Bounds for the strength of the graph minor theorem?

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Workshop on

“New Challenges in Reverse Mathematics"

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Proof-Theoretic Ordinals

| \text{RCA}_0 | = \omega^\omega \\
| \text{WKL}_0 | = \omega^\omega \\
| \text{ACA}_0 | = \varepsilon_0 \\
| \text{ATR}_0 | = \Gamma_0 \\
| (\Pi^1_1-\text{CA})_0 | = \psi_{\Omega_1} \Omega_\omega
A *finite tree* is a finite partially ordered set

\[ \mathbb{B} = (B, \leq) \]

such that:

(i) \( B \) has a smallest element (called the *root* of \( \mathbb{B} \));
(ii) for each \( s \in B \) the set \( \{ t \in B : t \leq s \} \) is a totally ordered subset of \( B \).
Combinatorial Independence Results

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- For finite trees \( \mathbb{B}_1 \) and \( \mathbb{B}_2 \), an embedding of \( \mathbb{B}_1 \) into \( \mathbb{B}_2 \) is a one-to-one mapping

\[ f : \mathbb{B}_1 \rightarrow \mathbb{B}_2 \]

such that

\[ f(a \wedge b) = f(a) \wedge f(b) \]

for all \( a, b \in \mathbb{B}_1 \), where \( a \wedge b \) denotes the infimum of \( a \) and \( b \).
• **Kruskal’s Theorem.** For every infinite sequence of trees $(B_k : k < \omega)$, there exist $i$ and $j$ such that $i < j < \omega$ and $B_i$ is embeddable into $B_j$.
(In particular, there is no infinite set of pairwise nonembeddable trees.)
• **Kruskal’s Theorem.** For every infinite sequence of trees \( (B_k : k < \omega) \), there exist \( i \) and \( j \) such that \( i < j < \omega \) and \( B_i \) is embeddable into \( B_j \). (In particular, there is no infinite set of pairwise nonembeddable trees.)

• **Theorem (H. Friedman, D. Schmidt)** Kruskal’s Theorem is not provable in \( \text{ATR}_0 \).
• **Kruskal’s Theorem.** For every infinite sequence of trees \((B_k : k < \omega)\), there exist \(i\) and \(j\) such that \(i < j < \omega\) and \(B_i\) is embeddable into \(B_j\).

(In particular, there is no infinite set of pairwise nonembeddable trees.)

• **Theorem (H. Friedman, D. Schmidt)** Kruskal’s Theorem is not provable in \(\mathbf{ATR}_0\).

• The proof utilizes that Kruskal’s Theorem implies that \(\Gamma_0\) is well-founded.
The Extended Kruskal Theorem

• For $n < \omega$, let $\mathcal{B}_n$ be the set of all finite trees with labels from $n$, i.e. $(\mathcal{B}, \ell) \in \mathcal{B}_n$ if $\mathcal{B}$ is a finite tree and

$$\ell : \mathcal{B} \rightarrow \{0, \ldots, n - 1\}.$$ 

The set $\mathcal{B}_n$ is quasiordered by putting $(\mathcal{B}_1, \ell_1) \leq (\mathcal{B}_2, \ell_2)$ if there exists an embedding

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such that:

- $\ell_1(b) = \ell_2(f(b))$ for each $b \in B_1$;
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  The set $\mathcal{B}_n$ is quasiordered by putting $(B_1, \ell_1) \leq (B_2, \ell_2)$ if there exists an embedding
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  such that:

  - $\ell_1(b) = \ell_2(f(b))$ for each $b \in B_1$;
  - if $b$ is an immediate successor of $a \in B_1$, then for each $c \in B_2$ in the interval $f(a) < c < f(b)$,
    $$\ell_2(c) \geq \ell_2(f(b)).$$

  This condition is called a **gap condition**.
The Extended Kruskal Theorem

**Theorem.** (Friedman) For each $n < \omega$, $\mathcal{B}_n$ is a **well quasi ordering** (abbreviated $\text{WQO}(\mathcal{B}_n)$), i.e. there is no infinite set of pairwise non-embeddable trees.
Theorem. (Friedman) For each $n < \omega$, $\mathcal{B}_n$ is a well quasi ordering (abbreviated $WQO(\mathcal{B}_n)$), i.e. there is no infinite set of pairwise non-embeddable trees.

