A Tutorial on Weihrauch Complexity

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New Challenges in Reverse Mathematics

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Outline

1. A Calculus of Mathematical Problems
2. Choice
3. The Classification of Theorems
4. Jumps
5. Ramsey’s Theorem
6. Lowness
7. Genericity
8. Randomness
Some History on Weihrauch Reducibility

- **1992** Klaus Weihrauch introduced the concept of his reducibility for single-valued functions $f : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ and for sets of such functions (in two unpublished technical reports).
- **1989-2007** he supervised 6 MSc/PhD theses on this topic, mostly unpublished (von Stein, Mylatz, B., Hertling, Pauly).
- The reducibility was also considered for single-valued functions $f : \subseteq X \rightarrow Y$ on other topological/represented spaces.
- **2008** Guido Gherardi and Alberto Marcone noticed that this reducibility for multi-valued functions can be used to classify the computational content of $\Pi_2$ theorems.
- **2009** Akitoshi Kawamura (and Stephen Cook) rediscovered a polynomial-time version of Weihrauch reducibility and used it for the study of uniform computational time complexity.
- **2012** Dorais, Dzhafarov, Hirst, Mileti, Shafer rediscovered Weihrauch reducibility directly for the special case of $\Pi_2^1$ statements (work extended by Hirschfeldt and Jockusch).
Currently there are 89 entries in this bibliography. Please help to update it!
A Calculus of Mathematical Problems
Mathematical Problems and Solutions

Definition

A mathematical problem is a partial multi-valued \( f : \subseteq X \Rightarrow Y \).

- There are a certain sets of potential inputs \( X \) and outputs \( Y \).
- \( D = \text{dom}(f) \) contains the valid instances of the problem.
- \( f(x) \) is the set of solutions of the problem \( f \) for instance \( x \).

Definition

\( g : \subseteq X \Rightarrow Y \) solves \( f : \subseteq X \Rightarrow Y \), if \( \text{dom}(f) \subseteq \text{dom}(g) \) and \( g(x) \subseteq f(x) \) for all \( x \in \text{dom}(f) \). We write \( g \sqsubseteq f \) in this situation.

Definition

For \( f : \subseteq X \Rightarrow Y \), \( g : \subseteq Y \Rightarrow Z \) we define the composition \( g \circ f : \subseteq X \Rightarrow Z \) by

\[
(g \circ f)(x) := \{ z \in Z : (\exists y \in Y) \ y \in f(x) \text{ and } z \in g(y) \}
\]

and \( \text{dom}(g \circ f) := \{ x \in X : f(x) \subseteq \text{dom}(g) \} \).
Mathematical Problems and Solutions

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Examples of Mathematical Problems

- **The Zero Problem** \( Z_X : \subseteq \mathcal{C}(X) \Rightarrow X, h \mapsto h^{-1}\{0\} \).

- **The Limit Problem** is the mathematical problem

  \[ \lim : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, ... \rangle \mapsto \lim_{i \rightarrow \infty} p_i \]

  with \( \text{dom}(\lim) := \{ \langle p_0, p_1, ... \rangle : (p_i)_i \text{ is convergent} \} \).

- **Martin-Löf Randomness** is the mathematical problem

  \( \text{MLR} : 2^{\mathbb{N}} \Rightarrow 2^{\mathbb{N}} \) with

  \[ \text{MLR}(x) := \{ y \in 2^{\mathbb{N}} : y \text{ is Martin-Löf random relative to } x \} \].

- **The Cohesiveness Problem** is the mathematical problem

  \( \text{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \Rightarrow 2^{\mathbb{N}} \) where \( \text{COH}(R_i) \) contains all infinite \( X \subseteq \mathbb{N} \) such that for all \( i \in \mathbb{N} \) one of the sets

  \[ X \cap R_i \text{ or } X \cap (\mathbb{N} \setminus R_i) \]

  is finite.
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- The cohesiveness problem is the mathematical problem $\text{COH} : (2^\mathbb{N})^\mathbb{N} \Rightarrow 2^\mathbb{N}$ where $\text{COH}(R_i)$ contains all infinite $X \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$ one of the sets

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Theorems as Problems

**Definition**

Any theorem $T$ of the $\Pi_2$ form

$$(\forall x \in X)(x \in D \implies (\exists y \in Y) P(x, y))$$

is identified with $F : \subseteq X \Rightarrow Y$ with $\text{dom}(F) := D$ and

$$F(x) := \{y \in Y : P(x, y)\}.$$  

**Examples:** Weak Weak Kőnig’s Lemma is the mathematical problem

$$\text{WWKL} : \subseteq \text{Tr} \Rightarrow 2^\mathbb{N}, T \mapsto [T]$$

with $\text{dom}(\text{WWKL}) := \{T \in \text{Tr} : \mu([T]) > 0\}.$

The Intermediate Value Theorem is the mathematical problem

$$\text{IVT} : \subseteq C[0, 1] \Rightarrow \mathbb{R}, f \mapsto f^{-1}\{0\}$$

where $\text{dom}(\text{IVT}) := \{f \in C[0, 1] : f(0) \cdot f(1) < 0\}.$
Theorems as Problems

