Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA

Ulrich Kohlenbach
Department of Mathematics

TECHNISCHE UNIVERSITAT DARMSTADT

NUS, Singapore, January 5, 2016
Let \((x_n)\) be a Cauchy sequence in a complete metric space \((X, d)\), i.e.

\[
\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall i, j \geq n \left( d(x_i, x_j) \leq \frac{1}{k + 1} \right) \in \forall \exists
\]
Let \((x_n)\) be a Cauchy sequence in a complete metric space \((X, d)\), i.e.

\[
\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall i, j \geq n \ (d(x_i, x_j) \leq \frac{1}{k + 1}) \in \forall \exists \forall
\]

Often no computable rate of convergence: computable decreasing \((a_n) \subset [0, 1] \cap \mathbb{Q}\) with no computable Cauchy rate (Specker 1949).
Usually: Cauchyness of \((x_n)\) (Cau) implies (intuit.) \(\Sigma^0_1\)-LEM.

Convergence statements and reverse mathematics

- Usually: Cauchyness of \((x_n)\) (Cau) implies (intuit.) \(\Sigma^0_1\)-LEM.

Since \(\Sigma^0_2\)-DNE \(\vdash (Cau)' \rightarrow (Cau)\) and systems s.a. HA+\(\Sigma^0_1\)-LEM are closed under \(\Sigma^0_2\)-DNE rule, \(\Sigma^0_1\)-LEM is also sufficient (Hayashi-Nakata 2002, Safarik/K.2014).
Convergence statements and reverse mathematics

- Usually: Cauchyness of \((x_n)\) (Cau) implies (intuit.) \(\Sigma^0_1\)-LEM.

  Since \(\Sigma^0_2\)-DNE \(\vdash (\text{Cau})' \rightarrow (\text{Cau})\) and systems s.a. HA+\(\Sigma^0_1\)-LEM are closed under \(\Sigma^0_2\)-DNE rule, \(\Sigma^0_1\)-LEM is also sufficient (Hayashi-Nakata 2002, Safarik/K.2014).

- Since elements in \(X\) are represented as fast Cauchy sequences, the convergence (Con) usually implies ACA, which conversely always suffices to infer Con from Cau.

Logical analysis of proofs in convex optimization and nonlinear semigroup the
Convergence statements and reverse mathematics

- Usually: Cauchyness of \((x_n)\) (Cau) implies (intuit.) \(\Sigma^0_1\)-LEM.

  Since \(\Sigma^0_2\)-DNE \(\vdash (\text{Cau})' \rightarrow (\text{Cau})\) and systems s.a. HA+\(\Sigma^0_1\)-LEM are closed under \(\Sigma^0_2\)-DNE rule, \(\Sigma^0_1\)-LEM is also sufficient (Hayashi-Nakata 2002, Safarik/K.2014).

- Since elements in \(X\) are represented as **fast Cauchy sequences**, the convergence (Con) usually implies ACA, which conversely always suffices to infer Con from Cau.

- So calibration of Con only requires to know the **strength** of the arithmetic Cau. When is **Cau finitistically provable**?
The Cauchy property \textit{noneffectively} is equivalent to

\[ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n+g(n)] (d(x_i, x_j) < \frac{1}{k+1}) \in \forall \exists \]
The Cauchy property noneffectively is equivalent to

\[ \forall k \in \mathbb{IN} \forall g \in \mathbb{IN}^\mathbb{IN} \exists n \in \mathbb{IN} \forall i, j \in [n; n+g(n)] (d(x_i, x_j) < \frac{1}{k + 1}) \in \forall \exists \]

Herbrand normal form or metastability (Tao).
The Cauchy property noneffectively is equivalent to

\[ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n+g(n)] \ (d(x_i, x_j) < \frac{1}{k + 1}) \in \forall \exists \]

Herbrand normal form or metastability (Tao).

A bound \( \Phi(k, g) \) on ‘\( \exists n \)’ in the latter formula is a rate of metastability (introduced by Kreisel in 1951 as no-counterexample interpretation).
The Cauchy property noneffectively is equivalent to

\[ \forall k \in \mathbb{N} \ \forall g \in \mathbb{N}^{\mathbb{N}} \ \exists n \in \mathbb{N} \ \forall i, j \in [n; n+g(n)] \ (d(x_i, x_j) < \frac{1}{k+1}) \in \forall \exists \]

Herbrand normal form or metastability (Tao).

A bound \( \Phi(k, g) \) on ‘\( \exists n \)’ in the latter formula is a rate of metastability (introduced by Kreisel in 1951 as no-counterexample interpretation). Methods from logic extract such and other rates (proof mining)!
The Cauchy property \textit{noneffectively} is equivalent to

\[ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^\mathbb{N} \exists n \in \mathbb{N} \forall i, j \in [n; n+g(n)] (d(x_i, x_j) < \frac{1}{k + 1}) \in \forall \exists \]

\textit{Herbrand normal form} or \textit{metastability} (Tao).

A bound \( \Phi(k, g) \) on ‘\( \exists n \)’ in the latter formula is a \textit{rate of metastability} (introduced by Kreisel in 1951 as \textit{no-counterexample interpretation}).

Methods from logic \textit{extract} such and other rates (\textit{proof mining})!

It is \( \Phi \) which \textit{measures} the \textit{specific computational content} of a \textit{specific} convergence theorem!
Usually, convergence theorems are of the form: \((x_n)\) converges to some solution \(x\) with \(F(x) =_{\mathbb{R}} 0\), where \(F: X \rightarrow \mathbb{R}\) is continuous.
Usually, convergence theorems are of the form: \((x_n)\) converges to some solution \(x\) with \(F(x) =_{\mathbb{R}} 0\), where \(F : X \rightarrow \mathbb{R}\) is continuous.

Then metastability should be written as

\[
(\ast) \quad \forall k, \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \quad \left( d(x_i, x_j), |F(x_i)| < \frac{1}{k + 1} \right).
\]
Properties of

\((*)\) \(\forall k, g \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \left( d(x_i, x_j), |F(x_i)| \leq \frac{1}{k + 1} \right)\).
Properties of

\((*) \forall k, g \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \left( d(x_i, x_j), |F(x_i)| \leq \frac{1}{k+1} \right)\).

