Reverse mathematics and the strong Tietze extension theorem

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New Challenges in Reverse Mathematics
Singapore
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A conjecture of Giusto & Simpson

Conjecture (Giusto & Simpson)

The following are equivalent over RCA₀:

(1) WKL₀.

(2) Let \( \hat{X} \) be a compact complete separable metric space, let \( C \) be a closed subset of \( \hat{X} \), and let \( f : C \to \mathbb{R} \) be a continuous function with a modulus of uniform continuity. Then there is a continuous function \( F : \hat{X} \to \mathbb{R} \) with a modulus of uniform continuity such that \( F \upharpoonright C = f \).

(3) Same as (2) with ‘closed’ replaced by ‘closed and separably closed.’

(4) Special case of (2) with \( \hat{X} = [0, 1] \).

(5) Special case of (3) with \( \hat{X} = [0, 1] \).

Let \( \text{sTET}_{[0,1]} \) denote statement (5).
Definitions in RCA\(_0\) (metric spaces)

Let’s remember what all the words in the conjecture mean in RCA\(_0\).

A **real number** is coded by a sequence \(\langle q_k : k \in \mathbb{N} \rangle\) of rationals such that \(\forall k \forall i (|q_k - q_{k+i}| \leq 2^{-k})\).

A **complete separable metric space** \(\hat{A}\) is coded by a non-empty set \(A\) and a metric \(d : A \times A \to \mathbb{R}^\geq\).

A **point** in \(\hat{A}\) is coded by a sequence \(\langle a_k : k \in \mathbb{N} \rangle\) of members of \(A\) such that \(\forall k \forall i (d(a_k, a_{k+i}) \leq 2^{-k})\).

A complete separable metric space \(\hat{A}\) is **compact** if there are finite sequences \(\langle \langle x_{i,j} : j \leq n_i \rangle : i \in \mathbb{N} \rangle\) with each \(x_{i,j} \in \hat{A}\) such that

\[
(\forall z \in \hat{A})(\forall i \in \mathbb{N})(\exists j \leq n_i)(d(x_{i,j}, z) < 2^{-i}).
\]
The interval [0, 1] is a complete separable metric space coded by the set 
\( \{ q \in \mathbb{Q} : 0 \leq q \leq 1 \} \) (with the usual metric).

The sequence \( \langle \langle j2^{-i} : j \leq 2^i \rangle : i \in \mathbb{N} \rangle \) witnesses that [0, 1] is compact 
according to the definition on the previous slide.

So RCA\(_0\) proves that [0, 1] is a compact complete separable metric space.

Contrast this to the following facts (Friedman):

- The Heine-Borel compactness of [0, 1] is equivalent to WKL\(_0\) over 
  RCA\(_0\).

- The sequential compactness of [0, 1] is equivalent to ACA\(_0\) over 
  RCA\(_0\).
Definitions in RCA$_0$ (closed and separably closed)

An open set in a metric space $\hat{A}$ is coded by a set $U \subseteq \mathbb{N} \times A \times \mathbb{Q}^{>0}$ (thought of as an enumeration of open balls).

A point $x \in \hat{A}$ is in the open set coded by $U$ if $(\exists \langle n, a, r \rangle \in U)(d(x, a) < r)$.

A closed set in a metric space is the complement of an open set.

A separably closed set in a metric space $\hat{A}$ is coded by a sequence $\langle x_n : n \in \mathbb{N} \rangle$ of points in $\hat{A}$. A point $x \in \hat{A}$ is in the separably closed set if $(\forall q \in \mathbb{Q}^{>0})(\exists n \in \mathbb{N})(d(x, x_n) < q)$.

In RCA$_0$, a closed set need not be separably closed, and a separably closed set need not be closed.

In ACA$_0$, a subset of a compact metric space is closed if and only if it is separably closed. Both implications require ACA$_0$ (Brown).
Definitions in RCA\(_0\) (continuous functions)

A **continuous partial function** from a metric space \(\widehat{A}\) to a metric space \(\widehat{B}\) is coded by a set \(\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^{>0} \times B \times \mathbb{Q}^{>0}\) (thought of as an enumeration of pairs of open balls \(\mathcal{B}(a, r)\) and \(\mathcal{B}(b, s)\)).