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Any theorem \( T \) of the \( \Pi_2 \) form

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(\forall x \in X)(x \in D \implies (\exists y \in Y) \ P(x, y))
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Let $f : \subseteq X \Rightarrow Y$ and $g : \subseteq Z \Rightarrow W$ be two mathematical problems.

$f$ is Weihrauch reducible to $g$, $f \leq_W g$, if there are computable $H : \subseteq X \times W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $H(\text{id}_X, gK) \subseteq f$.

$f$ is strongly Weihrauch reducible to $g$, $f \leq_{sW} g$, if there are computable $H : \subseteq W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $HgK \subseteq f$.

Equivalences $f \equiv_W g$ and $f \equiv_{sW} g$ are defined as usual.

Theorem (Tavana and Weihrauch 2011)

$f \leq_W g \iff$ there is a Turing machine that computes $f$ and uses $g$ as an oracle exactly once during its infinite computation.
Weihrauch Reducibility

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\[
\begin{array}{c}
\text{x} \\
\downarrow \\
K \\
\downarrow \\
g \\
\downarrow \\
H \\
\downarrow \\
f(x)
\end{array}
\]

- \( f \) is **Weihrauch reducible** to \( g \), \( f \leq_W g \), if there are computable \( H : \subseteq X \times W \Rightarrow Y \), \( K : \subseteq X \Rightarrow Z \) such that \( H(\text{id}_X, gK) \sqsubseteq f \).
- \( f \) is **strongly Weihrauch reducible** to \( g \), \( f \leq_{sW} g \), if there are computable \( H : \subseteq W \Rightarrow Y \), \( K : \subseteq X \Rightarrow Z \) such that \( HgK \sqsubseteq f \).
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\text{f} \\
\text{g} \\
\text{H} \\
\end{array} \quad \xrightarrow{\text{x}} \quad \xrightarrow{\text{f(x)}} \\
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\[ f \leq_W g \iff \text{there is a Turing machine that computes } f \text{ and uses } g \text{ as an oracle exactly once during its infinite computation.} \]
A representation of $X$ is a surjective map $\delta_X : \subseteq \mathbb{N}^\mathbb{N} \to X$. 

$F : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $f : \subseteq X \Rightarrow Y$, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \text{dom}(f\delta_X)$.

$f$ is continuous, computable, polynomial-time computable or Borel measurable, if it admits a corresponding realizer $F$.

$f \leq_w g $ $\iff$ there are computable $H, K : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $H(\text{id}, GK) \vdash f$ whenever $G \vdash g$. 
Realizers and Representations

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\mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\
\downarrow{\delta_X} & & \downarrow{\delta_Y} \\
X & \xrightarrow{f} & Y \\
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**Definition**

$(X, d, \alpha)$ is called **computable metric space** if

1. $d : X \times X \to \mathbb{R}$ is a metric on $X$,
2. $\alpha : \mathbb{N} \to X$ is a sequence with a dense range,
3. $d \circ (\alpha \times \alpha) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is computable.

**Definition**

$\delta_X : \subseteq \mathbb{N}^\mathbb{N} \to X$ is called **Cauchy representation**, if

$$\delta_X(p) = x : \iff (\forall k) \ d(\alpha p(k), x) < 2^{-k}.$$
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Realizer Version of Problems

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Let \((X, \delta_X)\) and \((Y, \delta_Y)\) be represented spaces and \(f : \subseteq X \Rightarrow Y\) a mathematical problem. Then we define the realizer version \(f^r : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N}\) of \(f\) by \(f^r := \delta_Y^{-1} \circ f \circ \delta_X\).

Proposition

\(f \equiv_{sW} f^r\).

- This means that properties of \(\leq_W\) and \(\leq_{sW}\) can be studied by considering only problems of type \(f : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N}\).
- Arbitrary represented spaces \(X, Y\) are used as types in order to classify practical problems and theorems, which are most naturally expressed in such types.
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Cylinders and Strong Weihrauch Reducibility

By \( id : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \) we denote the identity of Baire space \( \mathbb{N}^\mathbb{N} \). We always have \( f \leq_{sW} id \times f \), the inverse is not necessarily true.