- \((*)\) is purely universal: real statement in the sense of Hilbert.
Properties of

\((\ast) \forall k, g \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \left( d(x_i, x_j), |F(x_i)| \leq \frac{1}{k + 1} \right).\)

- \((\ast)\) is purely universal: **real statement** in the sense of Hilbert.
- By negative translation, \((\ast)\) always has a **constructive** proof.
Properties of

\((\ast)\, \forall k, g \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \left( d(x_i, x_j), |F(x_i)| \leq \frac{1}{k+1} \right) \).

- \((\ast)\) is purely universal: real statement in the sense of Hilbert.
- By negative translation, \((\ast)\) always has a constructive proof.
- By classical logic (and closure under recursion), \((\ast)\) mathematically trivially implies (by a fixed piece of proof) that \((x_n)\) is Cauchy.
Properties of

\((\ast) \ \forall k, g \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \left( d(x_i, x_j), |F(x_i)| \leq \frac{1}{k+1} \right) \).

- \((\ast)\) is purely universal: real statement in the sense of Hilbert.
- By negative translation, \((\ast)\) always has a constructive proof.
- By classical logic (and closure under recursion), \((\ast)\) mathematically trivially implies (by a fixed piece of proof) that \((x_n)\) is Cauchy.
- By ACA, (+) mathematically trivially implies (by a fixed piece of proof) that \((x_n)\) is convergent and \(F(\lim x_n) = 0\).
Properties of

\((\ast) \ \forall k, g \exists n \leq \Phi(k, g) \forall i, j \in [n; n+g(n)] \left( d(x_i, x_j), |F(x_i)| \leq \frac{1}{k+1} \right)\).

- \((\ast)\) is purely universal: **real statement** in the sense of Hilbert.
- By negative translation, \((\ast)\) always has a **constructive** proof.
- By classical logic (and closure under recursion), \((\ast)\) **mathematically trivially implies** (by a fixed piece of proof) that \((x_n)\) is Cauchy.
- By ACA, \((+)^\prime\) **mathematically trivially implies** (by a fixed piece of proof) that \((x_n)\) is convergent and \(F(\lim x_n) = 0\).
- The structure of \(\Phi\) yields information on the **learnability** of a convergence rate and sometimes **oscillation bounds** (Safarik/K., APAL 2014).
Interesting when **already** the proof of the **Cauchy property uses ACA (or stronger principles)** though being **eliminable**.

Existing proof mining has treated proofs based on: sequential compactness, weak compactness, Banach limits, Zorn's lemma.

Reverse mathematics informs on the strength of the principles used (and the extraction means to be used).

All (but one) rates extracted by us have been **primitive recursive**: Hilbert vindicated!

From this: \( \text{Cau}(x_n) \) provable with \( \Sigma^0_1 \)-IA.

Under certain conditions (e.g. uniqueness or monotonicity properties), metastability even yields rates of convergence.

Below: recent examples of metastability and rates of convergence based on sequential compactness (ACA) or Heine-Borel compactness (WKL).

Logical analysis of proofs in convex optimization and nonlinear semigroup theory.
Interesting when already the proof of the Cauchy property uses ACA (or stronger principles) though being eliminable.

Existing proof mining has treated proofs based on: sequential compactness, weak compactness, Banach limits, Zorn’s lemma.
Interesting when already the proof of the Cauchy property uses ACA (or stronger principles) though being eliminable.

Existing proof mining has treated proofs based on: sequential compactness, weak compactness, Banach limits, Zorn’s lemma.

Reverse mathematics informs on the strength of the principles used (and the extraction means to be used).
Proof mining, metastability and reverse mathematics

- Interesting when *already* the proof of the **Cauchy property uses ACA (or stronger principles)** though being **eliminable**.
- Existing proof mining has treated proofs based on: **sequential compactness, weak compactness, Banach limits, Zorn’s lemma**.
- **Reverse mathematics** informs on the **strength** of the principles used (and the extraction means to be used).
- All (but one) rates extracted by us have been **primitive recursive**: **Hilbert vindicated!** From this: \( \text{Cau}(x_n) \) provable with \( \Sigma^0_1 \text{-IA} \).

Under certain conditions (e.g. uniqueness or monotonicity properties), metastability even yields rates of convergence. Below: recent examples of metastability and rates of convergence based on sequential compactness (ACA) or Heine-Borel compactness (WKL).

Logical analysis of proofs in convex optimization and nonlinear semigroup theory.
Interesting when already the proof of the Cauchy property uses ACA (or stronger principles) though being eliminable.

Existing proof mining has treated proofs based on: sequential compactness, weak compactness, Banach limits, Zorn’s lemma.

Reverse mathematics informs on the strength of the principles used (and the extraction means to be used).

All (but one) rates extracted by us have been primitive recursive: Hilbert vindicated! From this: Cau($x_n$) provable with $\Sigma^0_1$-IA.

Under certain conditions (e.g. uniqueness or monotonicity properties), metastability even yields rates of convergence.
Proof mining, metastability and reverse mathematics

- Interesting when **already** the proof of the **Cauchy property uses ACA (or stronger principles)** though being **eliminable**.

- Existing proof mining has treated proofs based on: **sequential compactness, weak compactness, Banach limits, Zorn’s lemma**.

- **Reverse mathematics** informs on the strength of the principles used (and the extraction means to be used).

- All (but one) rates extracted by us have been **primitive recursive**: **Hilbert vindicated!** From this: \( \text{Cau}(x_n) \) provable with \( \Sigma^0_1 \)-IA.

- Under certain conditions (e.g. **uniqueness** or **monotonicity properties**), metastability even yields **rates of convergence**.

Below: recent examples of metastability and rates of convergence based on sequential compactness (**ACA**) or Heine-Borel compactness (**WKL**).
Fejér monotone sequences

(Leuştean, Nicolae, K. 2015)
Definition \((X, d)\) metric space, \(F_k \subseteq X\), where \(F = \bigcap_{k \in \mathbb{N}} F_k\).
Points of \(AF_k := \bigcap_{i \leq k} F_i\) are called approximate \(F\)-points.
Approximate points and explicit closedness

Definition \((X, d)\) metric space, \(F_k \subseteq X\), where \(F = \bigcap_{k \in \mathbb{N}} F_k\).

Points of \(AF_k := \bigcap_{i \leq k} F_i\) are called approximate \(F\)-points.

A sequence \((x_n) \subset X\) has approximate \(F\)-points if

\[
\forall k \in \mathbb{N} \exists n \in \mathbb{N} \ (x_n \in AF_k).
\]
Definition $(X, d)$ metric space, $F_k \subseteq X$, where $F = \bigcap_{k \in \mathbb{N}} F_k$. Points of $AF_k := \bigcap_{i \leq k} F_i$ are called approximate $F$-points.