If the pair \(\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle\) is enumerated, it means that every element of \(\mathcal{B}(a, r)\) is mapped into the closure of \(\mathcal{B}(b, s)\).

The enumeration must satisfy:

- If \(\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle\) and \(\langle \mathcal{B}(a, r), \mathcal{B}(b', s') \rangle\) are enumerated, then \(\mathcal{B}(b, s) \cap \mathcal{B}(b', s') \neq \emptyset\).
- \(\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle\) is enumerated and \(\mathcal{B}(a', r') \subseteq \mathcal{B}(a, r)\), then \(\langle \mathcal{B}(a', r'), \mathcal{B}(b, s) \rangle\) is enumerated.
- \(\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle\) is enumerated and \(\mathcal{B}(b, s) \subseteq \mathcal{B}(b', s')\), then \(\langle \mathcal{B}(a, r), \mathcal{B}(b', s') \rangle\) is enumerated.
Definitions in RCA₀ (continuous functions)

A point \( x \in \hat{A} \) is in the domain of the function coded by \( \Phi \) if

\[(\forall \epsilon > 0)(\Phi \text{ lists some } \langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle \text{ with } x \in \mathcal{B}(a, r) \text{ and } s < \epsilon), \]

in which case the value of the function at \( x \) is the \( y \in \hat{B} \) such that \( y \in \mathcal{B}(b, s) \) for every enumerated \( \langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle \) with \( x \in \mathcal{B}(a, r) \).

Today we mostly care about functions that are piecewise constant and whose domains are unions of disjoint closed intervals.
Definitions in RCA\(_0\) (modulus of uniform continuity)

A **modulus of uniform continuity** for a continuous function \( f : \hat{A} \to \hat{B} \) is a function \( h : \mathbb{N} \to \mathbb{N} \) such that

\[
(\forall n \in \mathbb{N})(\forall x, y \in \hat{A})(d(x, y) < 2^{-h(n)} \to d(f(x), f(y)) < 2^{-n}).
\]

Over RCA\(_0\), the following are equivalent (Brown, Simpson):

- WKL\(_0\).
- Every continuous function on a compact complete separable metric space has a modulus of uniform continuity.
- Every continuous function on \([0, 1]\) has a modulus of uniform continuity.

Also, in RCA\(_0\), a continuous function \( f : [0, 1] \to \mathbb{R} \) has a modulus of uniform continuity if and only if it has a Weierstraß approximation (more on this later).
Remember the conjecture?

**Conjecture (Giusto & Simpson)**

The following are equivalent over RCA₀:

1. WKL₀.

2. Let \( \hat{X} \) be a compact complete separable metric space, let \( C \) be a closed subset of \( \hat{X} \), and let \( f : C \to \mathbb{R} \) be a continuous function with a modulus of uniform continuity. Then there is a continuous function \( F : \hat{X} \to \mathbb{R} \) with a modulus of uniform continuity such that \( F \upharpoonright C = f \).

3. Same as (2) with ‘closed’ replaced by ‘closed and separably closed.’

4. Special case of (2) with \( \hat{X} = [0, 1] \).

5. Special case of (3) with \( \hat{X} = [0, 1] \) (sTET\([0,1]\)).

Need (1) \( \Rightarrow \) (2) and sTET\([0,1]\) \( \Rightarrow \) (1).
The Tietze extension theorem in RCA₀

If we give up on uniform continuity, then the Tietze extension theorem is provable in RCA₀.

**Theorem (in RCA₀; Brown, Simpson)**

Let \( \hat{X} \) be a complete separable metric space, let \( C \) be a closed subset of \( \hat{X} \), and let \( f : C \to [a, b] \subseteq \mathbb{R} \) be a continuous function. Then there is a continuous function \( F : \hat{X} \to [a, b] \) such that \( F \upharpoonright C = f \).