Definition

\( f : \subseteq X \Rightarrow Y \) is called a cylinder if \( id \times f \equiv_{sW} f \) and \( id \times f \) is called the cylindrification of \( f \).

Examples: \( \text{lim}, WKL \) are cylinders, \( WWKL, COH, MLR \) are not.

Proposition (B. and Gherardi 2011)

\[ f \leq_W g \iff f \leq_{sW} id \times g. \]

Corollary (B. and Gherardi 2011)

\[ (\forall f)(f \leq_W g \iff f \leq_{sW} g) \iff g \text{ is a cylinder.} \]

Remark: The relation between strong and ordinary Weihrauch reducibility has formal similarities to the relation between one-one and many-one reducibility.
Cylinders and Strong Weihrauch Reducibility

By $\text{id} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we denote the identity of Baire space $\mathbb{N}^\mathbb{N}$. We always have $f \leq_{\text{sW}} \text{id} \times f$, the inverse is not necessarily true.

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Cylinders and Strong Weihrauch Reducibility

By \( \text{id} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) we denote the identity of Baire space \( \mathbb{N}^\mathbb{N} \). We always have \( f \leq_{SW} \text{id} \times f \), the inverse is not necessarily true.

**Definition**

\( f : \subseteq X \Rightarrow Y \) is called a **cylinder** if \( \text{id} \times f \equiv_{SW} f \) and \( \text{id} \times f \) is called the **cylindrification** of \( f \).

**Examples:** \( \text{lim}, \text{WKL} \) are cylinders, \( \text{WWKL}, \text{COH}, \text{MLR} \) are not.

**Proposition (B. and Gherardi 2011)**

\[ f \leq_W g \iff f \leq_{SW} \text{id} \times g. \]

**Corollary (B. and Gherardi 2011)**

\[ (\forall f)(f \leq_W g \iff f \leq_{SW} g) \iff g \text{ is a cylinder}. \]

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Algebraic Operations in the Weihrauch Lattice

**Definition**

Let $f, g$ be two mathematical problems. We consider:

- $f \times g$: both problems are available in parallel (Product)
- $f \sqcup g$: both problems are available, but for each instance one has to choose which one is used (Coproduct)
- $f \sqcap g$: given an instance of $f$ and $g$, only one of the solutions will be provided (Sum)
- $f \ast g$: $f$ and $g$ can be used consecutively (Comp. Product)
- $g \rightarrow f$: this is the simplest problem $h$ such that $f$ can be reduced to $g \ast h$ (Implication)
- $f^*$: $f$ can be used any given finite number of times in parallel (Star)
- $\hat{f}$: $f$ can be used countably many times in parallel (Parallelization)
- $f'$: $f$ can be used on the limit of the input (Jump)
Definitions of Algebraic Operations

Definition

For \( f : \subseteq X \Rightarrow Y \) and \( g : \subseteq W \Rightarrow Z \) we define:

- \( f \times g : \subseteq X \times W \Rightarrow Y \times Z \), \((x, w) \mapsto f(x) \times g(w)\) (Product)

- \( f \sqcup g : \subseteq X \sqcup W \Rightarrow Y \sqcup Z \), \(z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}\) (Coproduct)

- \( f \sqcap g : \subseteq X \times W \Rightarrow Y \sqcup Z \), \((x, w) \mapsto f(x) \sqcup g(w)\) (Sum)

- \( f^* : \subseteq X^* \Rightarrow Y^* \), \(f^* = \bigsqcup_{i=0}^\infty f^i\) (Star)

- \( \hat{f} : \subseteq X^N \Rightarrow Y^N, \hat{f} = \bigcup_{i=0}^\infty f \) (Parallelization)

Here

- \( Y \times Z \) denotes the usual Caresian product,
- \( Y \sqcup Z := (\{0\} \times Y) \cup (\{1\} \times Z) \) denotes the disjoint union,
- \( X^* := \{f : \mathbb{N} \to X : \text{dom}(f) = n \text{ for some } n \in \mathbb{N}\} \) denotes the set of words over \( X \), where \( n = \{0, \ldots, n - 1\} \),
- \( X^N := \{f : \mathbb{N} \to X\} \) denotes the set of sequences over \( X \).
The Algebraic Structure of the Weihrauch Lattice

Proposition (B., Gherardi 2011, Pauly 2010)

Weihrauch reducibility induces a distributive lattice with the coproduct $\sqcup$ as supremum and $\sqcap$ as infimum. Parallelization $\hat{\wedge}$ and star operation $\ast$ are closure operators in the Weihrauch lattice.

- With $\sqcup, \times, \ast$ one obtains a Kleene algebra (B., Pauly).
- The Weihrauch lattice is neither a Brouwer nor a Heyting algebra (Higuchi und Pauly 2012).