A sequence $(x_n) \subset X$ has approximate $F$-points if

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} (x_n \in AF_k).$$

$F$ is explicitly closed (w.r.t. $(F_k)$) if

$$\forall p \in X \ (\forall N, M \in \mathbb{N} (AF_M \cap \bar{B}(p, 1/(N + 1)) \neq \emptyset) \rightarrow p \in F).$$
Let $C \subseteq X$ and $T : C \rightarrow C$.

$$F_k := \left\{ x \in C : d(x, Tx) \leq \frac{1}{k+1} \right\}.$$

$F$- (resp. $AF_k$)-points $\hat{=} \text{fixed points (resp. } 1/(k+1)-\text{approx. fixed points).} \)
Let $C \subseteq X$ and $T : C \rightarrow C$.

$$F_k := \left\{ x \in C : d(x, Tx) \leq \frac{1}{k + 1} \right\}.$$ 

$F$- (resp. $AF_k$)-points $\stackrel{\triangle}{=} \text{fixed points (resp. } 1/(k + 1)\text{-approx. fixed points).}$

If $T$ is continuous, then $F$ is explicitly closed.
Example II: zero sets of maximally monotone operators

\[ J_A = (\text{Id} + \gamma A)^{-1} \] resolvent of \( \gamma A \) for a \textit{maximally monotone} operator \( A : H \to 2^H \) in a real Hilbert space \( H \).
Example II: zero sets of maximally monotone operators

\[ J_A = (\text{Id} + \gamma A)^{-1} \] resolvent of \( \gamma A \) for a \textbf{maximally monotone} operator \( A : H \to 2^H \) in a real Hilbert space \( H \).

Let \( (\gamma_n) \subset (0, \infty) \) and define

\[ F_k := \bigcap_{i \leq k} \left\{ x \in H : \|x - J_{\gamma_i A}x\| \leq \frac{1}{k + 1} \right\}. \]
Example II: zero sets of maximally monotone operators

\[ J_A = (\text{Id} + \gamma A)^{-1} \] resolvent of \( \gamma A \) for a \textbf{maximally monotone} operator \( A : H \to 2^H \) in a real Hilbert space \( H \).

Let \( (\gamma_n) \subset (0, \infty) \) and define

\[ F_k := \bigcap_{i \leq k} \left\{ x \in H : \| x - J_{\gamma_i A} x \| \leq \frac{1}{k + 1} \right\}. \]

Then \( F = \text{zeroes}(A) \).
Definition: \((x_n) \subset X\) is Fejér monotone w.r.t. \(F(\neq \emptyset)\) if

\[
\forall n \in \mathbb{N} \forall p \in F \ (d(x_{n+1}, p) \leq d(x_n, p)).
\]
**Definition:** $(x_n) \subset X$ is Fejér monotone w.r.t. $F(\neq \emptyset)$ if

$$\forall n \in \mathbb{IN} \forall p \in F \ (d(x_{n+1}, p) \leq d(x_n, p)).$$

**Remark** Leuştean/Nicolae/K. consider actually a much more general form of Fejér monotonicity.
**Definition:** \((x_n) \subset X\) is **Fejér monotone** w.r.t. \(F (\neq \emptyset)\) if

\[
\forall n \in \mathbb{N} \forall p \in F \ (d(x_{n+1}, p) \leq d(x_n, p)).
\]

**Remark** Leuştean/Nicolae/K. consider actually a much more general form of Fejér monotonicity.

In this definition one can also incorporate **error terms** \(\delta_n \geq 0\) with \(\sum \delta_n < \infty\) : **quasi-Fejér monotonicity**.
Theorem (Leuştean, Nicolae, K. 2015)

Let \( X \) be a compact metric space, \( F \) explicitly closed. If \( (x_n) \subset X \) has approximate \( F \)-points and is Fejér monotone, then it converges to a point \( x \in F \).
Theorem (Leuştean, Nicolae, K. 2015)

$X$ compact metric space, $F$ explicitly closed. If $(x_n) \subset X$ has approximate $F$-points and is Fejér monotone, then it converges to a point $x \in F$.

Proof uses that sequences in $X$ have convergent subsequences.

Using results due to E. Neumann (LMCS 2015): for most of the usual iterations $(x_n)$, the convergence (Cauchy-ness) above already in the case $X = [0, 1]$ implies the convergence (Cauchy-ness) of monotone sequences in $(x_n)$ and hence ACA ($\Sigma^0_1$-IA).
Theorem (Leuștean, Nicolae, K. 2015)

Let $X$ be a compact metric space, $F$ an explicitly closed set. If $(x_n) \subset X$ is an approximate $F$-point and is Fejér monotone, then it converges to a point $x \in F$.

**Proof uses** that sequences in $X$ have **convergent subsequences**.

Using results due to E. Neumann (LMCS 2015): for most of the usual iterations $(x_n)$, the convergence (Cauchyness) above already in the case $X = [0, 1]$ implies the convergence (Cauchyness) of monotone sequences in $(x_n)$ and hence ACA ($\Sigma^0_1$-IA).

Next we will give a **quantitative version** of the above theorem!
Quantitative uniform explicit closedness

\( F \) is **uniformly closed** with moduli \( \delta_F, \omega_F : \mathbb{N} \rightarrow \mathbb{N} \) if

\[
\forall k \in \mathbb{N} \forall p, q \in X \left( q \in AF_{\delta_F(k)} \text{ and } d(p, q) \leq \frac{1}{\omega_F(k) + 1} \rightarrow p \in AF_k \right).
\]
F is **uniformly closed** with moduli $\delta_F, \omega_F : \mathbb{N} \to \mathbb{N}$ if

$$\forall k \in \mathbb{N} \forall p, q \in X \left( q \in AF_{\delta_F(k)} \text{ and } d(p, q) \leq \frac{1}{\omega_F(k) + 1} \rightarrow p \in AF_k \right).$$

$\omega_F, \delta_F$ solve functional interpretation of **instance of extensionality**

$$q \in F \land p = q \rightarrow p \in F.$$
Quantitative uniform explicit closedness

