Formal Reasoning for Quantum Programs

Yuxin Deng

East China Normal University

Thanks to Yuan Feng and Mingsheng Ying
Outline

Background

Preliminaries on quantum mechanics

Equivalences for quantum processes

Symbolic semantics

An algorithm for ground bisimulation

Hoare logic

Summary
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- A 2000-km quantum communication main network between Beijing and Shanghai will be fully operational later this year.
Quantum computation

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- In December 2015, Google announced that, in solving a specific optimization problem, their 512-qubit D-Wave 2X is 100 million times faster than conventional single-core computers.
Quantum programming

“the real challenge will be the software .... Programming this thing [D-Wave] is ridiculously hard; it can take months to work out how to phrase a problem so that the computer can understand it.”

— G. Rose
Founder and CTO at D-Wave Systems

Quantum programming languages

- “Quantum data, classical control” [Selinger]
- Sequential languages
  - Quipper [Dalhousie Univ.]
  - LIQUi|> [Microsoft]
  - Scaffold [Princeton]
  - ...
- Concurrent languages (quantum process algebras) Aiming to specify and verify quantum protocols.
  - QPAlg [Jorrand and Lalire]
  - CQP [Gay and Nagarajan]
  - qCCS [Feng et al.]
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- Coinduction for quantum processes
- Hoare logic for quantum programs
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Dirac-notation

Let $\mathcal{H}$ be a Hilbert space.
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- ‘ket’ $|\psi\rangle$ stands for a (normalized) vector in $\mathcal{H}$. 
- In particular, $|\psi\rangle^\dagger = \langle \psi |$. 
- Generally, $(A^\dagger |\psi\rangle, |\phi\rangle) = (|\psi\rangle, A |\phi\rangle)$. 

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- ‘bra’ $\langle \psi |$ stands for the adjoint (dual vector) of $|\psi\rangle$.
- Generally, $A^\dagger$ stands for the adjoint of $A$, such that
  \[(A^\dagger |\psi\rangle, |\phi\rangle) = (|\psi\rangle, A |\phi\rangle).\]

In particular, $(|\psi\rangle)^\dagger = \langle \psi |$.
Quantum states

- Associated to any quantum system is a Hilbert space known as the state space.
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- The state of a closed quantum system is described by a unit vector, say $|\psi\rangle$, in its state space.
Quantum states (Cont’d)

- $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ : lies in the state $|\psi_k\rangle$ with probability $p_k$, $\sum_k p_k = 1$.

  - $\rho$ is a positive operator
  - $\text{tr}(\rho) = 1$
Quantum states (Cont’d)

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  - $\rho$ is a positive operator
  - $\text{tr}(\rho) = 1$
- These two conditions characterize exactly the set of density operators.
Quantum dynamics

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- $\mathcal{E}$ is **trace-preserving**, if $\text{tr}(\mathcal{E}(A)) = \text{tr}(A)$ for any positive operator $A$.

- $\mathcal{E}$ is **completely positive**, if for any auxiliary space $\mathcal{H}'$ and any positive operator $\sigma$ on the tensor Hilbert space $\mathcal{H}' \otimes \mathcal{H}$, $(\mathcal{I}_{\mathcal{H}'} \otimes \mathcal{E})(\sigma)$ is also a positive operator on $\mathcal{H}' \otimes \mathcal{H}$. 
Quantum dynamics

- The evolution of a quantum system is described by a super-operator

\[ \rho' = \mathcal{E}(\rho) \]
Quantum measurements

- An observable $A$ is a Hermitian operator, $A^\dagger = A$. Let

$$A = \sum_k \lambda_k P_k,$$

where $P_k$ is the eigenspace associated with $\lambda_k$. 
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Quantum measurements

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Quantum measurements

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- If we measure $\rho$ by the observable $A$, then we obtain the result $k$ with probability
  \[ p_k = \text{tr}(P_k \rho) \]