Theorem $\forall n < \omega$ $WQO(\mathcal{B}_n)$ is not provable in $\Pi^1_1$ – CA$_0$. 
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Theorem. (Friedman) For each $n < \omega$, $B_n$ is a well quasi ordering (abbreviated $WQO(B_n)$), i.e. there is no infinite set of pairwise non-embeddable trees.

Theorem $\forall n < \omega \ WQO(B_n)$ is not provable in $\Pi^1_1 - CA_0$.

- The proof employs an ordinal representation system for the proof-theoretic ordinal of $\Pi^1_1 - CA_0$. The ordinal is $\psi_{\Omega_1}(\Omega_\omega)$. 
**The Graph Minor Theorem**

- **G**, **H** graphs. **H** is a **minor** of **G** if **H** is obtained from **G** by a succession of the following operations:
  (a) single-edge deletion
  (b) removal of an isolated vertex
  (c) single-edge contraction.

If only steps (a) and (b) are applied, **H** is a **subgraph** of **G**.
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**GMT Theorem.** (Robertson and Seymour 1986-1997 (resp. 2004)) If \( \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots \) is an infinite sequence of finite graphs, then there exist \( i < j \) so that \( \mathcal{G}_i \) is isomorphic to a minor of \( \mathcal{G}_j \).
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GMT Theorem. (Robertson and Seymour 1986-1997 (resp. 2004)) If G_0, G_1, G_2, ... is an infinite sequence of finite graphs, then there exist i < j so that G_i is isomorphic to a minor of G_j.

- Corollary. (Vázsonyi’s conjecture) All G_k trivalent, then G_i is embeddable into G_j for some i < j.
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- Corollary. (Wagner’s conjecture) For any 2-manifold \( M \) there are only finitely many graphs which are not embeddable in \( M \) and are minimal with this property.
Forbidden Minors

If $H$ is a class of graphs, then the class $\text{Forb} \leq (H) := \{G | G \preceq H \text{ for all } H \in H\}$ of all graphs without a minor in $H$ is a graph property, and it is closed under taking minors.

**Proposition.** A graph property $P$ can be expressed by forbidden minors if and only if it is closed under taking minors.

**Proof.** $P = \text{Forb} \leq \bar{P}$, where $\bar{P}$ is the complement of $P$.

**Proposition.** Let $P$ be a minor closed graph property. There is a unique smallest set of forbidden minors, $K_P := \{H | H$ is $\leq$-minimal in $\bar{P}\}$, called the Kuratowski set for $P$, such that $P = \text{Forb}(K_P)$.
Forbidden Minors

If $\mathcal{H}$ is a class of graphs, then the class

$$\text{Forb}_{\leq}(\mathcal{H}) := \{ G \mid G \not\cong H \text{ for all } H \in \mathcal{H} \}$$

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**Proposition.** A graph property $\mathcal{P}$ can be expressed by forbidden minors if and only if it is closed under taking minors.

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Forbidden Minors

If $\mathcal{H}$ is a class of graphs, then the class

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**Proposition.** Let $\mathcal{P}$ be a minor closed graph property. There is a unique smallest set of forbidden minors,

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called the Kuratowski set for $\mathcal{P}$, such that $\mathcal{P} = \text{Forb}(\mathcal{K}_\mathcal{P})$. 
Proposition (Kuratowski 1930)

Planar graphs = \text{Forb}\{K^5, K_{3,3}\}.
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Corollary. For every minor closed graph property \(\mathcal{P}\), membership in \(\mathcal{P}\) can be decided by an algorithm in polynomial (even cubic) time.
Tree-decomposition

A tree-decomposition of a graph $G$ is a pair $(T, (V_t)_{t \in T})$ that satisfies three conditions:

(T1) $T$ is a tree and $V(G) = \bigcup_{t \in T} V_t$;

(T2) for every edge $e \in G$ there exists a $t \in T$ such that both ends of $e$ lie in $V_t$;

(T3) $V_a \cap V_b \subseteq V_c$ whenever $a, b, c \in T$ and $c$ lies on the path from $a$ to $b$.

