The state equation for Petri Nets with Unordered Data

Piotr Hofman, Jerome Leroux, Patrick Totzke
LSV, CNRS & ENS de Cachan
Université de Bordeaux, France
University of Edinburgh, UK
Outline

1. Petri Nets and State equation.
2. Petri Nets with Unordered Data.
3. State equations for UD PN.
4. Ideas from the proof.
5. Future work.
Petri Nets
Petri nets

- Places
- Transitions

Definition (VAS - equivalent formalism)
- Places - Dimensions.
- Transitions - Vectors $t_i \in \mathbb{Z}^n$.
- The Marking - a Vector $m_0 \in \mathbb{N}^n$.
- Firing a transition - Adding a vector to the marking, $m' = m + t_k$ (the effect has to be positive i.e. $m' \in \mathbb{N}^n$).
Petri nets

- Places
- Transitions
- Tokens, a Marking
Places
Transitions
Tokens, a Marking
Firing a transition

Petri nets

Places
Transitions
Tokens, a Marking
Firing a transition

Definition (VAS - equivalent formalism)

Places - Dimensions.

Transitions - Vectors $t_i$ in $\mathbb{Z}^n$.

the Marking - a Vector $m_0$ in $\mathbb{N}^n$.

Firing a transition - Adding a vector to the marking, $m' = m + t_k$ (the effect has to be positive i.e. $m' \in \mathbb{N}^n$).
Petri nets

- Places
- Transitions
- Tokens, a Marking
- Firing a transition

Definition (VAS - equivalent formalism)

Places - Dimensions.

Transitions - Vectors $t_i$ in $\mathbb{Z}^n$.

the Marking - a Vector $m_0$ in $\mathbb{N}^n$.

Firing a transition - Adding a vector to the marking, $m' = m + t_k$ (the effect has to be positive i.e. $m'$ in $\mathbb{N}^n$).
Petri nets

- Places
- Transitions
- Tokens, a Marking
- Firing a transition

Definition (VAS - equivalent formalism)
- Places - Dimensions.
- Transitions - Vectors $t_i \in \mathbb{Z}^n$.
- The Marking - a Vector $m_0 \in \mathbb{N}^n$.
- Firing a transition - Adding a vector to the marking, $m' = m + t_k$ (the effect has to be positive i.e. $m'$ in $\mathbb{N}^n$).
Petri nets

Places
Transitions
Tokens, a Marking
Firing a transition

Definition (VAS - equivalent formalism)

Places
Transitions
Tokens, a Marking
Firing a transition
Petri nets

Definition (VAS - equivalent formalism)

- Places - Dimensions.
- Transitions
- Tokens, a Marking
- Firing a transition
Definition (VAS - equivalent formalism)

- **Places** - Dimensions.
- **Transitions** - Vectors $t_i$ in $\mathbb{Z}^n$.
- **Tokens, a Marking**
- **Firing a transition**
Petri nets

Definition (VAS - equivalent formalism)

- Places
- Transitions
- Tokens, a Marking
- Firing a transition

Places - Dimensions.

Transitions - Vectors $t_i$ in $\mathbb{Z}^n$.

the Marking - a Vector $m_0$ in $\mathbb{N}^n$. 

August 23, 2016 4 / 19
Petri nets

Definition (VAS - equivalent formalism)

- **Places** - Dimensions.
- **Transitions** - Vectors $t_i$ in $\mathbb{Z}^n$.
- **Tokens, a Marking** - a Vector $m_0$ in $\mathbb{N}^n$.
- **Firing a transition** - Adding a vector to the marking, $m' = m + t_k$ (the effect has to be positive i.e. $m'$ in $\mathbb{N}^n$).
\[
\begin{pmatrix}
0 & 1 & 2 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

- \( m_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \)

- \( \text{Dim} = 7 \) (4 places + 3 auxiliary)
\[
\begin{pmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{pmatrix}
\]

\[
m_0 =
\begin{pmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{pmatrix}
\]

\[
T =
\begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
0 \\
-1 \\
-1 \\
0 \\
-1 \\
0 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

\[
m_0 = \begin{pmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{pmatrix}
\]
$$\begin{bmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
0 \\
1 \\
1
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0 \\
0 \\
1 \\
1 \\
0 \\
-1
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
2 \\
1 \\
1 \\
0 \\
0
\end{bmatrix}$$

$m_0 = \begin{pmatrix}
0 \\
1 \\
2 \\
1 \\
1 \\
0 \\
0
\end{pmatrix}$

$T = \begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}$
Reachability

**Definition (Reachability)**

$m \rightarrow m'$ if there is $t \in T$ such that $m' = m + t$.

