always exist.
Using Topology to Define Negligible Sets

The set of branches in a tree as a topological space.

- Basic open sets: the cones.
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- A set of branches $B$ is **nowhere dense** if for all node $u$, there exists a descendant $v$ of $u$ such that no branch of $B$ goes through $v$.

- A set of branches is **meagre** if it is the countable union of nowhere dense sets.
Using Topology to Define Negligible Sets

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→ Go back to a non-deterministic semantics for Nature and look whether the set of loosing plays is meager.

**Definition.** $\varphi_E$ is topologically good if and only if for any strategy $\varphi_A$ of Abélard the set $\text{Outcomes}_{\varphi_E, \varphi_A} \setminus \Omega$ is meager.
What Kind of Results Did We Prove?

We want to decide, for a given game with nature $G$, the following problems:

(1) Does one has $\text{LeakVal}(G) \leq \aleph_0$?

(2) Has Éloïse a topologically good strategy?
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We give transformations that associate with any game with Nature $G = (A, \Omega, v_0)$ a game **without** Nature $\hat{G} = (\hat{A}, \hat{\Omega}, \hat{v}_0)$ on which question (1)/(2) is restated as whether Éloïse has a winning strategy.

- $\hat{A}$: $A$ + gadgets
- $\hat{\Omega}$: Simple Boolean combination of $\Omega$ with an $\omega$-regular condition
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Extra hypothesis:

- $\text{LeakVal}(G) \leq \aleph_0$? No extra hypothesis on $G$
- Existence of topo. good strategies: Éloïse + Nature only
The gadget:

Game $\mathcal{G}$

Game $\widehat{\mathcal{G}}$

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The gadget:

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**Theorem.** \( \text{LeakVal}(G) \leq \aleph_0 \) iff Éloïse has a winning strategy in \( \hat{G} \).

**Proposition.** If \( G \) is a CPDA game then so is \( \hat{G} \).
Topo. Good Strategies | Perfect-Inf. Setting

A set of nodes $W \subseteq \{0, 1\}^*$ is dense if for every node $u \in \{0, 1\}^*$ there exists $v \in W$ with $u \sqsubseteq v$. With such a $W$ associate the set $\mathcal{B}(W)$ of branches that have infinitely many prefixes in $W$.

**Lemma.** A Borel set $B$ of branches is large if and only if there exists $W$ dense such that $\mathcal{B}(W) \subseteq B$. 
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A direction mapping is a map $d : \{0, 1\}^* \rightarrow \{0, 1\}$. It points to $W$ if following the directions indicated by the mapping starting from any node one ends up in a node in $W$.

**Lemma.** A set of nodes $W$ is dense if and only if there exists a direction mapping that points to $W$. 

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Starting from an Éloïse-Nature game $G$ we build an Éloïse-Abélard game $\hat{G}$ where they mimics a play in $G$ and where

- Éloïse provides a direction mapping to a set of nodes $W$
- Abélard simulates the moves of Nature and try to either prove that $W$ is not dense or that there is a loosing play in $B(W)$.

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Consequences for Higher-Order Programs

Higher-order systems composed of: Program + Environment + Nature:

- For $\omega$-regular specification everything (local/global model-checking + synthesis) works fine w.r.t the counting semantics.
- For $\omega$-regular specification with no environment everything works fine w.r.t the topological semantics.
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Is this relevant? Yes because...

- For finite MDP with $\omega$-regular objective the probabilistic and the topological semantics coincide.
The Imperfect Information Setting

The Setting

- Set of vertices comes with an equivalence class $\sim$
- Éloïse only observes the equivalence class of the current states
- Éloïse plays actions and Nature resolves the non-determinism
- Abélard is perfectly informed: he plays as previously
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- $\hat{\Omega}$: Simple Boolean combination of $\Omega$ and an $\omega$-regular condition
- Extra hypothesis:
  - $\text{LeakVal}(G) \leq \aleph_0$? Abélard perfect + $\Omega$ parity
  - Existence of topo. good strategies: Éloïse-Nature only + $\Omega$ parity
Summary

Main contributions

• New semantics for games with Nature.
• Leads to decidability results in many situations where the stochastic semantics was undecidable, including when handling higher-order programs.
• Several consequences.

Perspectives

• Applications? Find a convincing example...
• Logical counterparts?
• Extensions (in particular in the topological setting)?