This immediately gives the strong Tietze extension theorem in WKL₀ (i.e., (1) \( \Rightarrow \) (2) on the previous slide) because in WKL₀, continuous functions on compact spaces have moduli of uniform continuity.
The strong Tietze extension theorem for located sets

Giusto & Simpson obtained a version of the strong Tietze extension theorem in RCA\(_0\) by assuming the closed set \(C\) is also located.

A closed or separably closed subset \(C\) of a metric space \(\hat{X}\) is located if there is a continuous distance function \(f : \hat{X} \to \mathbb{R}\) such that
\[
(\forall x \in \hat{X})(f(x) = \inf\{d(x, y) : y \in C\}).
\]

**Theorem (in RCA\(_0\); Giusto & Simpson)**

Let \(\hat{X}\) be a compact complete separable metric space, let \(C\) be a closed and located subset of \(\hat{X}\), and let \(f : C \to \mathbb{R}\) be a continuous function with a modulus of uniform continuity. Then there is a continuous function \(F : \hat{X} \to \mathbb{R}\) with a modulus of uniform continuity such that \(F \mid C = f\).
Digression on located sets

The following is in the context of a compact metric space \( \hat{X} \). \( \hat{X} \) may be taken to be \([0, 1]\). All results are due to Giusto & Simpson.

\( \text{RCA}_0 \) proves the following:

- If \( C \subseteq \hat{X} \) is closed and located, then it is separably closed.
- If \( C \subseteq \hat{X} \) is separably closed and located, then it is closed.

The following are equivalent to \( \text{ACA}_0 \) over \( \text{RCA}_0 \):

- Every closed \( C \subseteq \hat{X} \) is separably closed.
- Every separably closed \( C \subseteq \hat{X} \) is closed.
- Every closed \( C \subseteq \hat{X} \) is located.
- Every separably closed \( C \subseteq \hat{X} \) is located.

Over \( \text{RCA}_0 \), \( \text{WKL}_0 \) is equivalent to “every closed and separably closed \( C \subseteq \hat{X} \) is located.”
This version is **without** uniform continuity.

**Theorem (Giusto & Simpson)**

The following are equivalent over \( \text{RCA}_0 \):

1. \( \text{ACA}_0 \).

2. Let \( \hat{X} \) be a compact complete separable metric space, let \( C \) be a separably closed subset of \( \hat{X} \), and let \( f : C \to \mathbb{R} \) be a continuous function. Then there is a continuous function \( F : \hat{X} \to \mathbb{R} \) such that \( F \upharpoonright C = f \).

3. Special case of (2) with \( \hat{X} = [0, 1] \).
Strong Tietze extension theorems for separably closed sets

This version is with uniform continuity.

Theorem (Giusto & Simpson)

The following are equivalent over \( \text{RCA}_0 \):

1. \( \text{WKL}_0 \).
2. Let \( \hat{X} \) be a compact complete separable metric space, let \( C \) be a separably closed subset of \( \hat{X} \), and let \( f : C \to \mathbb{R} \) be a continuous function with a modulus of uniform continuity. Then there is a continuous function \( F : \hat{X} \to \mathbb{R} \) with a modulus of uniform continuity such that \( F \upharpoonright C = f \).
3. Special case of (2) with \( \hat{X} = [0, 1] \).
Brass tacks

Remember one more time that $\text{sTET}_{[0,1]}$ is the following statement:

Let $C$ be a closed and separably closed subset of $[0,1]$, and let $f : C \to \mathbb{R}$ be a continuous function with a modulus of uniform continuity. Then there is a continuous function $F : [0,1] \to \mathbb{R}$ with a modulus of uniform continuity such that $F \upharpoonright C = f$.

We want to show that $\text{RCA}_0 + \text{sTET}_{[0,1]} \vdash \text{WKL}_0$.

First, we give Giusto & Simpson’s proof that $\text{RCA}_0 \nvdash \text{sTET}_{[0,1]}$.

They show that REC is not a model of $\text{sTET}_{[0,1]}$ by building $C$ and $f$ to diagonalize against every possible Weierstraß approximation of an extension $F$ of $f$. 
Weierstraß approximations in RCA₀

In RCA₀, having a modulus of uniform continuity is the same as having a Weierstraß approximation.