The *reachability* relation is a transitive closure of $\rightarrow$. 

---

**State equation (for this talk)**

Can $m_f - m_i$ be expressed as a sum $\sum t_i$ where $t_i \in T$?

It is an invariant of the reachability relation.
Reachability

**Definition (Reachability)**

\[ m \rightarrow m' \text{ if there is } t \in T \text{ such that } m' = m + t. \]

The *reachability* relation is a transitive closure of \( \rightarrow \).

**Definition (State equation (for this talk))**

Can \( m_f - m_i \) be expressed as a sum \( \sum_i t_i \)

where \( t_i \in T \)?
Reachability

**Definition (Reachability)**

\[ m \rightarrow m' \text{ if there is } t \in T \text{ such that } m' = m + t. \]

The *reachability* relation is a transitive closure of \( \rightarrow \).

**Definition (State equation (for this talk))**

Can \( m_f - m_i \) be expressed as a sum \( \sum_i t_i \) where \( t_i \in T \)?

It is an invariant of the reachability relation.
Definition (Reachability)

\( m \rightarrow m' \) if there is \( t \in T \) such that \( m' = m + t \).

The reachability relation is a transitive closure of \( \rightarrow \).

Definition (State equation (for this talk))

Can \( m_f - m_i \) be expressed as a sum \( \sum_i t_i \)

where \( t_i \in T \)?

Lemma

If there is no solution for the state equation then \( m_f \) is not reachable from \( m_i \).
Nets with Data
decidable coverability and termination

Ordered Data Petri Nets $F_{\omega^\omega}$-complete

$\nu$-Petri Nets $F_{\omega_2}$-complete

Unordered Data Petri Nets $F_3 \subseteq \cdots \subseteq F_{\omega_2}$

Petri Nets EXPSPACE-complete

Ordered Data Nets $F_{\omega^\omega}$-complete

$\nu$-Petri Nets $F_{\omega_2}$-complete

Unordered Data Nets $F_{\omega^\omega}$-complete

Affine Nets $F_{\omega}$-complete

Nets with Tokens Which Carry Data, by Ranko Lazic, Thomas Newcomb, Joël Ouaknine, Andrew Roscoe, James Worrell.
Petri Nets

Unordered Data Petri Nets

Ordered Data Petri Nets

$\nu$-Petri Nets

decidable
place boundedness

+whole-place

+fresh

+order

Ordered Data Nets

Unordered Data Nets

Affine Nets

ExpSpace-complete

whole-place

data

+whole-place

reachability

undecidable

reachability

Nets with Tokens Which Carry Data, by Ranko Lazic, Thomas Newcomb, Joel Ouaknine, Andrew Roscoe, James Worrell.

August 23, 2016 8 / 19
Ordered Data Petri Nets

\[ \omega \]-complete

decidable coverability and termination

\[ \nu \]-Petri Nets

\[ \omega \]-complete

Unordered Data Nets

\[ \omega \_2 \]-complete

Affine Nets

Petri Nets

decidable reachability

\nu\text{-Petri Nets}

undecidable reachability

Ordered Data Petri Nets

+whole-place

Ordered Data Nets

+whole-place

Unordered Data Nets

+whole-place

Affine Nets

+whole-place

+data

+order

+fresh

+whole-place

Nets with Tokens Which Carry Data, by Ranko Lazic, Thomas Newcomb, Joël Ouaknine, Andrew Roscoe, James Worrell.

August 23, 2016 8 / 19
Let $\mathbb{D}$ be an infinite data domain.
Let $\mathcal{D}$ be an infinite data domain.

**Definition (Data VAS)**

- **Places** - Dimensions $n$.
- **Marking** - a function from $\mathcal{D}$ to $\mathbb{N}^n$. 