Open Problem

Does the strong Weihrauch reducibility induce a lattice structure?

- It is known that $\sqcap$ is an infimum for $\leq_{sW}$ and hence one obtains a lower semi-lattice (B., Gherardi).
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Constants of the Weihrauch Lattice

- **0**: the equivalence class of the *nowhere defined problems* is the bottom element of the Weihrauch lattice, and a neutral element with respect to $\sqcup$. It acts like a zero with respect to $\times$ and $\ast$.

- **1**: the equivalence class of the identity $\text{id} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a neutral element with respect to $\times$ and $\ast$.

- $0^* \equiv_W 1$.

- **$\infty$**: the equivalence class of all problems without realizer is the top element of the Weihrauch lattice and a neutral element with respect to $\sqcap$.

- $\infty$ exists if and only if the Axiom of Choice does not hold for Baire space $\mathbb{N}^\mathbb{N}$.

- We usually assume that the Axiom of Choice holds, but we can always add an artificial element $\infty$ on top of the Weihrauch lattice.
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The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

**Theorem (B. and Pauly 2013)**

For two mathematical problems \( f, g \) the following exist:

- \( f \star g := \max\{f_0 \circ g_0 : f_0 \leq_W f \text{ and } g_0 \leq_W g\} \) and
- \( g \rightarrow f := \min\{h : f \leq_W g \star h\}. \)

The maximum and minimum is understood with respect to \( \leq_W \).

**Proof.** (Sketch) For every \( f : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \) we consider the transpose \( f^t : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \) defined by

\[
f^t\langle p, q \rangle := \eta_p \circ f(q),
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where \( \eta \) is a standard representation of all continuous functions \( F : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \). For arbitrary \( f, g \) we obtain

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f \star g \equiv_W f^{rt} \circ g^{rt}.
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The case of \( g \rightarrow f \) can be treated similarly.
Compositional Product and Implication

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Relations Between Algebraic Operations

\( f \text{ pointed} : \iff 1 \leq_W f \iff (\exists x \in \text{dom}(f)) x \text{ computable.} \)

**Proposition**

For pointed \( f, g \) we obtain

\[
f \sqcap g \leq_W f \sqcup g \leq_W f \times g \leq_W f \ast g,
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where pointedness is needed only for \( f \sqcup g \leq_W f \times g \).

**Proof.** \( f \sqcap g \leq_W f \sqcup g \leq_W f \times g \) is clear. The last reduction follows since

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For pointed \( f \) we obtain \( f^* \leq_W \hat{f} \).
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Algebraic Closure Properties

- **f** is called **idempotent** if \( f \times f \equiv_W f \),
  for pointed \( f \) this holds if and only if \( f^* \equiv_W f \).

- **Examples**: \( \lim, \text{WKL, WWKL, MLR} \) are idempotent, \( \text{IVT} \) is not.

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Remark

There is a vague analogy between versions of Weihrauch reducibilities induced by closure operators and computability theoretic reducibilities:

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Embedding of the Medvedev Lattice

Proposition (B. and Gherardi 2011)

\[ A \leq_M B \iff c_A \leq_W c_B \iff \operatorname{id}|_B \leq_W \operatorname{id}|_A \text{ for } A, B \subseteq \mathbb{N}^\mathbb{N}. \]

- \( c_A : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, p \mapsto A \) is the constant multi-valued function.
- By \( \operatorname{id}|_A : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) denotes the identity restricted to \( A \).
- We note that \( \operatorname{id}|_A \leq_W 1 \leq_W c_A \).
- \( p \leq_T q \iff \{p\} \leq_M \{q\} \), hence also the Turing semi-lattice embeds into the Weihrauch lattice.

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Lemma (B., Hendtlass and Kreuzer 2015)

\[ f \leq_W g \implies (\forall \text{computable } p \in \text{dom}(f))(\exists \text{computable } q \in \text{dom}(g)) \]
\[ f(p) \leq_M g(q) \]

for \( f, g : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \).

- Hence, Weihrauch reducibility can be seen as a parameterized version of Medvedev reducibility.
- Computability theoretic problems such as MLR, where the input is just an oracle, can and have also been studied in the Medvedev lattice (for computable inputs).
- As long as the proofs relativize, one obtains corresponding results in the Weihrauch lattice.
- Other problems such as WKL, WWKL depend on inputs in a relevant way and can be compared to problems such as MLR in the Weihrauch lattice.
Lemma (B., Hendtlass and Kreuzer 2015)

\[ f \leq_W g \]

\[ \implies (\forall \text{computable } p \in \text{dom}(f))(\exists \text{computable } q \in \text{dom}(g)) \]

\[ f(p) \leq_M g(q) \]

for \( f, g : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \).