- The measurement disturbs the system, leaving it in a state $P_k \rho P_k / p_k$ determined by the outcome.
Syntax of qCCS

The syntax of qCCS:

\[ \text{nil} | \text{pref}\cdot P | P + Q | P \parallel Q | P \setminus L | \text{if } b \text{ then } P | A(\tilde{q}; \tilde{x}) \]

where

\[ \text{pref ::= } \tau | c?x | c!e | c?q | c!q | E[\tilde{q}] | M[\tilde{q}; x] \]
Further requirements

- $c?x.d!x.d!x.0$

$\nRightarrow c?r.d!r.d!r.0$

- Quantum no-cloning theorem!
Syntax of qCCS, cont’d

For a process to be legal, we require

1. \( q \not\in qv(P) \) in the process \( c!q.P \);

2. \( qv(P) \cap qv(Q) = \emptyset \) in the process \( P || Q \).
A pair of the form \( \langle P, \rho \rangle \)

is a configuration, where \( P \) is a closed quantum process and \( \rho \) is a density operator. The set of configurations is denoted by \( \text{Con} \). We let \( C, D, \ldots \) range over \( \text{Con} \).
Operational Semantics of qCCS

Let

\[ \text{Act} = \{ \tau \} \cup \{ c?\nu, c!\nu \mid c \text{ classical channel, } \nu \text{ real number} \} \cup \{ c?r, c!r \mid c \text{ quantum channel, } r \text{ quantum variable} \}, \]

and \( D(\text{Con}) \) be the set of finite-support probability distributions over \( \text{Con} \).

The semantics of qCCS is given by the probabilistic labeled transition system \( (\text{Con}, \text{Act}, \to) \), where \( \to \subseteq \text{Con} \times \text{Act} \times D(\text{Con}) \) is the smallest relation satisfying some rules.
An example: Teleportation

Quantum teleportation [Bennett, Brassard, Crepeau, Jozsa, Peres, and Wootters, PRL 1993] makes use of a maximally entangled state to teleport an unknown quantum state by sending only classical information.

It serves as a key ingredient in many other quantum communication protocols.
An example: Teleportation

Let

\[\begin{align*}
\text{Alice} & := \text{CNot}[q, q_1].H[q].M[q, q_1; x].c!x.\text{nil} \\
\text{Bob} & := c?x.U_x[q_2].\text{nil} \\
\text{Telep} & := (\text{Alice}\|\text{Bob})\{c\}
\end{align*}\]

Here \(M = \sum_{i=0}^{3} \lambda_i \langle i|i\rangle\), and

\[\begin{align*}
U_x[q_2].\text{nil} & := \text{if } x = \lambda_0 \text{ then } \sigma_0[q_2].\text{nil} + \text{if } x = \lambda_1 \text{ then } \sigma_1[q_2].\text{nil} \\
& \quad + \text{if } x = \lambda_2 \text{ then } \sigma_3[q_2].\text{nil} + \text{if } x = \lambda_3 \text{ then } \sigma_2[q_2].\text{nil}.
\end{align*}\]
\[
\langle Telep, [(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\rangle
\]

\[
\tau
\]

\[
\langle (H[q].M[q,q_1;x].c!x.nil||Bob)\{c\}, \frac{1}{\sqrt{2}}(\alpha(|000\rangle + |011\rangle) + \beta(|110\rangle + |101\rangle))\rangle
\]

\[
\tau
\]

\[
\langle (M[q,q_1;x].c!x.nil||Bob)\{c\}, \frac{1}{\sqrt{2}}(\alpha(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + \beta(|010\rangle - |110\rangle + |001\rangle - |101\rangle))\rangle
\]
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Lifted relation

Lift $\mathcal{R} \subseteq S \times S$ to $\mathcal{R}^\circ \subseteq \text{Dist}(S) \times \text{Dist}(S)$:
Lifted relation