**Transitions representation $T$** - a finite set of finitely supported functions from $\mathcal{D}$ to $\mathbb{Z}^n$.

**Firing an abstract transition $t \in T$**

1. Instantiate a transition - $t' = t \circ \pi$.
2. $m' = m + t'$ and $m'$ has to be a proper marking.
Let $\mathbb{D}$ be an infinite data domain.

**Definition (Data VAS)**

- **Places** - Dimensions $n$.
- **Marking** - a finitely supported function from $\mathbb{D}$ to $\mathbb{N}^n$. 
Let $\mathbb{D}$ be an infinite data domain.

**Definition (Data VAS)**

- **Places** - Dimensions $n$.
- **Marking** - a finitely supported function from $\mathbb{D}$ to $\mathbb{N}^n$.
- **Transitions** - a set of finitely supported functions from $\mathbb{D}$ to $\mathbb{Z}^n$. 

**Firing an abstract transition** $t \in T$.

1. Instantiate a transition - $t' = t \circ \pi$.
2. $m' = m + t'$ and $m'$ has to be a proper marking.
Let $\mathcal{D}$ be an infinite data domain.

**Definition (Data VAS)**

- **Places** - Dimensions $n$.
- **Marking** - a finitely supported function from $\mathcal{D}$ to $\mathbb{N}^n$.
- **Transitions** - a set of finitely supported functions from $\mathcal{D}$ to $\mathbb{Z}^n$ which is closed under data permutation.

**Firing an abstract transition $t \in T$**

1. **Instantiate a transition** - $t' = t \circ \pi$.
2. **Marking update** - $m' = m + t'$ and $m'$ has to be a proper marking.
Let $\mathcal{D}$ be an infinite data domain.

**Definition (Data VAS)**

- Places - Dimensions $n$.
- Marking - a finitely supported function from $\mathcal{D}$ to $\mathbb{N}^n$.
- Transitions representation $T$ - a finite set of finitely supported functions from $\mathcal{D}$ to $\mathbb{Z}^n$. 

Firing an abstract transition $t \in T$:

1. Instantiate a transition - $t' = t \circ \pi$.
2. $m' = m + t'$ and $m'$ has to be a proper marking.
Let \( \mathbb{D} \) be an infinite data domain.

**Definition (Data VAS)**

- **Places** - Dimensions \( n \).
- **Marking** - a finitely supported function from \( \mathbb{D} \) to \( \mathbb{N}^n \).

- Transitions representation \( T \) - a finite set of finitely supported functions from \( \mathbb{D} \) to \( \mathbb{Z}^n \).
- **Firing an abstract transition** \( t \in T \).
  
  1. Instantiate a transition - \( t' = t \circ \pi \).
  2. \( m' = m + t' \) and \( m' \) has to be a proper marking.
Example

Let $n = 3$
Example

Let $n = 3$

$$m_0 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix} \quad t_1 = \begin{pmatrix} -1\alpha \\ 1\alpha \\ 0 \end{pmatrix} \quad t_2 = \begin{pmatrix} \beta \\ \alpha \\ -2\gamma \end{pmatrix}$$
Let $n = 3$

$$m_0 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix} \quad t_1 = \begin{pmatrix} -1\alpha \\ 1\alpha \\ 0 \end{pmatrix} \quad t_2 = \begin{pmatrix} \beta \\ \alpha \\ -2\gamma \end{pmatrix}$$

Different instantiations: $t_1 \circ \pi_1 = \begin{pmatrix} -1\delta \\ 1\delta \\ 0 \end{pmatrix} \quad t_1 \circ \pi_2 = \begin{pmatrix} -1\beta \\ 1\beta \\ 0 \end{pmatrix}$
Example

Let \( n = 3 \)

\[
    m_0 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix}, \quad t_1 = \begin{pmatrix} -1\alpha \\ 1\alpha \\ 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} \beta \\ \alpha \\ -2\gamma \end{pmatrix}
\]

\[
    m_0 + t_1 \circ \pi_1 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix} + \begin{pmatrix} -1\delta \\ 1\delta \\ 0 \end{pmatrix} = \begin{pmatrix} 2\alpha + 3\beta - \delta \\ 1\gamma + 2\alpha + \delta \\ 4\beta \end{pmatrix}
\]