- Hence, Weihrauch reducibility can be seen as a parameterized version of Medvedev reducibility.
- Computability theoretic problems such as MLR, where the input is just an oracle, can and have also been studied in the Medvedev lattice (for computable inputs).
- As long as the proofs relativize, one obtains corresponding results in the Weihrauch lattice.
- Other problems such as WKL, WWKL depend on inputs in a relevant way and can be compared to problems such as MLR in the Weihrauch lattice.
Zoo of Reducibilities

Weihrauch complexity
(computable)

Reverse mathematics
(over $\text{RCA}_0$)

uniform
resource sensitive

refines

non-uniform
closed under composition

\[ f \leq_{\text{sW}} g \longrightarrow f \leq_{\text{W}} g \rightarrow f \leq_{gW} g \]

\[ f \leq_{\text{sc}} g \rightarrow f \leq_{c} g \rightarrow f \leq_{\omega} g \]

Diagram based on: Hirschfeldt and Jockusch, On Notions of Computability Theoretic Reduction Between $\Pi^1_2$ Principles, preprint 2015.
Can the slogan “Weihrauch complexity is a kind of a model of reverse mathematics with some form of (intuitionistic) linear logic” be converted into a theorem?
Choice
Let \((X, d, \alpha)\) be a computable metric space and \(A \subseteq X\) closed.

By \(B_{\langle n, k \rangle} := B(\alpha(n), \overline{k})\) we denote the ball with center \(\alpha(n)\) and rational radius \(\overline{k}\). Here \(\langle a, b, c \rangle := \frac{a-b}{c+1}\).

Then the following are equivalent to each other:

- \(A\) is co-c.e. closed,
- \(X \setminus A = \bigcup_{i=0}^{\infty} B_{n_i}\) for a computable sequence \((n_i)\) of natural numbers,
- \(A = f^{-1}\{0\}\) for a computable function \(f : X \rightarrow \mathbb{R}\).
We define a representation $\psi : \mathbb{N}^\mathbb{N} \to A_-(X)$ of the set $A_-(X)$ of all closed subsets of $X$ by

$$\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}.$$  

The computable points in the represented space $A_-(X)$ are exactly the co-c.e. closed subsets $A \subseteq X$.

There is also a natural representation of the set $C(X)$ of continuous functions $f : X \to \mathbb{R}$.

**Proposition**

$P : C(X) \to A_-(X), f \mapsto f^{-1}\{0\}$ is a computable isomorphism in the sense that $P$ and $P^{-1}$ are computable.
We define a representation \( \psi_- : \mathbb{N}^\mathbb{N} \to \mathcal{A}_-(X) \) of the set \( \mathcal{A}_-(X) \) of all closed subsets of \( X \) by

\[
\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_p(i).
\]

The computable points in the represented space \( \mathcal{A}_-(X) \) are exactly the co-c.e. closed subsets \( A \subseteq X \).

There is also a natural representation of the set \( \mathcal{C}(X) \) of continuous functions \( f : X \to \mathbb{R} \).

**Proposition**

\( P : \mathcal{C}(X) \to \mathcal{A}_-(X), f \mapsto f^{-1}\{0\} \) is a computable isomorphism in the sense that \( P \) and \( P^{-1} \) are computable.
We define a representation $\psi^- : N^N \to A_-(X)$ of the set $A_-(X)$ of all closed subsets of $X$ by

$$\psi^-(p) := X \setminus \bigcup_{i=0}^{\infty} B_p(i).$$

The computable points in the represented space $A_-(X)$ are exactly the co-c.e. closed subsets $A \subseteq X$.

There is also a natural representation of the set $C(X)$ of continuous functions $f : X \to \mathbb{R}$.

**Proposition**

$P : C(X) \to A_-(X), f \mapsto f^{-1}\{0\}$ is a computable isomorphism in the sense that $P$ and $P^{-1}$ are computable.
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**Proposition**

\( P : \mathcal{C}(X) \to \mathcal{A}_-(X), f \mapsto f^{-1}\{0\} \) is a computable isomorphism in the sense that \( P \) and \( P^{-1} \) are computable.
**Definition**

$C_X : \subseteq \mathcal{A}(X) \Rightarrow X, A \mapsto A$ is called the **choice problem** of a computable metric space $X$.

This is the problem that corresponds to the statement:

- Every non-empty closed set $A \subseteq X$ has a point $x \in A$.

**Corollary**

$C_X \equiv_{sW} Z_X$ for every computable metric space $X$.