**Theorem (in RCA₀; Simpson)**

If $F : [0, 1] \rightarrow \mathbb{R}$ is continuous, then $F$ has a modulus of uniform continuity if and only if there is a sequence of polynomials with rational coefficients $\langle p_n : n \in \mathbb{N} \rangle$ such that

$$\forall n \in \mathbb{N}, \forall x \in [0, 1], |F(x) - p_n(x)| < 2^{-n}.$$ 

So we want to define recursive codes for a $C$ and an $f : C \rightarrow \mathbb{R}$ with a modulus of uniform continuity such that there is no recursive Weierstraß approximation to an extension.
Preparing the domain

For each $e \in \omega$, let

$$I_e = \left[ \frac{1}{2^{2e+1}}, \frac{1}{2^{2e}} \right].$$

Let $D = \{0\} \cup \bigcup_{e \in \omega} I_e$.

We shrink $D$ to $C$ by taking advantage of the fact that $[0, 1]$ (and every $I_e$) is not Heine-Borel compact in REC.

Fix an enumeration of an open covering of $[0, 1] \cap \text{REC}$ that has no finite sub-covering.

Translate this covering to each $I_e$ by the appropriate linear function.
Shrinking $I_e$ and defining $f$

Plan: On interval $I_e$, diagonalize against $\Phi_e$ computing a Weierstraß approximation for an extension of $f$.

Implement the following strategy on $I_e$:

- Let $\langle (a_k, b_k) : k \in \omega \rangle$ enumerate the open cover of $I_e \cap \text{REC}$ with no finite sub-cover.
- Enumerate the intervals $(a_k, b_k)$ into the complement of $C$ while waiting for $\Phi_e(2e + 1)$ to converge.
- If $\Phi_e(2e + 1) \downarrow = p(x)$, stop and choose $q \in I_e \cap \mathbb{Q}$ not yet covered.
- If $p(q) \leq 0$, define $f(x) = 2^{-2e}$ on what’s left of $I_e$. Otherwise define $f(x) = -2^{-2e}$ on what’s left of $I_e$.
- If $\Phi_e(2e + 1) \uparrow$, then $I_e$ is erased and we don’t need to define $f$ there.
- $f$ has modulus of uniform continuity $n \mapsto 2n + 2$. 
Instead of building $f$ and $C$ to diagonalize against the recursive oracles, we could build $f$ and $C$ to diagonalize against (say) 90% of all oracles.

This proves that $\text{WWKL}_0 \nvdash \text{sTET}_{[0,1]}$.

This is worth mentioning because Giusto & Simpson improve their $\text{RCA}_0 \nvdash \text{sTET}_{[0,1]}$ proof to $\text{RCA}_0 + \text{sTET}_{[0,1]} \vdash \text{DNR}$.

This leads them to suggest that $\text{sTET}_{[0,1]}$ could be equivalent to $\text{WWKL}_0$ or DNR (if the conjecture were false).
Let $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ be injections with disjoint ranges. We want to find a separating set.

It would be nice if we could do what we did before:

- Use $I_e$ to code whether or not $e$ should be in the separating set.
- Chip away at $I_e$ until we see $e \in \text{ran } g_0$ or $e \in \text{ran } g_1$.
- If $e \in \text{ran } g_0$ ($g_1$), let $f(x) = 2^{-2e} (-2^{-2e})$ on the remainder of $I_e$.
- If $e$ is not in $\text{ran } g_0$ or $\text{ran } g_1$, then $I_e$ is disjoint from $\text{dom } f$.

(Note that this plan uses $\neg \text{WKL}_0$.)

Let $F : [0, 1] \to \mathbb{R}$ be an extension with modulus of uniform continuity $H$.

If we knew a point $q$ within $2^{-H(2e+2)}$ of a point in $I_e \cap C$ (if $C \neq \emptyset$), then we could decide whether or not to put $e$ in the separating set.
Let $F : [0, 1] \rightarrow \mathbb{R}$ be an extension with modulus of uniform continuity $H$.