Lift $\mathcal{R} \subseteq S \times S$ to $\mathcal{R}^\circ \subseteq \text{Dist}(S) \times \text{Dist}(S)$:

1. $s R t$ implies $\bar{s} R^\circ \bar{t}$;
Lifted relation

Lift $\mathcal{R} \subseteq S \times S$ to $\mathcal{R}^\circ \subseteq \text{Dist}(S) \times \text{Dist}(S)$:

1. $s \mathcal{R} t$ implies $\bar{s} \mathcal{R}^\circ \bar{t}$;

2. $\Delta_i \mathcal{R}^\circ \Theta_i$ for all $i \in I$ implies $(\sum_{i \in I} p_i \cdot \Delta_i) \mathcal{R}^\circ (\sum_{i \in I} p_i \cdot \Theta_i)$ for any $p_i \in [0, 1]$ with $\sum_{i \in I} p_i = 1$, where $I$ is a countable index set.

There are alternative formulations; related to the Kantorovich metric and the network flow problem. See e.g. http://www.springer.com/978-3-662-45197-7
Four criteria to judge equivalence

A relation \( \mathcal{R} \) is
Four criteria to judge equivalence

A relation $\mathcal{R}$ is

- **barb-preserving** if $\mathcal{CRD}$ implies that $\mathcal{C} \vdash^p_c \mathcal{D}$ iff $\mathcal{D} \vdash^p_c \mathcal{C}$ for any $p \in [0, 1]$ and any classical channel $c$, where $\mathcal{C} \vdash^p_c$ holds if $\mathcal{C} \xrightarrow{\hat{t} \tau} \Delta$ for some $\Delta$ with

$$\sum\{\Delta(C') \mid C' \xrightarrow{c!v} \text{ for some } v\} \geq p;$$
Four criteria to judge equivalence

A relation \( R \) is

- **barb-preserving** if \( CRD \) implies that \( C \Downarrow_c^{\geq p} \) iff \( D \Downarrow_c^{\geq p} \) for any \( p \in [0, 1] \) and any classical channel \( c \), where \( C \Downarrow_c^{\geq p} \) holds if \( C \xrightarrow{\hat{t}} \Delta \) for some \( \Delta \) with

\[
\sum \{ \Delta(C') \mid C' \xrightarrow{c!v} \text{ for some } v \} \geq p;
\]

- **reduction-closed** if \( CRD \) implies

  - whenever \( C \xrightarrow{\hat{t}} \Delta \), there exists \( \Theta \) such that \( D \xrightarrow{\hat{t}} \Theta \) and \( \Delta R^\circ \Theta \),
  - whenever \( D \xrightarrow{\hat{t}} \Theta \), there exists \( \Delta \) such that \( C \xrightarrow{\hat{t}} \Delta \) and \( \Delta R^\circ \Theta \);
Four criteria to judge equivalence, cont.

- **compositional** if \( CRD \) implies \((C \parallel R) R (D \parallel R)\) for any process \( R \) with \( qv(R) \) disjoint from \( qv(C) \cup qv(D) \),
Four criteria to judge equivalence, cont.

- **compositional** if $C RD$ implies $(C || R) R (D || R)$ for any process $R$ with $qv(R)$ disjoint from $qv(C) \cup qv(D)$,

- **closed under super-operator application**, if $C RD$ implies $\mathcal{E}(C) R \mathcal{E}(D)$ for any $\mathcal{E} \in SO(\mathcal{H}_{qv(C)})$. 

Reduction barbed congruence

Originated in [Honda & Tokoro 1995].

Let reduction barbed congruence, written $\approx_r$, be the largest relation over configurations which is
- barb-preserving,
- reduction-closed,
- compositional,
- closed under super-operator application,
- and furthermore, if $C \approx_r D$ then $qv(C) = qv(D)$ and $env(C) = env(D)$. 
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Open bisimulation

Inspired by [Sangorigi 1996].

