Higher reverse mathematics

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January 6, 2016
1 Background on higher reverse mathematics

2 Determinacy principles

3 Further $\mathsf{ATR}_0$ variants

4 Choice principles
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4 Choice principles
Some history

- Harnik (1987) introduces conservative extension of $RCA_0$ for studying reverse mathematics of stability theory
- Kohlenbach (2001) introduces $RCA_0$
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- Kohlenbach (2001) introduces $RCA_0^\omega$

Since then, substantial work has been done in the system $RCA_0^\omega$:

- Uniform versions of classical principles (Kohlenbach, Sakamoto/Yamazaki, Sanders)
- Topology and measure theory (Hunter, Kreuzer)
- Ultrafilters (Kreuzer, Towsner*)
- Interactions with NSA (Sanders)
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Today: interactions between higher reverse math and set theory (S., Hachtman)
Theorem (Grilliot’s Trick)

The following are “effectively equivalent”:

- The jump functional $x \mapsto x'$ exists.
- “Uniform Weak Konig’s Lemma”: There is a functional $F$ such that, if $T$ is an infinite binary tree, then $F(T)$ is a path through $T$.

“Proof”.

Let

- $T^0_n = \{ \sigma \in 2^\omega : (\forall i(\sigma(i) = 0)) \lor (|\sigma| < n \land \forall i(\sigma(i) = 1))\}$
- $T^1_n = \{ \sigma \in 2^\omega : (\forall i(\sigma(i) = 0)) \lor (|\sigma| < n \land \forall i(\sigma(i) = 1))\}$
- $T_\infty = \{ \sigma \in 2^\omega : \forall i, j(\sigma(i) = \sigma(j))\}$.

Then either $F(T_\infty)$ goes left (zero) or right (one).
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Then either $F(T_\infty)$ goes left (zero) or right (one). Suppose $F(T_\infty)$ goes left. Then, given real $x$ and natural $e$, let $T_{x,e}$ consist of “all ones” branch $\uparrow$ every all-zeros node of length $s$ such that $\varphi^x_e(e)[s] \uparrow$. 

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Then either $F(T_\infty)$ goes left (zero) or right (one). Suppose $F(T_\infty)$ goes left. Then, given real $x$ and natural $e$, let $T_{x,e}$ consist of “all ones” branch + every all-zeroes node of length $s$ such that $\varphi_{e}(x)[s] \uparrow$. Now ask, “$F(T_{x,e})(0) = ?$”
Kohlenbach: introduced $RCA_0^\omega$, a conservative extension of $RCA_0$ for all finite types. Different appearance from $RCA_0$. 
S.: base theory $RCA_0^3$ for types 0, 1, 2; similar form to $RCA_0$. 
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**Proposition (S.)**

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**Proposition (S.)**

$RCA_0^\omega$ is a conservative extension of $RCA_0^3$.

- Language: arithmetic, application symbols "$F(x)"$, and **coding**: \(\triangle, \ast\)
- Ordered semiring axioms + $\Sigma^0_1$ induction
- $\Delta^0_1$-comprehension for reals and functionals (with arbitrary parameters) in the language
- Coding operations defined as:
  - $n\triangle(a_0, a_1, a_2, \ldots) = (n, a_0, a_1, \ldots)$,
  - $F \ast r = (F(0\triangle r), F(1\triangle r), F(2\triangle r), \ldots)$
Δ₁⁰ comprehension

RCA₀³: ordered semiring axioms, \( \Sigma^0_1 \) induction, extensionality, and versions of \( \Delta^0_1 \)-comprehension for reals and functionals in the language of third-order arithmetic + "coding operations"

- "\( \Sigma^0_1 \)" has usual meaning: existential quantifier over naturals, matrix has bounded quantifiers over naturals only (and equality for naturals only)
- A \( \Delta^0_1 \)-definition of a real is a \( \Sigma^0_1 \) formula \( \varphi(x^\mathbb{N}, y^\mathbb{N}) \) such that
  \[
  \forall x \exists! y \varphi(x, y).
  \]
- A \( \Delta^0_1 \)-definition of a functional is a \( \Sigma^0_1 \) formula \( \varphi(x^\mathbb{R}, y^\mathbb{N}) \) such that
  \[
  \forall x \exists! y \varphi(x, y).
  \]

Note: arbitrary type parameters are allowed in \( \Sigma^0_1 \) formulas.
Models of $RCA_0^3$

A model of $RCA_0^3$ has form

$$(Nat, Rea, Fun; +, \times, 0, 1, \wedge, *, app)$$

$\wedge$ and $*$ are coding operations

$app$ is application — “$F(x)$” shorthand for "app($F, x$)"

Here: $\omega$-models only, so a model of $RCA_0^3$ is specified by $Rea$ and $Fun$.