The width of $(T, (V_t)_{t \in T})$ is the number $\max\{|V_t| - 1 : t \in T\}$.

The tree-width $tw(G)$ is the least width of any tree-decomposition of $G$.

Remark: Trees have tree-width 1.
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The Bounded Graph Minor Theorem, BGMT

Theorem. (Robertson & Seymour 1990, Graph Minors IV)

For every integer \( k > 0 \), the graphs of tree-width < \( k \) are well-quasi-ordered by the minor relation.

Idea: Graphs of bounded tree-width are sufficiently similar to trees so that an "iteration of the minimal bad sequence argument" (around \( tw(G) \) times) works.

Theorem. (RCA\(_0\))

For each \( k \), the graph minor theorem for tree-width \( \leq 12 \) \( (k + 1) \) implies Friedman's extended Kruskal theorem for \( k \) labels.
Theorem. (Robertson & Seymour 1990, Graph Minors IV) For every integer $k > 0$, the graphs of tree-width $< k$ are well-quasi-ordered by the minor relation.
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**Theorem.** (\( \text{RCA}_0 \)) For each \( k \), the graph minor theorem for tree-width \( \leq 12(k + 1) \) implies Friedman’s extended Kruskal theorem for \( k \) labels.
Theorem. TFAE over $\text{ACA}_0$

(a) BGMT,
(b) Friedman's extended Kruskal theorem EKT,
(c) the well-foundedness of $\psi_{\Omega^1_{\omega}}$,
(d) every $\Pi^1_1$-sentence provable in $\Pi^1_1$-$\text{CA}_0$ is true.

Remark. EKT and BGMT are provable in $\Pi^1_1$-$\text{CA}_0$ plus $\Pi^1_2$-induction on naturals.
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Remark. EKT and BGMT are provable in $\Pi^1_1$-$\text{CA}_0$ plus $\Pi^1_2$-induction on naturals.
Lemma A. \( (\Pi_1 \text{-CA}_0 + \text{BGMT}) \) For every surface \( S \) there exists a finite set of graphs \( H_1, ..., H_n \) such that a graph is embeddable in \( S \) if and only if it contains none of \( H_1, ..., H_n \) as minors. Proof uses \( \text{BGMT} \).

Lemma B. \( (\Pi_1 \text{-CA}_0 + \text{BGMT}) \) Given any surface \( S \), all graphs embeddable in \( S \) are well-quasi-ordered by the minor relation. Proof uses induction on the Euler genus of \( S \).
Lemma A. \((\Pi_1^1\text{-CA}_0 + \text{BGMT})\) For every surface \(S\) there exists a finite set of graphs \(H_1, \ldots, H_n\) such that a graph is embeddable in \(S\) if and only if it contains none of \(H_1, \ldots, H_n\) as minors.

Proof uses BGMT.
**Towards GMT**

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Proof uses BGMT.

**Lemma B.** \((\Pi^1_1\text{-CA}_0 + \text{BGMT})\) Given any surface \(S\), all graphs embeddable in \(S\) are well-quasi-ordered by the minor relation.

Proof uses **induction on the Euler genus** of \(S\).
The Final Step

Suppose $G_0, G_1, G_2, \ldots$ is an infinite sequence of graphs and $G_0$ is not a minor of any $G_i$ with $i > 0$.

If $G_0$ is planar then there exists $k$ such that $tw(G_i) < k$ for all $i > 0$. So done by BGMT.

Since $G_0$ is a minor of $K_n$ for some $n$. It suffices to look at the cases $G_0 = K_n$.

