Rado Path Decomposition

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Monochromatic paths

Definition
Let $c : [\omega]^2 \to r$. A monochromatic path of color $j$ is an ordered listing (possibly finite or empty) of integers $a_0, a_1, a_2 \ldots$ such that, for all $i \geq 0$, if $a_{i+1}$ exists then $c(\{a_i, a_{i+1}\}) = j$.

An empty listing can be a path of any color. A singleton can be a path of any color. The color is determined for paths of more than one node. Paths might be finite or infinite.

3 colors is more canonical than 2 color.
Improving on a result of Erdős, Rado published a theorem which implies:

**Theorem (Rado Path Decomposition or RPD$_r$)**

Let $c : [\omega]^2 \to r$. Then, for each $j < r$, there is a monochromatic path of color $j$ such that these $r$ paths (as sets) partition $\omega$ (so they are pairwise disjoint sets and their union is everything).

Ultrafilter Proof, Part I

Let neighbors of $m$ with color $i$ be

$$N(m, i) = \{n : c(\{m, n\}) = i\}.$$

Fix a non-principal ultrafilter on $\omega$. For all $m$, for some unique $j < r$, $N(m, j)$ is large (in the ultrafilter). Let $A_j = \{m : N(m, j) \text{ is large}\}$. The $A_j$ partition $\omega$.

Think of $m \in A_j$ as having color $j$. Each $m$ has a unique color.
Ultrafilter Proof, Part II

For any pair of points $m < n$ in $A_j$, $N(m, j) \cap N(n, j)$ is large. So there are infinitely many $v \in N(m, j) \cap N(n, j)$. For all such $v$, $c(m, v) = c(v, n) = j$. Note that any such $v$ is likely much larger than $m$ and $n$ and not necessary the same color.

Stagewise build finite paths such that the current end of the path of color $j$ has color $j$ and at stage $s$ use a $v$ like above to add $s$ to the path of it’s color (the path and $s$ have the same color).
One of the $A_j$ must be large. We can thin $A_j$ to get a homogenous set of color $j$. Given $a_i \in A_j$ choose $a_{i+1}$ in $A_j \cap \bigcap_{k \leq i} N(a_k, j)$.
The existence of an ultrafilter cannot be shown in ZF. But, by independent results by Enayat, Kreuzer, and Towsner, adding a non-principal ultrafilter is conservative over $ACA_0$. So $RPD_r$ follows from $ACA_0$.

**Question**

Does $RPD_r$ imply $ACA_0$?

**Question**

Does $RPD_r$ imply $RT^2_r$?
It is well known that there is a computable linear order, 
$(\omega, <_L)$ of type of $\omega + \omega^*$ with no computable ascending or 
descending sequence. For $x < y$, color the pair $(x, y)$ red iff 
$x \leq_L y$. Blue otherwise. A computable red (blue) path is an 
ascending (descending) sequence.

This can be improved to show that there is no uniform $\Delta^0_2$
path decomposition for 2-computable colorings.

**Question**

*Is there an $r$-coloring without a $\Delta^0_2$ path decomposition?*
Recall \( N(m, i) = \{ n : c(\{m, n\}) = i \} \) is the neighbors of \( m \) with color \( i \). Let \( C \) be cohesive w.r.t. to all \( N(m, i) \), so \( C \) is infinite and, for all \( m, i \), either \( C \subseteq^* N(m, i) \) or \( C \subseteq^* N(m, i) \).

Now a set \( X \) is large iff \( C \subseteq^* X \) and repeat ultrafilter proof with this notion of largeness.
A careful analysis of the last proof shows that the path decomposition is computable in $C'$.

Why the jump? Exactly one $N(m, j)$ is large (in our cohesive set $C$). It is $\Delta^C_2$ to determine which one.

By Jockusch and Stephan, $d \gg 0'$ iff there is an $r$-cohesive set $C$ such that $C' \leq_T d$.

For computable graphs a path decomposition is computable in $d$ if $d \gg 0'$.