$F$ is **uniformly closed** with moduli $\delta_F, \omega_F : \mathbb{N} \to \mathbb{N}$ if

$$\forall k \in \mathbb{N} \forall p, q \in X \left( q \in AF_{\delta_F(k)} \text{ and } d(p, q) \leq \frac{1}{\omega_F(k) + 1} \rightarrow p \in AF_k \right).$$

$\omega_F, \delta_F$ solve functional interpretation of **instance of extensionality**

$$q \in F \land p = q \rightarrow p \in F.$$  

**Example 1:** $T$ is uniformly continuous (with modulus $\omega_T$), then $F$ is uniformly closed with moduli

$$\omega_F(k) := \max\{4k + 3, \omega_T(4k + 3)\}, \quad \delta_F(k) := 2k + 1.$$ 

However: uniform continuity in general not necessary.
Quantitative uniform explicit closedness

$F$ is **uniformly closed** with moduli $\delta_F, \omega_F : \mathbb{N} \to \mathbb{N}$ if

$$\forall k \in \mathbb{N} \forall p, q \in X \left( q \in AF_{\delta_F(k)} \text{ and } d(p, q) \leq \frac{1}{\omega_F(k) + 1} \rightarrow p \in AF_k \right).$$

$\omega_F, \delta_F$ solve functional interpretation of instance of extensionality

$$q \in F \land p = q \rightarrow p \in F.$$

**Example 1:** $T$ is uniformly continuous (with modulus $\omega_T$), then $F$ is uniformly closed with moduli

$$\omega_F(k) := \max\{4k + 3, \omega_T(4k + 3)\}, \delta_F(k) := 2k + 1.$$  

However: uniform continuity in general not necessary.

**Example 2:** $F$ is uniformly closed with moduli

$$\omega_F(k) = 4k + 3, \delta_F(k) = 2k + 1.$$
Definition: \((x_n)\) is uniformly Fejér monotone w.r.t. \(F\) with modulus \(\chi: \mathbb{N}^3 \rightarrow \mathbb{N}\) if for all \(m, n, r \in \mathbb{N}\)

\[
\forall p \in X \left( p \in AF_{\chi(n,m,r)} \rightarrow \forall i \leq m \left( d(x_{n+i}, p) < d(x_n, p) + \frac{1}{r + 1} \right) \right).
\]
Quantitative uniform Fejér monotonicity

**Definition:** \( (x_n) \) is uniformly Fejér monotone w.r.t. \( F \) with modulus \( \chi : \mathbb{N}^3 \to \mathbb{N} \) if for all \( m, n, r \in \mathbb{N} \)

\[
\forall p \in X \left( p \in AF_{\chi(n,m,r)} \rightarrow \forall i \leq m \left( d(x_{n+i}, p) < d(x_n, p) + \frac{1}{r + 1} \right) \right).
\]

**Remark:** If \( X \) compact and \( F \) explicitly closed: Fejér monotone \( \iff \) uniformly Fejér monotone.
Quantitative compactness

**Definition:** Let \( \emptyset \neq A \subseteq X \). \( \gamma : \mathbb{N} \rightarrow \mathbb{N} \) is a **modulus of total boundedness** for \( A \) if for all \( k \in \mathbb{N} \) and any sequence \((x_n)\) in \( A \)

\[ \exists i < j \leq \gamma(k) \left( d(x_i, x_j) \leq \frac{1}{k + 1} \right). \]

**Remark:** \( A \) totally bounded iff \( A \) has modulus of total boundedness.
Quantitative compactness

**Definition:** Let $\emptyset \neq A \subseteq X$. $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ is a modulus of total boundedness for $A$ if for all $k \in \mathbb{N}$ and any sequence $(x_n)$ in $A$

$$\exists i < j \leq \gamma(k) \left( d(x_i, x_j) \leq \frac{1}{k + 1} \right).$$

**Remark:** $A$ totally bounded iff $A$ has modulus of total boundedness. The logarithm of the optimal $\gamma$ is also called the capacity of $A$. 

Logical analysis of proofs in convex optimization and nonlinear semigroup theory.
Quantitative approximate $F$-points

Let $(x_n)$ be a sequence with approximate $F$-points. $\Phi : \mathbb{N} \rightarrow \mathbb{N}$ is an approximate $F$-bound bound for $(x_n)$ if it is nondecreasing and

$$\forall k \in \mathbb{N} \exists N \leq \Phi(k) (x_N \in AF_k).$$
Let \((x_n)\) be a sequence with approximate \(F\)-points. \(\Phi : \mathbb{N} \to \mathbb{N}\) is an approximate \(F\)-bound bound for \((x_n)\) if it is nondecreasing and

\[
\forall k \in \mathbb{N} \exists N \leq \Phi(k) \ (x_N \in AF_k).
\]

\((x_n)\) has the \textit{lim inf-property} with bound \(\Phi\) if

\[
\forall k, n \in \mathbb{N} \exists N \leq \Phi(k, n) \ (N \geq n \land x_N \in AF_k).
\]
**Main quantitative theorem**

**Theorem (L. Leuştean, A. Nicolae, K. 2015)**

Let $X$ be totally bounded with modulus $\gamma$. $(x_n)$ is uniformly Fejér monotone w.r.t. $F$ with modulus $\chi$. Let $(x_n)$ have approximate $F$-points with bound $\Phi$. Then $(x_n)$ is Cauchy and for all $k \in \mathbb{N}$ and all $g : \mathbb{N} \to \mathbb{N}$

$$\exists N \leq \Psi \forall i, j \in [N, N + g(N)] \left( d(x_i, x_j) \leq \frac{1}{k + 1} \right),$$

where $\Psi(k, g, \Phi, \chi, \gamma) = \Psi_0(P, k, g, \Phi, \chi)$ with $\Psi_0(0, \ldots) = 0$.
Main quantitative theorem

Theorem (L. Leuştean, A. Nicolae, K. 2015)

$X$ totally bounded with modulus $\gamma$. $(x_n)$ uniformly Fejér monotone w.r.t. $F$ with modulus $\chi$. Let $(x_n)$ have approximate $F$-points with bound $\Phi$. Then $(x_n)$ is Cauchy and for all $k \in \mathbb{N}$ and all $g : \mathbb{N} \to \mathbb{N}$

$$\exists N \leq \Psi \ \forall i, j \in [N, N + g(N)] \left( d(x_i, x_j) \leq \frac{1}{k + 1} \right),$$

where $\Psi(k, g, \Phi, \chi, \gamma) := \Psi_0(P, k, g, \Phi, \chi)$.
Main quantitative theorem