BAD
Example

Let $n = 3$

$$m_0 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix} \quad t_1 = \begin{pmatrix} -1\alpha \\ 1\alpha \\ 0 \end{pmatrix} \quad t_2 = \begin{pmatrix} \beta \\ \alpha \\ -2\gamma \end{pmatrix}$$

$$m_0 + t_1 \circ \pi_2 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix} + \begin{pmatrix} -1\beta \\ 1\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 2\alpha + 2\beta \\ 1\gamma + 2\alpha + \beta \\ 4\beta \end{pmatrix}$$

GOOD
Example

Let $n = 3$

$$m_0 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix}, \quad t_1 = \begin{pmatrix} -1\alpha \\ 1\alpha \\ 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} \beta \\ \alpha \\ -2\gamma \end{pmatrix}$$

$$m_0 + t_2 \circ \pi_3 = \begin{pmatrix} 2\alpha + 3\beta \\ 1\gamma + 2\alpha \\ 4\beta \end{pmatrix} + \begin{pmatrix} 1\alpha \\ 1\delta \\ -2\beta \end{pmatrix} = \begin{pmatrix} 3\alpha + 3\beta \\ 1\gamma + 2\alpha + \delta \\ 2\beta \end{pmatrix}$$

GOOD
State equation for **UDPN**

\[ m_f - m_i = \sum_i t_i \circ \pi_i \]

where \( t_i \in T \).
State equation for **UDPN**

\[ m_f - m_i = \sum_i t_i \circ \pi_i \]

where \( t_i \in T \).

**INPUT:** a set of data vectors \( T \), and a data vector \( m \), binary encoded

**OUTPUT** If \( m \) can be expressed as \( \sum_i t_i \circ \pi_i \)?

**Theorem**

*State equation problem for UDNP is in NP.*
State equation for **UDPN**

\[ m_f - m_i = \sum_{i} t_i \circ \pi_i \]

where \( t_i \in T \).

**INPUT:** a set of data vectors \( T \), and a data vector \( m \), binary encoded

**OUTPUT** If \( m \) can be expressed as \( \sum_i t_i \circ \pi_i \)?

**Theorem**

*State equation problem for UDPN is in NP.*

Exactly the same like the state equation for Petri Nets.
Problems and Ideas
Definition (Support)

Let $\text{supp}(x) \overset{\text{def}}{=} \{\alpha : \alpha \in D \text{ and } x(\alpha) \neq 0\}$.

Problem 1 - unbounded number of data.
There is no bound on the number of data involved in the sum.
Idea: Problems.

Definition (Support)

Let $\text{supp}(x) \overset{\text{def}}{=} \{ \alpha : \alpha \in D \text{ and } x(\alpha) \neq 0 \}$.

Problem 1- unbounded number of data.

Suppose $m = \sum_{i=1}^{n} t_i \circ \pi_i$, and $S = \bigcup_{0<i<n} \text{supp}(t_i \circ \pi_i)$.

What is the bound on $|S|$?
Definition (Support)

Let $\text{supp}(x) \stackrel{\text{def}}{=} \{ \alpha : \alpha \in \mathbb{D} \text{ and } x(\alpha) \neq 0 \}.$

Problem 1- unbounded number of data.

Suppose $m = \sum_{i=1}^{n} t_i \circ \pi_i,$ and $S = \bigcup_{0<i<n} \text{supp}(t_i \circ \pi_i).$

What is the bound on $|S|$?

Lemma

$|S|$ can be bounded by polynomially.
Idea: Problems.

**Definition (Support)**

Let $\text{supp}(x) \overset{\text{def}}{=} \{ \alpha : \alpha \in D \text{ and } x(\alpha) \neq 0 \}$.

**Problem 1- unbounded number of data.**

Suppose $m = \sum_{i=1}^{n} t_i \circ \pi_i$, and $S = \bigcup_{0 < i < n} \text{supp}(t_i \circ \pi_i)$.

What is the bound on $|S|$?

**Lemma**

$|S|$ can be bounded by polynomially.

**Problem 2- compression.**

Suppose $|S|$ is already bounded by a polynomial and we reduce problem to the state equation for a Petri Net. What will be the size of the Petri Net?
Idea: Problems.