Example

If $X \subseteq \omega^\omega$ is a Turing ideal, there is a smallest model of $RCA_0^3$ with second-order part $X$: $$(\omega, X, \{s \mapsto \Phi_t \oplus s \in X, \Phi_t \oplus e\}$$

Example

Other models: $(\omega, R, \text{continuous functions})$ and $(\omega, R, \text{Borel functions})$
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Example

If $X \subseteq \omega^\omega$ is a Turing ideal, there is a smallest model of $RCA^3_0$ with second-order part $X$:

$$(\omega, X, \{s \mapsto \Phi_e^{t+s} : t \in X, \Phi_e^{t+s} \text{ total on } X\})$$

Example

Other models: $(\omega, \mathbb{R}, \text{continuous functions})$ and $(\omega, \mathbb{R}, \text{Borel functions})$
Background on higher reverse mathematics

Determinacy principles

Further $\text{ATR}_0$ variants

Choice principles
Every clopen game on $\omega$ has (relatively) hyperarithmetic winning strategy. Fails for open games.
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**Theorem (Steel)**

Over $\text{RCA}_0$, the following are equivalent:

- **Open determinacy.**
- **Clopen determinacy.**

Open and clopen determinancy are equivalent because “clopen” is $\Pi^1_1$-complete — more complex than principles involved
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Open and clopen determinancy are equivalent because “clopen” is $\Pi^1_1$-complete — more complex than principles involved

Question

Is this the only reason?
Determinacy on reals

Since \( \omega \)-sequences of reals can be coded by reals, “\( T \subseteq (\omega^\omega)^{<\omega} \) is well-founded” is \( \Pi^1_1 \).

**Definition**

- **Open determinacy for reals** (\( \Sigma^R_1 \)-Det): “Any open game of length \( \omega \) on \( \mathbb{R} \) is determined.” (Game tree \( \subseteq \mathbb{R}^{<\omega} \), I wins iff play leaves tree.)

- **Clopen determinacy for reals** (\( \Delta^R_1 \)-Det): “Any clopen game of length \( \omega \) on \( \mathbb{R} \) is determined.” (Game tree \( \text{wellfounded} \subseteq \mathbb{R}^{<\omega} \), first to leave tree loses.)
Determinacy on reals

Since $\omega$-sequences of reals can be coded by reals, “$T \subseteq (\omega^\omega)^{<\omega}$ is well-founded” is $\Pi^1_1$.

**Definition**

- Open determinacy for reals ($\Sigma^R_1$-Det): “Any open game of length $\omega$ on $\mathbb{R}$ is determined.” (Game tree $\subseteq \mathbb{R}^{<\omega}$, I wins iff play leaves tree.)
- Clopen determinacy for reals ($\Delta^R_1$-Det): “Any clopen game of length $\omega$ on $\mathbb{R}$ is determined.” (Game tree *wellfounded* $\subseteq \mathbb{R}^{<\omega}$, first to leave tree loses.)

**Theorem (S.)**

*Over $\text{RCA}_0^3$, $\Delta^R_1$-Det is strictly weaker than $\Sigma^R_1$-Det.*

Uses nontrivial countably closed higher-type forcings — counterpart of complexity of “clopenness” at second-order

Shortly afterwards: Hachtman, via analysis of Goedel’s $L$ (see later)
Let $\alpha$ be an ordinal.

- $C_{\alpha}$ is clopen game “walk down $\alpha$”: Players I and II (independently) build decreasing sequences in $\alpha$; first who cannot play, loses.
- $O_{\alpha}$ is open game “play $C_{\alpha}$ until I wins”: Players I and II play $\omega$-many games of $C_{\alpha}$ (in sequence). II wins iff she wins every game.

We let $T \subseteq (c^+)^{<\omega}$ be game tree for $O_{c^+}$.

Actually, right game is slight variation on this.
The separating model

- Start with ground model $V \models ZFC$
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- Force: generic copy $G$ of $\mathbb{T}$ as tree on reals \textbf{(countably closed)}
- Model $M$: Closure of $G$ (+ ground functionals) under “$(< c^+)$-many jumps” (a la Steel forcing)
The separating model

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The separating model

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- Properties:
  - $RCA_0^3$, "$\mathcal{G}$ is undetermined" are easy
The separating model

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  - Clopen games of rank $< c^+$ determined via ranking argument
The separating model

- Start with ground model $V \models ZFC$
- Force: generic copy $\mathcal{G}$ of $T$ as tree on reals (**countably closed**)
- Model $M$: Closure of $\mathcal{G}$ (+ ground functionals) under "($< c^+$)-many jumps" (a la Steel forcing)
- Properties:
  - $RCA_0^3$, "$\mathcal{G}$ is undetermined" are easy
  - Clopen games of rank $< c^+$ determined via ranking argument
  - Countable closure: no clopen games of rank $\geq c^+$ in $M$
Subsequent analysis

Shortly afterwards, Sherwood Hachtman drew a connection with his work on the constructible universe $L$:  

**Definition (Hachtman)**

θ is the least ordinal such that $L_\theta$ satisfies: “$\mathcal{P}(\omega)$ exists and for every height-$\omega$ tree $T$ with no path, there is $\rho : T \rightarrow ON$ such that $x \supseteq y \implies \rho(x) < \rho(y)$.”