Theorem (L. Leuştean, A. Nicolae, K. 2015)

\( X \) totally bounded with modulus \( \gamma \). \((x_n)\) uniformly Fejér monotone w.r.t. \( F \) with modulus \( \chi \). Let \((x_n)\) have approximate \( F \)-points with bound \( \Phi \). Then \((x_n)\) is Cauchy and for all \( k \in \mathbb{N} \) and all \( g : \mathbb{N} \to \mathbb{N} \)

\[
\exists N \leq \psi \forall i, j \in [N, N + g(N)] \left( d(x_i, x_j) \leq \frac{1}{k + 1} \right),
\]

where \( \psi(k, g, \Phi, \chi, \gamma) := \psi_0(P, k, g, \Phi, \chi) \) with

\[
\psi_0(0, \ldots) := 0,
\psi_0(n + 1, \ldots) := \Phi \left( \chi_g^{M}(\psi_0(n, \ldots), 4k + 3) \right),
\chi_g^{M}(n, k) := \max_{i \leq n} \{ \chi(i, g(i), k), P := \gamma(4k + 3) \}.
\]

By \( \text{Cau}(x_n) \to \text{Cau}_{\text{mon}} \to \Sigma^0_1\text{-IA} \) this essentially is optimal!
If $F$ is additionally **uniformly closed** with moduli $\delta_F, \omega_F$

$$\exists N \leq \tilde{\Psi} \forall i, j \in [N, N+g(N)] \left( d(x_i, x_j) \leq \frac{1}{k+1} \quad \text{and} \quad x_i \in AF_k \right),$$

where $\tilde{\Psi}$ results from $\Psi$ by replacing $k$ and $\chi$ by

$$k' := \max\{k, \lceil \omega_F(k)/2 \rceil \} \quad \text{and} \quad \chi'(n, m, r) := \max\{\delta_F(k), \chi(n, m, r)\}$$
If $F$ is additionally \textbf{uniformly closed} with moduli $\delta_F, \omega_F$

$$\exists N \leq \tilde{\Psi} \forall i, j \in [N, N + g(N)] \left( d(x_i, x_j) \leq \frac{1}{k + 1} \text{ and } x_i \in AF_k \right),$$

where $\tilde{\Psi}$ results from $\Psi$ by replacing $k$ and $\chi$ by

$$k' := \max\{k, \lceil (\omega_F(k)/2) \rceil\} \text{ and } \chi'(n, m, r) := \max\{\delta_F(k), \chi(n, m, r)\}$$

- Extends to \textbf{uniformly quasi-Fejér monotone} if $\Phi$ is a \textit{lim inf-bound}. 

Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA.
Applications (Leuştean/Nicolae/K. 2015)

- Picard iterations of firmly nonexpansive mappings in uniformly convex $W$-hyperbolic ('UCW'-)spaces.
- Ishikawa iterations of nonexpansive mappings in UCW-spaces.
- Mann iterations of strict pseudo-contractions in Hilbert spaces.
- The proximal point algorithm for the zeroes of maximally monotone operators in Hilbert space (see below).
- Mann iterations of mappings satisfying condition $(E)$.
- Minimization problems for two maps (see below).
I) Convex minimization problem for compositions of maps

Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $X$ be a CAT(0)-space and $T_1, T_2 : X \to X$ satisfy the condition

$$(P) : 2d(T_1x, T_1y)^2 \leq d(x, T_1y)^2 + d(y, T_1x)^2 - d(x, T_1x)^2 - d(y, T_1y)^2.$$
Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $X$ be a CAT(0)-space and $T_1, T_2 : X \to X$ satisfy the condition

$$(P) : 2d(T_i x, T_i y)^2 \leq d(x, T_i y)^2 + d(y, T_i x)^2 - d(x, T_i x)^2 - d(y, T_i y)^2.$$ 

Let $\text{Fix}(T_2 \circ T_1) \neq \emptyset$. Consider sequences $(x_n), (y_n)$ in $X$ with

$$d(y_n, T_1 x_n) \leq \varepsilon_n \text{ and } d(x_{n+1}, T_2 y_n) \leq \delta_n, \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.

Then $\lim d(y_{n+1}, y_n) = \lim d(x_{n+1}, x_n) = 0$.

Proof makes repeated use of convergence of mon. sequences (ACA)! 

Logical analysis of proofs in convex optimization and nonlinear semigroup theory.
Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $X$ be a CAT(0)-space and $T_1, T_2 : X \to X$ satisfy the condition

$$(P) : 2d(T_i x, T_i y)^2 \leq d(x, T_i y)^2 + d(y, T_i x)^2 - d(x, T_i x)^2 - d(y, T_i y)^2.$$ 

Let $\text{Fix}(T_2 \circ T_1) \neq \emptyset$. Consider sequences $(x_n), (y_n)$ in $X$ with

$$d(y_n, T_1 x_n) \leq \varepsilon_n \text{ and } d(x_{n+1}, T_2 y_n) \leq \delta_n, \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.

Then

$$\lim d(y_{n+1}, y_n) = \lim d(x_{n+1}, x_n) = 0.$$
I) Convex minimization problem for compositions of maps

Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $X$ be a CAT(0)-space and $T_1, T_2 : X \rightarrow X$ satisfy the condition

$$(P) : 2d(T_i x, T_i y)^2 \leq d(x, T_i y)^2 + d(y, T_i x)^2 - d(x, T_i x)^2 - d(y, T_i y)^2.$$

Let $\text{Fix}(T_2 \circ T_1) \neq \emptyset$. Consider sequences $(x_n), (y_n)$ in $X$ with

$$d(y_n, T_1 x_n) \leq \varepsilon_n \text{ and } d(x_{n+1}, T_2 y_n) \leq \delta_n, \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.