The choice problem is equivalent to the zero problem of finding a solution $x \in X$ of the equation

$$f(x) = 0$$

for a continuous function $f : X \rightarrow \mathbb{R}$. Formally, we consider the zero problem as $Z_X : \subseteq \mathcal{C}(X) \Rightarrow X, f \mapsto f^{-1}\{0\}$.
Choice

**Definition**

\( C_X : \subseteq A_\neg (X) \implies X, A \mapsto A \) is called the choice problem of a computable metric space \( X \).

This is the problem that corresponds to the statement:

- Every non-empty closed set \( A \subseteq X \) has a point \( x \in A \).

**Corollary**

\( C_X \equiv_{SW} Z_X \) for every computable metric space \( X \).

The choice problem is equivalent to the zero problem of finding a solution \( x \in X \) of the equation

\[ f(x) = 0 \]

for a continuous function \( f : X \to \mathbb{R} \). Formally, we consider the zero problem as \( Z_X : \subseteq C(X) \implies X, f \mapsto f^{-1}\{0\} \).
Choice

Definition

\( C_X : \subseteq A_-(X) \Rightarrow X, A \mapsto A \) is called the choice problem of a computable metric space \( X \).

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Mind-Changes and Choice

Proposition

\[ C_0 \equiv_W 0, \quad C_1 \equiv_W 1, \quad C_2 \equiv_W \text{LLPO}, \quad C_N \equiv_{sW} \lim_N. \]

Proposition (B. and Gherardi 2011)

\( f \leq_W g. \) If \( g \) is computable with \( n \) mind changes, then so is \( f \).

Proposition (B., de Brecht and Pauly 2012)

\( f \leq_W C_N \iff f \) is computable with finitely many mind changes.

Corollary

\( C_n <_W C_{n+1} <_W C_N \) for all \( n \in \mathbb{N} \).
Proposition

\[ C_0 \equiv_W 0, \; C_1 \equiv_W 1, \; C_2 \equiv_W \text{LLPO}, \; C_N \equiv_{sW} \text{lim}_N. \]

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Corollary

\[ C_n <_W C_{n+1} <_W C_N \text{ for all } n \in \mathbb{N}. \]
Theorem (B. and Gherardi 2011)

\[ \text{WKL} \equiv_{sW} \text{C}_2^N \equiv_{sW} \hat{\text{C}}_2. \]

**Proof.** The equivalence \( \text{WKL} \equiv_{sW} \text{C}_2^N \) follows since the map

\[ [\,] : \text{Tr} \rightarrow \mathcal{A}_-(2^N), \ T \mapsto [T] \]

which maps a binary tree to the set of its infinite paths is computable and has a computable right inverse. The equivalence proof for \( \text{C}_2^N \equiv_{sW} \hat{\text{C}}_2 \) exploits the fact that for finding an infinite path it is sufficient to make countably many binary decisions (regarding the question which subtree is infinite) and vice versa. \( \square \)

Proposition (B., Gherardi and Marcone 2012)

\( \text{C}_2^* \equiv_W \text{K}_N <_W \text{C}_N. \)

Here \( \text{K}_N \) denotes compact choice on \( \mathbb{N} \), where besides the negative information on the set \( A \subseteq \mathbb{N} \) also an upper bound is provided.
Theorem (B. and Gherardi 2011)

\[ \text{WKL} \equiv_{sW} C_{2^N} \equiv_{sW} \hat{C}_2. \]

**Proof.** The equivalence \( \text{WKL} \equiv_{sW} C_{2^N} \) follows since the map

\[ [\cdot] : \text{Tr} \to \mathcal{A}_-(2^N), \; T \mapsto [T] \]

which maps a binary tree to the set of its infinite paths is computable and has a computable right inverse. The equivalence proof for \( C_{2^N} \equiv_{sW} \hat{C}_2 \) exploits the fact that for finding an infinite path it is sufficient to make countably many binary decisions (regarding the question which subtree is infinite) and vice versa. □

Proposition (B., Gherardi and Marcone 2012)

\[ C^*_2 \equiv_W K_N <_W C_N. \]

Here \( K_N \) denotes compact choice on \( \mathbb{N} \), where besides the negative information on the set \( A \subseteq \mathbb{N} \) also an upper bound is provided.
Theorem (B. and Gherardi 2011)

\[ \text{WKL} \equiv_{\text{sW}} C_{2^\mathbb{N}} \equiv_{\text{sW}} \widehat{C}_2. \]

**Proof.** The equivalence \( \text{WKL} \equiv_{\text{sW}} C_{2^\mathbb{N}} \) follows since the map

\[ [\ ] : \text{Tr} \to A_-(2^\mathbb{N}), \ T \mapsto [T] \]

which maps a binary tree to the set of its infinite paths is computable and has a computable right inverse. The equivalence proof for \( C_{2^\mathbb{N}} \equiv_{\text{sW}} \widehat{C}_2 \) exploits the fact that for finding an infinite path it is sufficient to make countably many binary decisions (regarding the question which subtree is infinite) and vice versa. \( \square \)