A relation $R \subseteq \text{Con} \times \text{Con}$ is an open simulation if $CRD$ implies that

- $qv(C) = qv(D)$, and $env(C) = env(D)$,
- for any $E \in SO(H_{qv(C)})$, whenever $E(C) \xrightarrow{\alpha} \Delta$, there is some $\Theta$ with $E(D) \xrightarrow{\hat{\alpha}} \Theta$ and $\Delta R^\circ \Theta$.

A relation $R$ is an open bisimulation if both $R$ and $R^{-1}$ are open simulations. We let $\approx_o$ be the largest open bisimulation.
Theorem : Congruence

The relation $\equiv_0$ between processes is preserved by all the constructors of qCCS except for summation.

$C \equiv_0 D$ if and only if $C \equiv_r D$. 


Theorem: Congruence

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$C \approx_o D$ if and only if $C \approx_r D$. 
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Summary
An equivalence for super-operators

Let \( \sqsubseteq \) be the Löwner preorder defined on operators: \( A \sqsubseteq B \) if and only if \( B - A \) is positive semi-definite.

For two super-operators \( A, B \) on \( \mathcal{H} \), let \( A \preceq_V B \) if for any \( \rho \in D(\mathcal{H}) \), \( \text{tr}_{\overline{V}} (A(\rho)) \sqsubseteq \text{tr}_{\overline{V}} (B(\rho)) \), where \( V \) is the complement set of \( V \) in \( qVar \).

Let \( \sim_V \) be \( \preceq_V \cap \succeq_V \) and we abbreviate \( \preceq_{\emptyset} \) and \( \sim_{\emptyset} \) to \( \preceq \) and \( \sim \), respectively.
A super-operator valued distribution $\Delta$ over $S$ is a function from $S$ to $SO(H)$ such that $\sum_{s \in S} \Delta(s) \simeq I_H$.

Let $Dist_H(S)$ be the set of finite-support super-operator valued distributions over $S$. 
Symbolic semantics

Inspired by [Hennessy & Lin 1995]

A pair of the form \((t, E)\), where \(t \in \mathcal{T}\) and \(E \in SO_t(H)\), is called a snapshot. The set of snapshots is denoted by \(SN\).

The symbolic semantics of qCCS is given by the qLTS \((SN, BAct_s, \to)\) on snapshots, where \(\to \subseteq SN \times BAct_s \times Dist_H(SN)\) is the smallest relation satisfying a few rules.
Symbolic semantics

E.g.

\[ M = \sum_{i \in I} \lambda_i |\phi_i\rangle \langle \phi_i| \]

\[ \langle M[\bar{q}; x].t, \mathcal{E} \rangle \xrightarrow{tt, \tau} \sum_{i \in I} A_\tau^{\phi_i} \bullet (t\{\lambda_i/x\}, Set_\tau^{\phi_i} \mathcal{E}) \]

where

\[ A_\tau^{\phi_i} : \rho \mapsto |\phi_i\rangle_\tau \langle \phi_i| \rho |\phi_i\rangle_\tau \langle \phi_i| \quad (1) \]

\[ Set_\tau^{\phi_i} : \rho \mapsto \sum_{j \in I} |\phi_i\rangle_\tau \langle \phi_j| \rho |\phi_j\rangle_\tau \langle \phi_i| \quad (2) \]
Symbolic semantics

\( (P, \mathcal{I}_\mathcal{H}) \)

\( \text{tt, } \tau \)

\( \text{Set}_q^0 \)

\( (\mathcal{I}[q].\text{nil}, \text{Set}_q^0) \)

\( \text{tt, } \tau \)

\( (\text{nil}, \text{Set}_q^0) \)

\( (Q, \mathcal{I}_\mathcal{H}) \)

\( \text{tt, } \tau \)

\( \mathcal{A}_0 \)

\( \mathcal{A}_1 \)

\( (Q_0, \text{Set}_q^0) \)