Then

$$\lim d(y_{n+1}, y_n) = \lim d(x_{n+1}, x_n) = 0.$$ 

Proof makes repeated use of convergence of mon. sequences (ACA)!
Consider two convex and lower semi-continuous \( f, g : X \rightarrow (-\infty, +\infty] \)
and define (Bauschke, Combettes, Reich 2005)
\[
\Phi(x, y) := f(x) + g(y) + \frac{1}{2\lambda} d(x, y)^2.
\]
Then \( T_1 = J^g_\lambda, T_2 = J^f_\lambda \) (resolvents of \( f, g \)) satisfy (\( P \)).
Computing sequences \((x_n), (y_n)\) as above (which only requires to know the resolvents up to some error) provides \( \varepsilon \)-solutions for the minimization problem
\[
\arg\min_{(x, y) \in X \times X} \Phi(x, y).
\]
Rate of convergence in the theorem

Theorem (López-Acedo/Nicolae/K. 2015)

Let \( \alpha \) be a Cauchy-rate for \( \sum_{n=0}^{\infty} \gamma_n \) with \( \gamma_n := \varepsilon_n + \delta_n \) and \( \sum \gamma_n \leq B \in \mathbb{N} \) and \( d(x_0, u) \leq b \) for some \( u \in \text{Fix}(T_2 \circ T_1) \).
Rate of convergence in the theorem

Theorem (López-Acedo/Nicolae/K. 2015)

Let $\alpha$ be a Cauchy-rate for $\sum_{n=0}^{\infty} \gamma_n$ with $\gamma_n := \varepsilon_n + \delta_n$ and $\sum \gamma_n \leq B \in \mathbb{N}$ and $d(x_0, u) \leq b$ for some $u \in \text{Fix}(T_2 \circ T_1)$.

Then

$$\forall n \geq \Phi(\varepsilon, b, B, \alpha) \ (d(y_n, y_{n+1}), d(x_n, x_{n+1}) \leq \varepsilon),$$

where

$$\Phi := \alpha(\varepsilon/3) + k \left[ \frac{12(1 + 2^k)(b + B)}{\varepsilon} \right] + 1, \quad k := \left[ \frac{12(b + B)}{\varepsilon} \right].$$
(xₙ), (yₙ) uniformly quasi-Fejér monotone, hence rates of metastability for totally bounded X:
(x_n), (y_n) uniformly quasi-Fejér monotone, hence rates of metastability for totally bounded X:

Theorem (López-Acedo/Nicolae/K. 2015)

Let X additionally be totally bounded with a modulus γ. Then

∀k ∈ IN∀g ∈ IN IN ∃n ≤ Ψ(k, g) ∀i, j ∈ [n, n + g(n)](d(x_i, x_j) ≤ \frac{1}{k + 1}),
uniformly quasi-Fejér monotone, hence rates of metastability for totally bounded $X$:

**Theorem (López-Acedo/Nicolae/K. 2015)**

Let $X$ additionally be totally bounded with a modulus $\gamma$. Then

$$\forall k \in \mathbb{IN} \forall g \in \mathbb{IN}^\mathbb{IN} \exists n \leq \Psi(k, g) \forall i, j \in [n, n+g(n)] \left( d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Psi(k, g) := \Psi_0(P), \quad P := \gamma(8k + 7) + 1, \quad \xi(k) := \alpha(1/(k + 1)),$$

$$\chi_M^g(n, k) := (\max_{i \leq n} g(i)) \cdot (k + 1),$$

Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA
$(x_n), (y_n)$ uniformly quasi-Fejér monotone, hence rates of metastability for totally bounded $X$:

**Theorem (López-Acedo/Nicolae/K. 2015)**

Let $X$ additionally be totally bounded with a modulus $\gamma$. Then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^\mathbb{N} \exists n \leq \Psi(k, g) \forall i, j \in [n, n+g(n)] (d(x_i, x_j) \leq \frac{1}{k+1}),$$

where

$$\Psi(k, g) := \Psi_0(P), \ P := \gamma(8k + 7) + 1, \ \xi(k) := \alpha(1/(k + 1)),$$

$$\chi^M_g(n, k) := (\max_{i \leq n} g(i)) \cdot (k + 1),$$

and using $\hat{\Phi}(k, N) := \max\{N, \Phi(1/2(i + 1)) : i \leq k\}$

$$\Psi_0(0) := 0, \ \Psi_0(n + 1) := \hat{\Phi} \left( \chi^M_g(\Psi_0(n), 8k + 7), \xi(8k + 7) \right).$$
$(x_n), (y_n)$ uniformly quasi-Fejér monotone, hence rates of metastability for totally bounded $X$:

**Theorem (López-Acedo/Nicolae/K. 2015)**

Let $X$ additionally be **totally bounded** with a modulus $\gamma$. Then

$$\forall k \in \mathbb{IN} \forall g \in \mathbb{IN} \exists n \leq \Psi(k, g) \forall i, j \in [n, n+g(n)] \left( d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Psi(k, g) := \Psi_0(P), \ P := \gamma(8k + 7) + 1, \ \xi(k) := \alpha(1/(k + 1)),$$

$$\chi^{M}_g(n, k) := (\max_{i \leq n} g(i)) \cdot (k + 1),$$

and using $$\hat{\Phi}(k, N) := \max\{N, \Phi(1/2(i + 1)) : i \leq k\}$$

$$\Psi_0(0) := 0, \ \Psi_0(n + 1) := \hat{\Phi} \left( \chi^{M}_g(\Psi_0(n), 8k + 7), \xi(8k + 7) \right).$$

Similarly for $(y_n)$. 

Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA
II) Proximal Point Algorithm (Rockafellar 1976)

$H$ real Hilbert space, $A : H \to 2^H$ be a maximally monotone, $J_{\gamma A} = (Id + \gamma A)^{-1}$ be the resolvent of $\gamma A$ for $\gamma > 0$ and $(\gamma_n) \subset (0, \infty)$.
II) Proximal Point Algorithm (Rockafellar 1976)

$H$ real Hilbert space, $A : H \to 2^H$ be a maximally monotone, $J_{\gamma A} = (Id + \gamma A)^{-1}$ be the resolvent of $\gamma A$ for $\gamma > 0$ and $(\gamma_n) \subset (0, \infty)$.

**Proposition** (Leuștean/Nicolae/K.2015)

$x_{n+1} := J_{\gamma_n A} x_n$ is uniformly Fejér monotone with modulus

$$\chi(n, m, r) := \max\{n + m - 1, m(r + 1)\}.$$
II) Proximal Point Algorithm (Rockafellar 1976)

$H$ real Hilbert space, $A : H \to 2^H$ be a maximally monotone, $J_{\gamma A} = (Id + \gamma A)^{-1}$ be the resolvent of $\gamma A$ for $\gamma > 0$ and $(\gamma_n) \subset (0, \infty)$.