Definition (Support)
Let \( supp(x) \overset{\text{def}}{=} \{ \alpha : \alpha \in \mathbb{D} \text{ and } x(\alpha) \neq 0 \} \).

Problem 1- unbounded number of data.
Suppose \( m = \sum_{i=1}^{n} t_i \circ \pi_i \), and \( \mathbb{S} = \bigcup_{0<i<n} supp(t_i \circ \pi_i) \).

What is the bound on \( |\mathbb{S}| \)?

Lemma
\( |\mathbb{S}| \) can be bounded by polynomially.

Problem 2- compression.
Suppose \( |\mathbb{S}| \) is already bounded by a polynomial and we reduce problem to the state equation for a Petri Net. What will be the size of the Petri Net?

Exponential!
The main combinatorial insight.

**Definition (**\(n\)-**histogram)**

Suppose \(S \subset D\) is a finite set. A function \(H : S \times D \rightarrow \mathbb{N}\) is a \(n\)-histogram if:

1. \(\sum_{\beta \in D} H(\alpha, \beta) = n\) for all \(\alpha \in S\),
2. \(\sum_{\alpha \in S} H(\alpha, \beta) \leq n\) for all \(\beta \in D\).

Histograms over the same set \(S\) can be added pointwise.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\n\end{array}
\]

\[
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
\]

\(=\)
A relation between 1-histogram and $t_i \circ \pi_i$.

**Definition**

For a given $\pi$ and $t$ we define 1-histogram $H^\text{supp(t)}_\pi : \text{supp}(t) \times D \rightarrow \mathbb{N}$ such that

$$H_\pi(\alpha, \beta) = 1 \iff \pi(\beta) = \alpha.$$
A relation between 1-histogram and $t_i \circ \pi_i$.

**Definition**

For a given $\pi$ and $t$ we define 1-histogram $H_{\pi}^{\text{supp}(t)} : \text{supp}(t) \times \mathbb{D} \rightarrow \mathbb{N}$ such that

$$H_{\pi}(\alpha, \beta) = 1 \iff \pi(\beta) = \alpha.$$

Let

$$t = \begin{pmatrix} 2\alpha + 3\beta \\ 2\alpha \\ 4\beta \end{pmatrix}, \quad \pi(\phi) \overset{\text{def}}{=} \begin{cases} \beta & \text{if } \phi = \alpha \\ \delta & \text{if } \phi = \beta \\ \alpha & \text{if } \phi = \delta \\ \phi & \text{otherwise} \end{cases} \quad H_{\pi}^{\text{supp}(t)} = \begin{pmatrix} \alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & \gamma \\ 1 & 0 & \delta \end{pmatrix}$$
A relation between 1-histogram and $t_i \circ \pi_i$.

**Definition**

For a given $\pi$ and $t$ we define 1-histogram $H_{\pi}^{\text{supp}(t)} : \text{supp}(t) \times \mathbb{D} \rightarrow \mathbb{N}$ such that

$$H_{\pi}(\alpha, \beta) = 1 \iff \pi(\beta) = \alpha.$$ 

Let

$$t = \begin{pmatrix} 2\alpha + 3\beta \\ 2\alpha \\ 4\beta \end{pmatrix} \quad \pi(\phi) \stackrel{\text{def}}{=} \begin{cases} \beta & \text{if } \phi = \alpha \\ \delta & \text{if } \phi = \beta \\ \alpha & \text{if } \phi = \delta \\ \phi & \text{otherwise} \end{cases} \quad H_{\pi}^{\text{supp}(t)} = \begin{pmatrix} \alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & \gamma \\ 1 & 0 & \delta \end{pmatrix}$$

$$t \circ \pi = \begin{pmatrix} 2\delta + 3\alpha \\ 2\delta \\ 4\alpha \end{pmatrix}$$
A relation between 1-histogram and \( t_i \circ \pi_i \).