\( (Q_1, \text{Set}_q^1) \)

\( 0 = 0, \tau \)

\( 0 = 1, \tau \)

\( 1 = 0, \tau \)

\( 1 = 1, \tau \)

\( (\text{nil}, \text{Set}_q^0) \)

\( (\text{nil}, \text{Set}_q^1) \)

\( X_q \)

\( (\text{nil}, \text{Set}_q^0) \)

\( (\text{nil}, \text{Set}_q^1) \)

\( X_q \)
Symbolic bisimulation

Definition
Let $\mathcal{S} = \{S^b : b \in B\text{Exp}\}$ be a family of equivalence relations on $SN$. $\mathcal{S}$ is called a symbolic (strong open) bisimulation if for any $b \in B\text{Exp}$, $\langle t, \mathcal{E} \rangle S^b \langle u, \mathcal{F} \rangle$ implies that

1. $q\nu(t) = q\nu(u)$ and $\mathcal{E} \sim_{q\nu(t)} \mathcal{F}$, if $b$ is satisfiable;

2. for any $\mathcal{G} \in \mathcal{S}\mathcal{O}_t(\mathcal{H}_{q\nu(t)})$, whenever $\langle t, \mathcal{G}\mathcal{E} \rangle \xrightarrow{b_1, \gamma} \Delta$ with $b\nu(\gamma) \cap f\nu(b, t, u) = \emptyset$, there exists a collection of booleans $B$ such that $b \land b_1 \rightarrow \bigvee B$ and $\forall b' \in B$, $\exists b_2, \gamma'$ with $b' \rightarrow b_2$, $\gamma = b' \gamma'$, $\langle u, \mathcal{G}\mathcal{F} \rangle \xrightarrow{b_2, \gamma'} \Xi$, and $(\mathcal{G}\mathcal{E} \bullet \Delta) S^{b'} (\mathcal{G}\mathcal{F} \bullet \Xi)$. 
Definition
A family of equivalence relations \( \{ S^b : b \in B\text{Exp} \} \) is called a symbolic ground bisimulation if for any \( b \in B\text{Exp} \), \( \langle t, E \rangle S^b \langle u, F \rangle \) implies that

1. \( \mathcal{q}_v(t) = \mathcal{q}_v(u) \) and \( E \equiv_{\mathcal{q}_v(t)} F \), if \( b \) is satisfiable,

2. whenever \( \langle t, E \rangle \xrightarrow{b_1, \gamma} \Delta \) with \( \text{bv}(\gamma) \cap \text{fv}(b, t, u) = \emptyset \), there exists a collection of booleans \( B \) such that \( b \land b_1 \rightarrow \bigvee B \) and \( \forall b' \in B, \exists b_2, \gamma' \) with \( b' \rightarrow b_2, \gamma =_{b'} \gamma' \), \( \langle u, F \rangle \xrightarrow{b_2, \gamma'} \Xi \), and \( (E \bullet \Delta) S^{b'} (F \bullet \Xi) \).
Closure under super-operator application

Definition
A relation $S$ on $SN$ is said to be closed under super-operator application if $\langle t, E \rangle S \langle u, F \rangle$ implies $\langle t, G E \rangle S \langle u, G F \rangle$ for any $G \in S'O_t(\mathcal{H}_{qv(t)})$.

Theorem
A family of equivalence relations $\{S^b : b \in BExp\}$ is a symbolic bisimulation if and only if it is both a ground bisimulation and closed under super-operator application.
Special case

**Theorem**

If $t$ and $u$ are both free of quantum input, then $(t, \mathcal{E}) \sim^b_s (u, \mathcal{F})$ if and only if $(t, \mathcal{E}) \sim^b_g (u, \mathcal{F})$. 
Symbolic bisimilarity

Theorem

1. For each $b \in BExp$, $\sim_s^b$ is an equivalence relation.
2. The family $\{\sim_s^b: b \in BExp\}$ is a symbolic bisimulation.
Symbolic vs open bisimulation