Theorem. (Robertson & Seymour 2003, Graph Minors XVIII).

Roughly speaking, for every $n$ there exists a finite set $S$ of surfaces such that every graph without $K_n$ as a minor has a tree-decomposition into parts each 'nearly' embeddable in some $S \in S$.

Final step: Prove that the set of all the parts in these tree-decompositions is a wqo.
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- If $G_0$ is planar then there exists $k$ such that $\text{tw}(G_i) < k$ for all $i > 0$. So done by BGMT.
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"Bounds for the strength of the graph minor theorem?"
The Final Step

- Suppose \( G_0, G_1, G_2, \ldots \) is an infinite sequence of graphs and \( G_0 \) is not a minor of any \( G_i \) with \( i > 0 \).
- If \( G_0 \) is planar then there exists \( k \) such that \( \text{tw}(G_i) < k \) for all \( i > 0 \). So done by BGMT.
- Since \( G_0 \) is a minor of \( K^n \) for some \( n \). It suffices to look at the cases \( G_0 = K^n \).

**Theorem.** (Robertson & Seymour 2003, Graph Minors XVIII).
Roughly speaking, for every \( n \) there exists a finite set \( S \) of surfaces such that every graph without \( K^n \) as a minor has a tree-decomposition into parts each ‘nearly’ embeddable into in some \( S \in S \).

**Final step:** Prove that the set of all the parts in these tree-decompositions is a wqo.
"By a generalization of Theorem 12.3.7 [BGMT]–and hence of Kruskal’s theorem– it now suffices, essentially to prove that the set of all the parts in these tree-decompositions is well-quasi ordered: then the graphs decomposing into these parts are well-quasi-ordered, too."
Graph Minors. XIX.

• A surface $S$ is a compact, connected 2-manifold with (possibly null) boundary $\partial(S)$. The components of $\partial(S)$ are called cuffs.

• Main Theorem

For every surface $S$ and fixed wqo $Q$, if $H_1, H_2, H_3, \ldots$ is an infinite sequence of hypergraphs drawn in $S$ where each edge has 2 or 3 ends, and $\phi_i : E(H_i) \to Q$ is some function from the edges of $H_i$ to $Q$, then there exist $1 \leq i < j$ such that $H_i$ is a minor of $H_j$ and $\phi_i(e) \leq Q \phi_j(e)$ for all edges $e$ of $H_i$.

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**Bounds for the strength of the graph minor theorem?**

- Here a surface \( S \) is a compact, connected 2-manifold with (possibly null) boundary \( \partial(S) \). The components of \( \partial(S) \) are called cuffs.

- **Main Theorem** For every surface \( S \) and fixed wqo \( Q \), if \( H_1, H_2, H_3, \ldots \) is an infinite sequence of hypergraphs drawn in \( S \) where each edge has 2 or 3 ends, and \( \varphi_i : E(H_i) \to Q \) is some function from the edges of \( H_i \) to \( Q \), then there exist \( 1 \leq i < j \) such that \( H_i \) is a minor of \( H_j \) and \( \varphi_i(e) \leq_Q \varphi_j(e) \) for all edges \( e \) of \( H_i \).
Graph Minors XIX: Grand Induction

Main induction on the complexity of the surface $S$, that is the number of its handles and crosscaps. So a surface $S'$ is simpler if it has fewer handles and crosscaps, "even if its boundary has more cuffs and the wqo is bigger".

This is in effect an induction on natural numbers. Because of the quantification over wqo's the induction predicate is of complexity $\Pi^1_2$, thus beyond $\Pi^1_1 = CA^0_0$. 

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• This is in effect an induction on natural numbers. Because of the quantification over wqo’s the induction predicate is of complexity $\Pi^1_2$, thus beyond $(\Pi^1_1 - \text{CA})_0$. 