If we knew a point $q$ within $2^{-H(2e+2)}$ of a point in $I_e \cap C$ (if $C \neq \emptyset$), then we could decide whether or not to put $e$ in the separating set:

- Use $F$ to find a rational $2^{-(2e+2)}$-approximation $r$ of $F(q)$.
- If $e \in \text{ran } g_0$, there is $x \in I_e \cap C$ within $2^{-H(2e+2)}$ of $q$ such that $F(x) = 2^{-2e}$.
- This means $F(q)$ is within $2^{-(2e+2)}$ of $2^{-2e}$.
- So $r$ is within $2^{-(2e+1)}$ of $2^{-2e}$. So $r > 0$.
- Put $e$ in the separating set if $r > 0$. Otherwise leave $e$ out.

But how would you find $q$?
Reorganizing the pre-domain

We reorganize $f$’s pre-domain to arrange the $q$’s ahead of time.

Replace $I_e$ with infinitely many disjoint closed intervals $\langle I_{e,m} : m \in \mathbb{N} \rangle$ contained in the old $I_e$.

Ensure each $I_{e,m}$ has length at most $2^{-m}$.

Fix an open cover of $I_{e,m}$ with no finite subcover (and ensure that the cover of $I_{e,m}$ doesn’t intersect a different $I_{e',m'}$).

Choose $q_{e,m} \in I_{e,m}$ for each $e, m \in \mathbb{N}$.

Now run the plan from two slides ago on each $I_{e,m}$:

- Chip away at $I_{e,m}$ until we see $e \in \text{ran } g_0$ or $e \in \text{ran } g_1$.
- If $e \in \text{ran } g_0$ ($g_1$), let $f(x) = 2^{-2e} \cdot (-2^{-2e})$ on the remainder of $I_{e,m}$.
- If $e$ is not in $\text{ran } g_0$ or $\text{ran } g_1$, then every $I_{e,m}$ is disjoint from $\text{dom } f$. 
A lemma to find the $I_{e,m}$'s

**Lemma (in RCA$_0 + \neg$WKL$_0$)**

For each $e \in \mathbb{N}$, let $I_e = [2^{-(2e+1)}, 2^{-2e}]$. There are pairwise disjoint closed intervals with rational endpoints $\langle I_{e,m} : e, m \in \mathbb{N} \rangle$, rationals $\langle q_{e,m} : e, m \in \mathbb{N} \rangle$, and open intervals with rational endpoints $\langle (a_{k}^{e,m}, b_{k}^{e,m}) : e, m, k \in \mathbb{N} \rangle$ such that

(i) $\{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$ is closed;

(ii) $q_{e,m} \in I_{e,m}$;

(iii) $I_{e,m} \subseteq I_e$, and the length of $I_{e,m}$ is less than $2^{-m}$;

(iv) $\langle (a_{k}^{e,m}, b_{k}^{e,m}) : k \in \mathbb{N} \rangle$ is an open cover of $I_{e,m}$ with no finite subcover;

(v) if $\langle e, m \rangle \neq \langle e', m' \rangle$, then $I_{e,m}$ and $(a_{k}^{e',m'}, b_{k}^{e',m'})$ are disjoint.
The strong Tietze extension theorem and $\text{WKL}_0$

**Theorem**

The following are equivalent over $\text{RCA}_0$:

1. $\text{WKL}_0$.

2. Let $\hat{X}$ be a compact complete separable metric space, let $C$ be a closed subset of $\hat{X}$, and let $f : C \to \mathbb{R}$ be a continuous function with a modulus of uniform continuity. Then there is a continuous function $F : \hat{X} \to \mathbb{R}$ with a modulus of uniform continuity such that $F \upharpoonright C = f$.

3. Same as (2) with ‘closed’ replaced by ‘closed and separably closed.’

4. Special case of (2) with $\hat{X} = [0,1]$.

5. Special case of (3) with $\hat{X} = [0,1]$.
$sTET_{[0,1]}$ and the Weihrauch degrees
Thank you for coming to my talk! Do you have a question about it?