**Proposition** (Leuştean/Nicolae/K.2015)

1. $x_{n+1} := J_{\gamma_n A}x_n$ is uniformly Fejér monotone with modulus

   $$\chi(n, m, r) := \max\{n + m - 1, m(r + 1)\}.$$ 

2. If $\sum \gamma_n^2 = \infty$ with rate of divergence $\theta$, then

   $$\Phi_{b, \theta}(k) := \theta([b^2(M_k + 1)^2])[b^2(M_k + 1)^2] - 1,$$

   where $M_k := [(k + 1)(2 + \max_{0 \leq i \leq k} \gamma_i)] - 1$ and $\|x_0 - p\| \leq b$ for some $p \in \text{zer}(A)$ is an approximate $F$-point bound.
II) Proximal Point Algorithm (Rockafellar 1976)

$H$ real Hilbert space, $A : H \to 2^H$ be a maximally monotone,
$J_{\gamma A} = (\text{Id} + \gamma A)^{-1}$ be the resolvent of $\gamma A$ for $\gamma > 0$ and $(\gamma_n) \subset (0, \infty)$.

**Proposition** (Leuştean/Nicolae/K.2015)

1. $x_{n+1} := J_{\gamma_n A}x_n$ is uniformly Fejér monotone with modulus

   $\chi(n, m, r) := \max\{n + m - 1, m(r + 1)\}$.

2. If $\sum \gamma_n^2 = \infty$ with rate of divergence $\theta$, then

   $\Phi_{b, \theta}(k) := \theta(\lceil b^2(M_k + 1)^2 \rceil \lceil b^2(M_k + 1)^2 \rceil - 1$,

   where $M_k := \lceil (k + 1)(2 + \max_{0 \leq i \leq k} \gamma_i) \rceil - 1$ and $\|x_0 - p\| \leq b$ for some $p \in \text{zer}(A)$ is an approximate $F$-point bound.

Hence in the **finite dimensional case**: rate of metastability!
Metastability without Compactness

- Use **Halpern-versions** of usual iterations to get **strong convergence**.
Use **Halpern-versions** of usual iterations to get **strong convergence**.

Use **Halpern-versions** of usual iterations to get **strong convergence**.


Recently, this approach has been extended to cover Isao Yamada’s **algorithm for variational inequality problems** (Körnlein 2015).
I. Yamada’s Theorem

**Problem:** $H$ real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone and $C = Fix(T)$ for some nonexpansive $T : H \to H$. 

Theorem (I. Yamada 2001): Under suitable conditions on $(\lambda_n)$ the scheme (with $\mu := \eta / \kappa^2$)

$$u_{n+1} := T(u_n) - \lambda_{n+1} \mu FT(u_n)$$

converges strongly to a solution of VIP.

Theorem (D. Körnlein 2015): Explicit and highly uniform effective rate of metastability.

Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA
I. Yamada’s Theorem

Problem: $H$ real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone and $C = \text{Fix}(T)$ for some nonexpansive $T : H \to H$.

Equivalent formulation:

VIP: Find $u^* \in C$ s.t. $\langle v - u^*, F(u^*) \rangle \geq 0$ for all $v \in C$. 

Theorem (I. Yamada 2001): Under suitable conditions on $(\lambda_n)$ the scheme (with $\mu := \eta/\kappa^2$) 

$u_{n+1} := T(u_n) - \lambda_{n+1} \mu FT(u_n)$ converges strongly to a solution of VIP.

Theorem (D. Körlein 2015): Explicit and highly uniform effective rate of metastability.

Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA
I. Yamada’s Theorem

**Problem:** $H$ real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone and $C = \text{Fix}(T)$ for some nonexpansive $T : H \to H$.

**Equivalent formulation:**

VIP: Find $u^* \in C$ s.t. $\langle v - u^*, F(u^*) \rangle \geq 0$ for all $v \in C$.

**Theorem (I. Yamada 2001):** Under suitable conditions on $(\lambda_n)$ the scheme (with $\mu := \eta/\kappa^2$)

$$u_{n+1} := T(u_n) - \lambda_{n+1} \mu F T(u_n)$$

converges strongly to a solution of VIP.

Theorem (D. Körnlein 2015): Explicit and highly uniform effective rate of metastability.

Logical analysis of proofs in convex optimization and nonlinear semigroup theory that are based on WKL or ACA.
I. Yamada’s Theorem

**Problem:** $H$ real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone and $C = \text{Fix}(T)$ for some nonexpansive $T : H \to H$.

**Equivalent formulation:**

VIP: Find $u^* \in C$ s.t. $\langle v - u^*, F(u^*) \rangle \geq 0$ for all $v \in C$.

**Theorem (I. Yamada 2001):** Under suitable conditions on $(\lambda_n)$ the scheme (with $\mu := \eta/\kappa^2$)

$$u_{n+1} := T(u_n) - \lambda_{n+1} \mu FT(u_n)$$

converges strongly to a solution of VIP.

**Theorem** (D. Körnlein 2015): Explicit and highly uniform effective rate of metastability.
Proof Mining in nonlinear semigroup theory with WKL-elimination (with A. Koutsoukou-Argyraki)

Let $X$ be a Banach space, $C \subseteq X$ convex, $\lambda \in (0, 1)$.

**Definition:** A family $\{T(t) : t \geq 0\}$ of nonexpansive mappings $T(t) : C \rightarrow C$ is a **nonexpansive semigroup** if

(i) $T(s + t) = T(s) \circ T(t)$ $(s, t \geq 0)$,

(ii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous.
Let $X$ be a Banach space, $C \subseteq X$ convex, $\lambda \in (0,1)$.

**Definition:** A family $\{T(t) : t \geq 0\}$ of nonexpansive mappings $T(t) : C \to C$ is a **nonexpansive semigroup** if

1. $T(s + t) = T(s) \circ T(t)$ $(s, t \geq 0)$,
2. for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous.