Proposition (B., Gherardi and Marcone 2012)

\[ C_2^* \equiv_{\text{W}} K_{\mathbb{N}} <_{\text{W}} C_{\mathbb{N}}. \]

Here \( K_{\mathbb{N}} \) denotes compact choice on \( \mathbb{N} \), where besides the negative information on the set \( A \subseteq \mathbb{N} \) also an upper bound is provided.
The positive choice problem $PC_X :\subseteq \mathcal{A}_-(X) \Rightarrow X$, $A \mapsto A$ of a computable metric space $X$ with a Borel measure $\mu$ is the restriction of $C_X$ to $\text{dom}(PC_X) := \{ A \subseteq X : \mu(A) > 0 \}$.

We use the usual uniform measure on $2^\mathbb{N}$ and the Lebesgue measure on $[0, 1]$.

**Proposition (B., Gherardi and Hölzl 2015)**

$PC_{2^\mathbb{N}} \equivSW WWKL$.

**Proposition (B. and Pauly 2010)**

$WWKL <W WKL$.

We have $\text{id}_{2^\mathbb{N}} \not\leqSW WWKL$. Hence $WWKL$ is not a cylinder.

We have $C_2 \leqW WWKL$. Hence $WWKL \equivW WKL$. 
Positive Choice

- The positive choice problem $\text{PC}_X : \subseteq \mathcal{A}_+(X) \Rightarrow X, A \mapsto A$ of a computable metric space $X$ with a Borel measure $\mu$ is the restriction of $C_X$ to $\text{dom}(\text{PC}_X) := \{ A \subseteq X : \mu(A) > 0 \}$.
- We use the usual uniform measure on $2^\mathbb{N}$ and the Lebesgue measure on $[0, 1]$.

**Proposition (B., Gherardi and Hölzl 2015)**

$\text{PC}_{2^\mathbb{N}} \equiv_s \text{WWKL}$.  

**Proposition (B. and Pauly 2010)**

$\text{WWKL} <_W \text{WKL}$.  

- We have $\text{id}_{\mathbb{N}^\mathbb{N}} \not\leq_s \text{WWKL}$. Hence $\text{WWKL}$ is not a cylinder.
- We have $C_2 \leq_W \text{WWKL}$. Hence $\overline{\text{WWKL}} \equiv_W \text{WKL}$. 
Positive Choice

- The positive choice problem $\text{PC}_X : \subseteq \mathcal{A}_-(X) \Rightarrow X, A \mapsto A$ of a computable metric space $X$ with a Borel measure $\mu$ is the restriction of $C_X$ to $\text{dom}(\text{PC}_X) := \{A \subseteq X : \mu(A) > 0\}$.
- We use the usual uniform measure on $2^\mathbb{N}$ and the Lebesgue measure on $[0, 1]$.

**Proposition (B., Gherardi and Hölzl 2015)**

$\text{PC}_{2^\mathbb{N}} \equiv_{\text{sw}} \text{WWKL}$.

**Proposition (B. and Pauly 2010)**

$\text{WWKL} \prec \text{WKL}$.

- We have $\text{id}_{\mathbb{N}^\mathbb{N}} \not\leq_{\text{sw}} \text{WWKL}$. Hence $\text{WWKL}$ is not a cylinder.
- We have $C_2 \leq_{\text{w}} \text{WWKL}$. Hence $\text{WWKL} \equiv_{\text{w}} \text{WKL}$. 
Positive Choice

- The positive choice problem $PC_X : \subseteq A_-(X) \Rightarrow X, A \mapsto A$ of a computable metric space $X$ with a Borel measure $\mu$ is the restriction of $C_X$ to $\text{dom}(PC_X) := \{A \subseteq X : \mu(A) > 0\}$.

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Proposition (B., Gherardi and Hölzl 2015)

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$WWKL \prec_W WKL$.

- We have $\text{id}_{\mathbb{N}^\mathbb{N}} \not\leq_s WWKL$. Hence $WWKL$ is not a cylinder.

- We have $C_2 \leq_W WWKL$. Hence $\overline{WWKL} \equiv_W WKL$. 