Theorem

1. \( t \sim^b_s u \) if and only if for any evaluation \( \psi \), \( \psi(b) = tt \) implies \( t\psi \sim_o u\psi \).
2. \( t \sim_s u \) if and only if \( t \sim_o u \).
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The algorithm

\[
\text{Bisim}(t, u) = \text{Match}(t, u, tt, \emptyset)
\]

\[
\text{Match}(t, u, b, W) = \text{where } t = \langle t, \mathcal{E} \rangle \text{ and } u = \langle u, \mathcal{F} \rangle
\]

if \((t, u) \in W\) then

\[
(\theta, T) := (tt, \emptyset)
\]
else

for \(\gamma \in \text{Act}(t, u)\) do

\[
(\theta_\gamma, T_\gamma) := \text{MatchAction}(\gamma, t, u, b, W)
\]
end

\[
(\theta, T) := (\bigwedge \gamma \theta_\gamma, \biguplus \gamma (T_\gamma \cup \{(t, u) \mapsto (b \land \bigwedge \gamma \theta_\gamma)\}))
\]
end

return \((\theta \land (qv(t) = qv(u)) \land (\mathcal{E} \equiv qv(t) \mathcal{F}), T)\)

\[
\text{MatchAction}(\gamma, t, u, b, W) = ...
\]

case \(\tau\)

\[
\text{for } t \xrightarrow{b_i, \tau} \Delta_i \text{ and } u \xrightarrow{b'_j, \tau} \Theta_j \text{ do}
\]

\[
(\theta_{ij}, T_{ij}) := \text{MatchDistribution}(\Delta_i, \Theta_j, b \land b_i \land b'_j, \{(t, u) \cup W\})
\]
end

return \((\bigwedge i (b_i \rightarrow \bigvee j (b'_j \land \theta_{ij})) \land \bigwedge j (b'_j \rightarrow \bigvee i (b_i \land \theta_{ij})), \biguplus ij T_{ij})\)
endsw

...

\[
\text{MatchDistribution}(\Delta, \Theta, b, W) =
\]

for \(t_i \in [\Delta] \text{ and } u_j \in \Theta \) do

\[
(\theta_{ij}, T_{ij}) := \text{Match}(t_i, u_j, b, W)
\]
end

\[
\mathcal{R} := \{(t, u) | b \rightarrow (\biguplus ij T_{ij})(t, u)\}^*\]

return \((\text{Check}(\Delta, \Theta, \mathcal{R}), \biguplus ij T_{ij})\)

\[
\text{Check}(\Delta, \Theta, \mathcal{R}) =
\]

\[
\theta := tt
\]

for \(S \in [\Delta] \cup [\Theta] / \mathcal{R} \) do

\[
\theta := \theta \land (\Delta(S) \equiv \Theta(S))
\]
end
Correctness

**Theorem**

*For two snapshots \( t \) and \( u \), the function \( \text{Bisim}(t, u) \) terminates. Moreover, if \( \text{Bisim}(t, u) = (\theta, T) \) then \( T(t, u) = \theta = mgb(t, u) \).*
Assume the ability of real computation, the worst case time complexity of executing $\text{Bisim}(t, u)$ is $O(n^5 / \log n)$. To implement the algorithm, we have to approximate super-operators using matrices of algebraic or even rational numbers, thus increase the complexity.
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Summary
Quantum while-language [Ying 2011]

- Fix the alphabet of quantum while-language: A countably infinite set \( q\text{Var} \) of quantum variables. Symbols \( q, q', q_0, q_1, q_2, \ldots \) denote quantum variables.
- Each quantum variable \( q \in q\text{Var} \) has a type \( \mathcal{H}_q \) (a Hilbert space).
- For simplicity, we only consider two basic types:

\[
\text{Boolean} = \mathcal{H}_2, \quad \text{integer} = \mathcal{H}_\infty.
\]