**Definition**

For a given \( \pi \) and \( t \) we define 1-histogram \( H_{\pi}^{\text{supp}(t)} : \text{supp}(t) \times D \rightarrow \mathbb{N} \) such that

\[
H_{\pi}(\alpha, \beta) = 1 \iff \pi(\beta) = \alpha.
\]

Let

\[
t = \begin{pmatrix}
2\alpha + 3\beta \\
2\alpha \\
4\beta
\end{pmatrix}
\]

\[
\pi(\phi) \overset{\text{def}}{=} \begin{cases}
\beta & \text{if } \phi = \alpha \\
\delta & \text{if } \phi = \beta \\
\alpha & \text{if } \phi = \delta \\
\phi & \text{otherwise}
\end{cases}
\]

\[
H_{\pi}^{\text{supp}(t)} = \begin{pmatrix}
\alpha & \beta \\
0 & 1 & \alpha \\
0 & 0 & \beta \\
0 & 0 & \gamma \\
1 & 0 & \delta
\end{pmatrix}
\]

\[
t \circ \pi = \begin{pmatrix}
2\delta + 3\alpha \\
2\delta \\
4\alpha
\end{pmatrix}
\]
A relation between 1-histogram and $t_i \circ \pi_i$.

**Definition**

For a given $\pi$ and $t$ we define 1-histogram $H_{\pi}^{supp(t)} : supp(t) \times D \rightarrow \mathbb{N}$ such that

$$H_{\pi}(\alpha, \beta) = 1 \iff \pi(\beta) = \alpha.$$ 

Let

$$t = \begin{pmatrix} 2\alpha + 3\beta \\ 2\alpha \\ 4\beta \end{pmatrix} \quad t \circ \pi = \begin{pmatrix} 2\delta + 3\alpha \\ 2\delta \\ 4\alpha \end{pmatrix} \quad H_{\pi}^{supp(t)} = \begin{array}{ccc} \alpha & \beta & \alpha \\ 0 & 1 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & \gamma \\ 1 & 0 & \delta \end{array}$$

Observe:

$$t \circ \pi(\phi) = \sum_{\eta \in supp(t)} t(\eta) \cdot H_{\pi}(\eta, \phi) \text{ for all } \phi \in D.$$
Observe:

\[ t \circ \pi(\phi) = \sum_{\eta \in \text{supp}(t)} t(\eta) \cdot H(\eta, \phi) \quad \text{for all } \phi \in D. \]
Definition: Homomorphism Eval

\[ \text{Eval}(t, H)(\phi) \overset{\text{def}}{=} \sum_{\eta \in \text{supp}(t)} t(\eta) \cdot H(\eta, \phi) \quad \text{for all } \phi \in \mathbb{D}. \]
Definition: Homomorphism $Eval$

$$Eval(t, H)(\phi) \overset{\text{def}}{=} \sum_{\eta \in \text{supp}(t)} t(\eta) \cdot H(\eta, \phi) \quad \text{for all } \phi \in D.$$ 

Theorem

A function $H : S \times D \to \mathbb{N}$ is an $n$-histogram if, and only if, $H$ is the sum of $n$ 1-histograms over $S$.

Lemma

$m_t$ can be expressed as $\sum_{i=1}^{n} t \circ \pi_i$ iff there is a $n$-histogram $H$ such that $m_t = Eval(t, H)$. 

August 23, 2016 16 / 19
Lemma

\( m \) can be expressed as \( \sum_{t \in T} \sum_{i=1}^{n_t} t \circ \pi_{t,i} \) iff there is a sequence of \( n_t \)-histogram \( H_t \) for any \( t \in T \) such that

\[
m = \sum_{t \in T} \text{Eval}(t, H_t).
\]
Lemma

\( m \) can be expressed as \( \sum_{t \in T} \sum_{i=1}^{n_t} t \circ \pi_{t,i} \) iff

there is a sequence of \( n_t \)-histogram \( H_t \) for any \( t \in T \) such that

\[ m = \sum_{t \in T} \text{Eval}(t, H_t). \]

Lemma (recall)

The set of data that appear in the solution, \( T \equiv \bigcup_{t \in T, i \in \mathbb{N}} \text{supp}(t \circ \pi_{t,i}) \), can be bounded by polynomially.
Lemma

\[ m \text{ can be expressed as } \sum_{t \in T} \sum_{i=1}^{n_t} t \circ \pi_{t,i} \iff \text{there is a sequence of } n_t\text{-histogram } H_t \text{ for any } t \in T \text{ such that} \]
\[ m = \sum_{t \in T} \text{Eval}(t, H_t). \]

Lemma (recall)

The set of data that appear in the solution, \[ T \overset{\text{def}}{=} \bigcup_{t \in T, i \in \mathbb{N}} \text{supp}(t \circ \pi_{t,i}) \], can be bounded by polynomially.