**Theorem (Hachtman)**

$(\omega, \omega^\omega \cap L_\theta, \omega^\omega \cap L_\theta)$ separates clopen and open determinacy for reals.

That is, Hachtman found a set-theoretic canonical model of the separation. This θ is also connected to $\Sigma^0_4$ determinacy on naturals, and reflection principles.

**Question**

Are there other canonical models? (Hyperanalytic functionals?)
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Definition

- **TR**: $\Sigma^1_1$ recursion along a well-ordering with domain $\subseteq \mathbb{R}$.
- **RR**: $\Sigma^1_1$ recursion along a well-founded tree with domain $\subseteq \mathbb{R}$.

Definition

- **WO**: The reals are well-orderable. (Role: Kleene-Brouwer ordering of tree)
- **SF**: Real-indexed families of nonempty sets of reals have choice functionals. (Role: quasistrategy $\rightarrow$ strategy)

Proposition (S.)

Over $RCA_0^3$, we have:

- **RR + SF** is equivalent to clopen determinacy for reals.
- **TR + WO + SF** implies clopen determinacy for reals.
\( \Sigma^2_1 \)-Separation

**Definition**

\( \Sigma^2_1 \)-Sep is the statement: “Given \( \varphi, \psi \in \Sigma^2_1 \), if at most one holds for each real \( r \), then have separating functional.”

**Proposition (S.)**

*Over \( RCA_0^3 + SF \), \( \Sigma^2_1 \)-Sep implies clopen determinacy for reals.*
Σ²¹-Separation

Definition

Σ²¹-Sep is the statement: “Given φ, ψ ∈ Σ²¹, if at most one holds for each real r, then have separating functional.”

Proposition (S.)

Over RCA³₀ + SF, Σ²¹-Sep implies clopen determinacy for reals.

Proof sketch.

Suppose G is a clopen game. For each node σ ∈ G, at most one of the following hold:

- There is a witness to σ being a win for player I.
- There is a witness to σ being a win for player II.

Applying Σ²¹-Sep yields a winning quasistrategy.
**Definition**

$\Sigma^2_1$-Sep is the statement: “Given $\varphi, \psi \in \Sigma^2_1$, if at most one holds for each real $r$, then have separating functional.”

**Proposition (S.)**

Over $\text{RCA}_0^3 + \text{SF}$, $\Sigma^2_1$-Sep implies clopen determinacy for reals.

**Proof sketch.**

Suppose $G$ is a clopen game. For each node $\sigma \in G$, at most one of the following hold:

- There is a witness to $\sigma$ being a win for player I.
- There is a witness to $\sigma$ being a win for player II.

Applying $\Sigma^2_1$-Sep yields a winning quasistrategy. . . after analysis.

What is the relationship between $\Sigma^2_1$-Sep and open determinacy for reals?
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4 Choice principles
Comparing choice principles, I/II

Two natural choice principles:

\[ SF \]: 'Every family \( S_r (r \in \mathbb{R}) \) of nonempty sets of reals has a choice function'

\[ WO \]: 'The reals are well-orderable'

Proposition \( SF \) does not imply \( WO \) over \( RCA_0^3 \).

Proofs.

In \( ZF + AD_R \), projective functionals give separating model.

Over \( ZF \), Truss 1978 provided a forcing argument.

Over \( RCA_0^3 \), set of continuous functionals is model of \( SF + \neg WO \).

What about other direction? Note that choice functions are definable from a well-ordering...
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**Proposition**

$SF$ does not imply $WO$ over $\text{RCA}_0^3$.

**Proofs.**
- In $ZF + AD_{\mathbb{R}}$, projective functionals give separating model
- Over $ZF$, Truss 1978 provided a forcing argument
- Over $\text{RCA}_0^3$, set of continuous functionals is model of $SF + \neg WO$
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**Proofs.**

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- Over \( RCA_0^3 \), set of continuous functionals is model of \( SF + \neg WO \)

What about other direction? Note that choice functions are *definable* from a well-ordering . . .
Two natural choice principles:

- **SF** = “Every family $S_r \ (r \in \mathbb{R})$ of nonempty sets of reals has a choice function”
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**Theorem (S.)**

Over $\text{RCA}_0^3$, WO does not imply SF.
Comparing choice principles, II/II

Two natural choice principles:
- $SF = \text{“Every family } S_r \ (r \in \mathbb{R}) \text{ of nonempty sets of reals has a choice function”}$
- $WO = \text{“The reals are well-orderable”}$

Theorem (S.)

Over $RCA^3_0$, $WO$ does not imply $SF$.

Proof sketch.
- Force with countable partial injections $\mathbb{R} \rightarrow \omega_1$; call generic induced ordering “$\prec_G$.”
- Take functionals which are definable from $\prec_G$ via truth tables of “countable depth”
- Let $S_r = \{ s : r \prec_G s \}$. This family has no choice function.
Thanks!