- A quantum register is a finite sequence \( \bar{q} = q_1, \ldots, q_n \) of distinct quantum variables. Its state Hilbert space:

\[
\mathcal{H}_{\bar{q}} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i}.
\]
Quantum programs

\[ S ::= \textbf{skip} \mid q := \vert 0 \rangle \mid \bar{q} := U[\bar{q}] \mid S_1; S_2 \]
\[ \mid \textbf{if} \ (\Box m \cdot M[\bar{q}] = m \rightarrow S_m) \ 	extbf{fi} \]
\[ \mid \textbf{while} \ M[\bar{q}] = 1 \ 	extbf{do} \ S \ 	extbf{od}. \]
Notations

- A positive operator $\rho$ is called a partial density operator if $\text{tr}(\rho) \leq 1$.
- Write $D(\mathcal{H})$ for the set of partial density operators in $\mathcal{H}$.
- Write $\mathcal{H}_{all}$ for the tensor product of the state Hilbert spaces of all quantum variables:

$$\mathcal{H}_{all} = \bigotimes_{q \in q\text{Var}} \mathcal{H}_q.$$

- Let $\bar{q} = q_1, ..., q_n$ be a quantum register. An operator $A$ in the state Hilbert space $\mathcal{H}_{\bar{q}}$ of $\bar{q}$ has a cylindrical extension $A \otimes I$ in $\mathcal{H}_{all}$.
- We will use $E$ to denote the empty program; i.e. termination.
- A configuration is a pair $\langle S, \rho \rangle$, where:
  1. $S$ is a quantum program or the empty program $E$;
  2. $\rho \in D(\mathcal{H}_{all})$, denoting the (global) state of quantum variables.
- A transition between quantum configurations:
Operational semantics (selected rules)

(SC) \[
\frac{\langle S_1, \rho \rangle \rightarrow \langle S'_1, \rho' \rangle}{\langle S_1; S_2, \rho \rangle \rightarrow \langle S'_1; S_2, \rho' \rangle}
\]

where \( E; S_2 = S_2 \).

(IF) \[
\frac{\text{if} (\Box m \cdot M[\bar{q}] = m \rightarrow S_m) \mathbf{fi}, \rho \rightarrow \langle S_m, M_m \rho M_m^\dagger \rangle}{\text{for each possible outcome } m \text{ of measurement } M = \{M_m\}.}
\]

(L0) \[
\frac{\langle \text{while } M[\bar{q}] = 1 \text{ do } S \mathbf{od}, \rho \rangle \rightarrow \langle E, M_0 \rho M_0^\dagger \rangle}{\text{(L1)}}
\]

(L1) \[
\frac{\langle \text{while } M[\bar{q}] = 1 \text{ do } S \mathbf{od}, \rho \rangle \rightarrow \langle S; \text{while } M[\bar{q}] = 1 \text{ do } S \mathbf{od}, M_1 \rho M_1^\dagger \rangle}{\text{(L1)}}
\]
Let $S$ be a quantum program. Then its semantic function

$$[S] : \mathcal{D}(\mathcal{H}_{all}) \rightarrow \mathcal{D}(\mathcal{H}_{all})$$

$$[S](\rho) = \sum |\rho' : \langle S, \rho \rangle \rightarrow^* \langle E, \rho' \rangle|$$
Quantum Predicates

- What is a quantum predicate?
- A quantum predicate should be a physical observable!
- A quantum predicate in a Hilbert space $\mathcal{H}$ is a Hermitian operator $M$ in $\mathcal{H}$ with all its eigenvalues lying within the unit interval $[0, 1]$.
- The set of predicates in $\mathcal{H}$ is denoted $\mathcal{P}(\mathcal{H})$.