**Question**

*Can this be improved?*
Consider \((\tau_0, \tau_1 \ldots \tau_{r-1}, X)\) such that \(X\) is infinite and if \(\tau_j = \sigma^* m\) then \(X \subseteq^* N(m, j)\) (so \(m\) has color \(j\)) as our forcing conditions. A generic \(G\) for this forcing is a path decomposition. Forcing \(\Sigma_1^G\) statements (like does \(\Phi^G(w) \downarrow\)) is \(\Sigma_2^X\). So this forcing cannot be used for cone avoidance.
Stable Colorings

A coloring \( c \) is \textit{stable} iff for all \( m \), \( \lim_n c(m, n) \) exists. Fix a stable coloring and now let \textit{large} mean almost all and repeat our ultrafilter proof.

Stable computable colorings have \( \Delta_2^0 \) path decompositions.

A path decomposition restricted for the coloring \( c : [X]^2 \to r \) (where \( X \subset \omega \)) does not help find a path decomposition for the coloring \( c : [\omega]^2 \to r \). So COH does not help reduce the problem.
Finite versions of $\text{RPD}_r$

Pokrovskiy showed that given any $r > 2$ and $M$ there is an $r$-coloring of $[M]^2$ (this graph is just $K_M$) which does not partition into $r$ many paths, one of each color. For $r = 3$, 3 paths is enough but two of them might have the same color. $r = 2$ is special and will be dealt with shortly.

The normal proof of the finite version from the infinite version using compactness breaks down because the paths linking numbers below $M$ might also involve some very large numbers.
Proof for 2-colorings of $K_M$

Assume the colors are RED and BLUE. Inductively assume we have two paths of color RED and BLUE. Let $x$ be the least integer not in any path. Let $x_R$ be the end of RED path and similarly with $x_B$.

If there is any RED path between $x_R$ and $x$ avoiding our partially constructed paths, add that path to the end of the RED path. (Since finite, this is a computable question.) Similarly for BLUE.

Otherwise look at the color of $(x_R, x_B)$. If this is RED add $x_B, x$ (in that order) to the end of the RED path and remove $x_B$ from the end of the BLUE path. So $x_B$ switches to RED. If this is BLUE add $x_R, x$ (in that order) to the end of the BLUE path and remove $x_R$ from the end of the RED path. Since there are only finitely many $x$’s we settle on our final paths.

This proof fails for $r = 3$. 
Path Decompositions for 2-colorings

Theorem
If $c : [\omega]^2 \to 2$ is computable then there is a $\Delta^0_2$ Path Decomposition and the proof is nonuniform.
A Key Observation about Switching

Assume $x_b$ switches to RED. Only the ends of the paths switch so if $x_b$ switches again back to BLUE $x$ also must switch back to BLUE but there no BLUE path from $x_b$ to $x$. If $x_b$ switches from the blue path to the red path, it cannot switch again.

If there are infinitely many BLUE and RED switches then both paths stabilized and are infinite. If there are only finitely many switches then again the path stabilized but one might be finite.

But otherwise our algorithm breaks down. We used this failure to create another algorithm which works within the environment of this failure. Hence the end result is nonuniform.
Questions, II

Question
Let $I$ be a Turing ideal where $RPD_r$ holds. Is $0' \in I$? Does $RT^2_r$ hold in $I$? WKL? …

Question
What is the Medvedev degree of $RPD_r$? The Muchnik degree?
Take a path decomposition for an $r$-coloring. A coloring restricted to $k$ of the paths is not necessarily a $k$ coloring.

Question
Does $RPD_r$ imply $RPD_{r+1}$? Is this computably true?
Some More References

Daniel T Soukup.
Decompositions of edge-coloured infinite complete graphs into monochromatic paths II.

Henry Towsner.
Ultrafilters in reverse mathematics.

Alexey Pokrovskiy.
Partitioning edge-coloured complete graphs into monochromatic cycles and paths.