**Theorem (Suzuki, LMS 2006)** Let $0 < \alpha < \beta$ such that $\alpha/\beta$ is irrational. Then any fixed point of $p \in C$

$$S := \lambda T(\alpha) + (1 - \lambda)T(\beta)$$

is a common fixed point of $T(t)$ for all $t \geq 0$, i.e.

$$\text{Fix}(S) \subseteq \bigcap_{t \geq 0} \text{Fix}(T(t)).$$
General logical metatheorems guarantee:

let $t \mapsto T(t)x$ be equicontinuous on norm-bounded subsets of $X$ with modulus $\omega$, let $f$ be a modulus of irrationality for $\alpha/\beta, \Lambda, N, D \in \mathbb{IN}$ be s.t. $1/\Lambda \leq \lambda, 1 - \lambda$ and $1/N \leq \beta \leq D$. Then one can extract a bound $\Phi(\varepsilon, M, d, N, \Lambda, D, f, \omega)$ s.t. for all $M, d \in \mathbb{IN}$, $\forall p \in C \forall \varepsilon > 0 \parallel p \parallel \leq d \wedge \parallel S(p) - p \parallel \leq \Phi(\varepsilon, M, d)$ $\rightarrow \forall t \in [0, M]$ $(\parallel T(t)x - p \parallel \leq \varepsilon)$. Let $x_{n+1} := 1/2 x_n + 1/2 Sx_n$ be a $d$-bounded Krasnoselski iteration of $S$ with rate of asymptotic regularity $\Psi$, then $\forall n \geq \Psi(\Phi(\varepsilon, M, d)) \forall t \in [0, M]$ $(\parallel T(t)x_n - x_n \parallel \leq \varepsilon)$. Note: Suzuki's proof uses WKL: WKL-elimination!
General logical metatheorems guarantee:

let \( t \mapsto T(t)x \) be equicontinuous on norm-bounded subsets of \( X \) with modulus \( \omega \), let \( f \) be a modulus of irrationality for \( \alpha/\beta, \Lambda, N, D \in \mathbb{N} \) be s.t. \( 1/\Lambda \leq \lambda, 1 - \lambda \) and \( 1/N \leq \beta \leq D \). Then one can extract a bound \( \Phi(\varepsilon, M, d, N, \Lambda, D, f, \omega) \) s.t. for all \( M, d \in \mathbb{N} \) \( \forall p \in C \forall \varepsilon > 0 \)

\[
\|p\| \leq d \land \|S(p) - p\| \leq \Phi(\varepsilon, M, d) \rightarrow \forall t \in [0, M] (\|T(t)p - p\| \leq \varepsilon).
\]
General logical metatheorems guarantee:

let \( t \mapsto T(t)x \) be equicontinuous on norm-bounded subsets of \( X \) with modulus \( \omega \), let \( f \) be a \textit{modulus of irrationality} for \( \alpha/\beta, \Lambda, N, D \in \mathbb{N} \) be s.t. \( 1/\Lambda \leq \lambda, 1 - \lambda \) and \( 1/N \leq \beta \leq D \). Then one can extract a bound \( \Phi(\varepsilon, M, d, N, \Lambda, D, f, \omega) \) s.t. for all \( M, d \in \mathbb{N} \) \( \forall p \in C \forall \varepsilon > 0 \)

\[
\|p\| \leq d \wedge \|S(p) - p\| \leq \Phi(\varepsilon, M, d) \rightarrow \forall t \in [0, M] (\|T(t)p - p\| \leq \varepsilon).
\]

Let \( x_{n+1} := \frac{1}{2}x_n + \frac{1}{2}Sx_n \) be a \( d \)-bounded Krasnoselski iteration of \( S \) with rate of asymptotic regularity \( \Psi \), then

\[
\forall n \geq \Psi(\Phi(\varepsilon, M, d)) \forall t \in [0, M] (\|T(t)x_n - x_n\| \leq \varepsilon).
\]
General logical metatheorems guarantee:

let $t \mapsto T(t)x$ be equicontinuous on norm-bounded subsets of $X$ with modulus $\omega$, let $f$ be a **modulus of irrationality** for $\alpha/\beta$, $\Lambda, N, D \in \mathbb{N}$ be s.t. $1/\Lambda \leq \lambda, 1 - \lambda$ and $1/N \leq \beta \leq D$. Then one can extract a bound $\Phi(\varepsilon, M, d, N, \Lambda, D, f, \omega)$ s.t. for all $M, d \in \mathbb{N}$ $\forall p \in C \forall \varepsilon > 0$

\[ \|p\| \leq d \wedge \|S(p) - p\| \leq \Phi(\varepsilon, M, d) \rightarrow \forall t \in [0, M] (\|T(t)p - p\| \leq \varepsilon). \]

Let $x_{n+1} := \frac{1}{2}x_n + \frac{1}{2}Sx_n$ be a $d$-bounded Krasnoselski iteration of $S$ with rate of asymptotic regularity $\Psi$, then

\[ \forall n \geq \Psi(\Phi(\varepsilon, M, d)) \forall t \in [0, M] (\|T(t)x_n - x_n\| \leq \varepsilon). \]

Note: Suzuki’s proof uses WKL: **WKL-elimination!**

Under the previous assumptions:

\[ \forall M \in \mathbb{N} \ \forall m \in \mathbb{N} \ \forall n \geq \Phi(M, m) \ \forall t \in [0, M] \ (\| T(t)x_n - x_n \| < 2^{-m}) \]

Under the previous assumptions:

$$\forall M \in \mathbb{N} \ \forall m \in \mathbb{N} \ \forall n \geq \Phi(M, m) \ \forall t \in [0, M] \ (\|T(t)x_n - x_n\| < 2^{-m})$$

with

$$\Phi(M, m) = \frac{2^{2m+8}d^2((\sum_{i=1}^{\phi(k,f)}} - 1) \Lambda^i + 1)(1 + MN))^2}{\pi}$$

where $d \geq \|x_0 - Sx_n\|$ for all $n$, $k := D2^{\omega_{D,b}(3+\lfloor \log_2(1+MN) \rfloor + m) + 1}$ and

$$\phi(k, f) := \max\{2f(i) + 6 : 0 < i \leq k\}.$$

Under the previous assumptions:

\[ \forall M \in \mathbb{N} \ \forall m \in \mathbb{N} \ \forall n \geq \Phi(M, m) \ \forall t \in [0, M] \ (\|T(t)x_n - x_n\| < 2^{-m}) \]

with

\[ \Phi(M, m) = \frac{2^{2m+8}d^2((\sum_{i=1}^{\phi(k,f) - 1} \Lambda^i + 1)(1 + MN))^2}{\pi} \]

where \( d \geq \|x_0 - Sx_n\| \) for all \( n, k := D2^{\omega_{D,b}(3 + [\log_2(1+MN)] + m) + 1} \) and

\[ \phi(k, f) := \max\{2f(i) + 6 : 0 < i \leq k\}. \]

Example:

\[ \alpha = \sqrt{2}, \beta = 2, \lambda = 1/2. \] Then \( \Lambda = 2, N = 1, D = 2, f_\gamma(p) = 4p^2. \]