Satisfaction of Quantum Predicates

- $\text{tr}(M \rho)$ may be interpreted as the degree to which quantum state $\rho$ satisfies quantum predicate $M$. 
Correctness Formulas

- A correctness formula is a statement of the form:

\[ \{P\}S\{Q\} \]

where:
- $S$ is a quantum program;
- $P, Q \in \mathcal{P}(\mathcal{H}_{all})$ are quantum predicates in $\mathcal{H}_{all}$.
- $P$ is called the precondition, $Q$ the postcondition.

Partial Correctness, Total Correctness

- Two interpretations of Hoare logical formula $\{P\}S\{Q\}$:
  - **Partial correctness**: If an input to program $S$ satisfies the precondition $P$, then either $S$ does not terminate, or it terminates in a state satisfying the postcondition $Q$.
  - **Total correctness**: If an input to program $S$ satisfies the precondition $P$, then $S$ must terminate and it terminates in a state satisfying the postcondition $Q$. 
Partial Correctness, Total Correctness (Continued)

- The correctness formula $\{P\} S \{Q\}$ is true in the sense of total correctness, written
  \[
  \models_{\text{tot}} \{P\} S \{Q\},
  \]
  if:
  \[
  tr(P\rho) \leq tr(Q\llbracket S \rrbracket(\rho))
  \]
  for all $\rho \in \mathcal{D}(\mathcal{H}_{\text{all}})$, where $\llbracket S \rrbracket$ is the semantic function of $S$.

- The correctness formula $\{P\} S \{Q\}$ is true in the sense of partial correctness, written
  \[
  \models_{\text{par}} \{P\} S \{Q\},
  \]
  if:
  \[
  tr(P\rho) \leq tr(Q\llbracket S \rrbracket(\rho)) + [tr(\rho) - tr(\llbracket S \rrbracket(\rho))]
  \]
  for all $\rho \in \mathcal{D}(\mathcal{H}_{\text{all}})$.
Hoare logic for partial correctness (selected rules)

\[(R - SC) \quad \frac{\{P\}S_1\{Q\} \quad \{Q\}S_2\{R\}}{\{P\}S_1; S_2\{R\}}\]

\[(R - IF) \quad \frac{\{P_m\}S_m\{Q\} \text{ for all } m \quad \{\sum_m M^\dagger_m P_m M_m\} \text{ if } (\varpi m \cdot M[\vec{q}] = m \rightarrow S_m) \quad \text{fi}\{Q\}}{\{P\}}\]

\[(R - LP) \quad \frac{\{Q\}S \{M^\dagger_0 PM_0 + M^\dagger_1 QM_1\}}{\{M^\dagger_0 PM_0 + M^\dagger_1 QM_1\} \text{while } M[\vec{q}] = 1 \text{ do } S \text{ od}\{P\}}\]

\[(R - Or) \quad \frac{P \sqsubseteq P' \quad \{P'\}S\{Q'\} \quad Q' \sqsubseteq Q}{\{P\}S\{Q\}}\]
Soundness Theorem
For any quantum while-program $S$ and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$:

$$
\vdash_{qPD} \{P\}S\{Q\} \text{ implies } \models_{par} \{P\}S\{Q\}.
$$

(Relative) Completeness Theorem
For any quantum while-program $S$ and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$:

$$
\models_{par} \{P\}S\{Q\} \text{ implies } \vdash_{qPD} \{P\}S\{Q\}.
$$
Theorem prover for quantum programs

- A theorem prover for quantum Hoare logic based on Isabelle/HOL has been implemented by Liu et al.
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Outline

Background

Preliminaries on quantum mechanics

Equivalences for quantum processes

Symbolic semantics

An algorithm for ground bisimulation

Hoare logic

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- Hoare logic for quantum programs
Future work

- Symbolic weak bisimulation?
- Apply the open bisimulation to analyze quantum cryptographic protocols, e.g. BB84 quantum key distribution protocol
- Model checking for quantum protocols
- Termination analysis
- Invariant generation
- Fully abstract denotational semantics
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(Incomplete) references


Thank you!