**Bounds for the strength of the graph minor theorem?**
Subsidiary induction

• Second, for a fixed number of handles and crosscaps, we assume the result for sets of labelled hypergraphs in which the labels of the "internal" edges (that is, not on the boundary of $S$) all come from some proper subideal of our well-quasi-order, even if the labels on the boundary come from some larger well-quasi-order. This is an induction along a wqo. The induction predicate, however, contains a quantification over wqo's and again this is $\Pi_2$, thus even beyond $(\Pi_1 - \text{CA})$. This is covered, though, by what's called $\Pi_1$ induction.

• Third, we proceed by induction on the number of cuffs, and there is a fourth of the same kind.
“Second, for a fixed number of handles and crosscaps, we assume the result for sets of labelled hypergraphs in which the labels of the “internal” edges (that is, not on the boundary of $S$) all come from some proper subideal of our well-quasi-order, even if the labels on the boundary come from some \textit{larger well-quasi-order}.”
Subsidiary induction

- “Second, for a fixed number of handles and crosscaps, we assume the result for sets of labelled hypergraphs in which the labels of the “internal” edges (that is, not on the boundary of \( S \)) all come from some proper subideal of our well-quasi-order, even if the labels on the boundary come from some larger well-quasi-order.”

- This is an induction along a wqo. The induction predicate, however, contains a quantification over wqo’s and again this is \( \Pi^1_2 \), thus even beyond \( (\Pi^1_1 \rightarrow \text{CA}) \). This is covered, though, by what’s called \( \Pi^1_2 \) bar induction.
Subsidiary induction

- “Second, for a fixed number of handles and crosscaps, we assume the result for sets of labelled hypergraphs in which the labels of the “internal” edges (that is, not on the boundary of $S$) all come from some proper subideal of our well-quasi-order, even if the labels on the boundary come from some larger well-quasi-order.”

- This is an induction along a wqo. The induction predicate, however, contains a quantification over wqo’s and again this is $\Pi^1_2$, thus even beyond ($\Pi^1_1$–$\text{CA}$). This is covered, though, by what’s called $\Pi^1_2$ bar induction.

- “Third, we proceed by induction on the number of cuffs, and there is a fourth of the same kind.”
“To make this induction work, we find it necessary to divide the boundary of our surface into segments, and then have different restrictions on the labels of edges bordering each segment; and also, some edges drawn on the boundary have to be regarded as fixed."
Bounds for the strength of the Graph Minor Theorem
Bounds

\[ |\Pi_1^1 - \mathbf{CA}_0| = \psi_{\Omega_1}(\Omega_\omega). \]
Bounds

\[ |\Pi_1^1 - \text{CA}_0| = \psi_{\Omega_1}(\Omega \omega). \]
\[ |\Pi_1^1 - \text{CA}_0 + \Pi_2^1\text{-IND}| = \psi_{\Omega_1}(\Omega_\omega \cdot \omega^\omega). \]
Bounds

\[ |\Pi_1^1 - CA_0| = \psi_{\Omega_1}(\Omega_\omega). \]
\[ |\Pi_1^1 - CA_0 + \Pi_2^1 - \text{IND}| = \psi_{\Omega_1}(\Omega_\omega \cdot \omega^\omega). \]
\[ |(\Pi_1^1 - CA)| = \psi_{\Omega_1}(\Omega_\omega \cdot \varepsilon_0). \]
Bounds for the strength of the graph minor theorem?

Bounds

\[ |\Pi_1^1 - CA_0| = \psi_{\Omega_1}(\Omega_\omega). \]
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Bounds

\[ |\Pi_1^1 - \text{CA}_0| = \psi_{\Omega_1}(\Omega_\omega). \]
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\[ |(\Pi_1^1 - \text{CA})| = \psi_{\Omega_1}(\Omega_\omega \cdot \varepsilon_0). \]
\[ |\Pi_1^1 - \text{CA}_0 + \Pi_2^1 - \text{BI}| = \psi_{\Omega_1}(\Omega_\omega^\omega). \]
\[ \psi_{\Omega_1}(\Omega_\omega) \leq \text{ordinal of GMT} < \psi_{\Omega_1}(\Omega_\omega^\omega). \]
The End

Thanks!