- We can restrict histograms to \( S \times T \rightarrow \mathbb{N} \).
Lemma

\[ m \text{ can be expressed as } \sum_{t \in T} \sum_{i=1}^{n_t} t \circ \pi_{t,i} \text{ iff there is a sequence of } n_t\text{-histogram } H_t \text{ for any } t \in T \text{ such that} \]

\[ m = \sum_{t \in T} \text{Eval}(t, H_t). \]

Lemma (recall)

The set of data that appear in the solution, \( T \overset{\text{def}}{=} \bigcup_{t \in T, i \in \mathbb{N}} \text{supp}(t \circ \pi_{t,i}) \), can be bounded by polynomially.

- We can restrict histograms to \( S \times T \longrightarrow \mathbb{N} \).
- Thus histograms \( H_t \) can be described using polynomially many numbers.
Lemma

\( m \) can be expressed as \( \sum_{t \in T} \sum_{i=1}^{n_t} t \circ \pi_{t,i} \) iff there is a sequence of \( n_t \)-histogram \( H_t \) for any \( t \in T \) such that

\[ m = \sum_{t \in T} \text{Eval}(t, H_t). \]

Lemma (recall)

The set of data that appear in the solution, \( T \overset{\text{def}}{=} \bigcup_{t \in T, i \in \mathbb{N}} \text{supp}(t \circ \pi_{t,i}) \), can be bounded by polynomially.

- We can restrict histograms to \( S \times T \rightarrow \mathbb{N} \).
- Thus histograms \( H_t \) can be described using polynomially many numbers.
- Guess values and check the sum. Incorrect as we don’t have bound on numbers.
**Lemma**

\( m \) can be expressed as \( \sum_{t \in T} \sum_{i=1}^{n_t} t \circ \pi_{t,i} \) iff there is a sequence of \( n_t \)-histogram \( H_t \) for any \( t \in T \) such that

\[
    m = \sum_{t \in T} \text{Eval}(t, H_t).
\]

**Lemma (recall)**

The set of data that appear in the solution, \( T \overset{\text{def}}{=} \bigcup_{t \in T, i \in \mathbb{N}} \text{supp}(t \circ \pi_{t,i}) \), can be bounded by polynomially.

- We can restrict histograms to \( S \times T \rightarrow \mathbb{N} \).
- Thus histograms \( H_t \) can be described using polynomially many numbers.
- We need to build a system of linear equations which incorporates conditions for being a histogram and proper Eval-uation.
Future.
Reachability for *UDPN*. 
- Reachability for \textit{UDPN}.
- Which algebraic techniques can be transferred to \textit{UDPN}.
- Reachability for *UDPN*.
- Which algebraic techniques can be transferred to *UDPN*.
- What with the free-choice and the conflict-free Nets?
• Reachability for \textit{UDPN}.
• Which algebraic techniques can be transferred to \textit{UDPN}.
• What with the free-choice and the conflict-free Nets?
• What is the impact of a state equation if used to improve the coverability algorithm for \textit{UDPN}.

How a state equation should look like for Petri Nets with ordered data, or a fresh datum operator.

And many other questions...
Reachability for $UDPN$.
Which algebraic techniques can be transfered to $UDPN$.
What with the free-choice and the conflict-free Nets?
What is the impact of a state equation if used to improve the coverability algorithm for $UDPN$.
How a state equation should look like for Petri Nets with ordered data, or a fresh datum operator.
- Reachability for *UDPN*.
- Which algebraic techniques can be transferred to *UDPN*.
- What with the free-choice and the conflict-free Nets?
- What is the impact of a state equation if used to improve the coverability algorithm for *UDPN*.
- How a state equation should look like for Petri Nets with ordered data, or a fresh datum operator.
- And many other questions...
- Reachability for *UDPN*.
- Which algebraic techniques can be transferred to *UDPN*.
- What with the free-choice and the conflict-free Nets?
- What is the impact of a state equation if used to improve the coverability algorithm for *UDPN*.
- How a state equation should look like for Petri Nets with ordered data, or a fresh datum operator.
- And many other questions...

Thank You.