Basic Complexity Classes

\[
C_{NN} \supseteq \text{lim } sW C_N
\]

\[
C_R \equiv w C_N \times C_{2^N}
\]

\[
WKL \equiv_{sW} C_{2^N} \equiv_{sW} \hat{C}_2
\]

\[
WWKL \equiv_{sW} PC_{2^N}
\]

\[
K_N \equiv_{sW} C_{2}^*
\]

\[
\text{LLPO} \equiv_{sW} C_2
\]
Turing Machines with Advice

Condition: \((\forall x \in \text{dom}(f)) \{r \in R : r \text{ does not fail with } x\} \neq \emptyset\)
Las Vegas Turing Machines

Condition: \((\forall x \in \text{dom}(f)) \mu\{ r \in R : r \text{ does not fail with } x \} > 0\)
Calibrating Computability with Choice

Theorem (B., de Brecht and Pauly 2012)

For \( R \subseteq \mathbb{N}^\mathbb{N} \) and \( f : \subseteq X \Rightarrow Y \) the following are equivalent:

- \( f \leq_W C_R \),
- \( f \) is computable on a Turing machine with advice from \( R \).

Corollary

- \( f \leq C_{\{0\}} \iff f \) is computable,
- \( f \leq_W C_N \iff f \) comp. with finitely many mind changes,
- \( f \leq_W C_{2^N} \iff f \) is non-deterministically computable,
- \( f \leq_W PC_{2^N} \iff f \) is Las Vegas computable,
- \( f \leq_W \widehat{C}_N \iff f \) is limit computable,
- \( f \leq_W C_{\mathbb{N}^\mathbb{N}} \iff f \) is effectively Borel measurable.

In the last case \( f \) is single-valued on computable Polish spaces.
### Theorem (B., de Brecht and Pauly 2012)

For $R \subseteq \mathbb{N}^\mathbb{N}$ and $f : \subseteq X \Rightarrow Y$ the following are equivalent:

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### Corollary

- $f \leq C_{\{0\}} \iff f$ is computable,
- $f \leq_w C_\mathbb{N} \iff f$ comp. with finitely many mind changes,
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Computational Classes

\[ C_{\mathbb{N}} \]

- Effective Borel measurability

\[ \text{lim} \equiv_{sW} \widehat{C_{\mathbb{N}}} \]

- Limit computation

\[ \text{WKL} \equiv_{sW} C_{2^{\mathbb{N}}} \]

- Non-deterministic computation

\[ \text{WWKL} \equiv_{sW} PC_{2^{\mathbb{N}}} \]

- Las Vegas computation

\[ C_{\mathbb{N}} \]

- Finite mind change computation
Independent Choice Theorem

**Theorem (B., de Brecht and Pauly 2012)**

\[ C_R \ast C_S \leq_W C_{R\times S} \text{ for all } R, S \subseteq \mathbb{N}^\mathbb{N}. \]

**Proof.** Run a Turing machine that simulates upon advice \((r, s)\) two consecutive machines with advice \(r\) and \(s\), respectively. □

**Proposition**

If \(s : R \to S\) is a computable surjection, then \(C_S \leq_W C_R\).

**Corollary**

\(C_R\) is closed under composition for \(R \in \{\mathbb{N}, 2^\mathbb{N}, \mathbb{N} \times 2^\mathbb{N}, \mathbb{N}^\mathbb{N}\}\).

**Corollary (Gherardi and Marcone 2009, B. and Gherardi 2011)**

\(WKL\) is closed under composition.
Theorem (B., de Brecht and Pauly 2012)

\[ C_R \times C_S \leq_W C_{R \times S} \text{ for all } R, S \subseteq \mathbb{N}^\mathbb{N}. \]

**Proof.** Run a Turing machine that simulates upon advice \((r, s)\) two consecutive machines with advice \(r\) and \(s\), respectively. \(\square\)

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### Independent Choice Theorem

**Theorem (B., de Brecht and Pauly 2012)**

\[ C_R \ast C_S \leq_W C_{R \times S} \text{ for all } R, S \subseteq \mathbb{N}^N. \]

**Proof.** Run a Turing machine that simulates upon advice \((r, s)\) two consecutive machines with advice \(r\) and \(s\), respectively. \(\square\)

**Proposition**

*If \( s : R \rightarrow S \) is a computable surjection, then \( C_S \leq_W C_R \).*

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**Corollary (Gherardi and Marcone 2009, B. and Gherardi 2011)**

*WKL is closed under composition.*
Theorem (B., de Brecht and Pauly 2012)

\[ C_R \times C_S \leq^W C_{R \times S} \text{ for all } R, S \subseteq \mathbb{N}^\mathbb{N}. \]

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**Corollary (Gherardi and Marcone 2009, B. and Gherardi 2011)**

\(WKL\) is closed under composition.
Theorem (B., Gherardi and Hölzl 2015)

\[ \text{PC}_R \times \text{PC}_S \leq_W \text{PC}_{R \times S} \] for \( R, S \subseteq \mathbb{N}^\mathbb{N} \) with \( \sigma \)-finite Borel measures and their product measure.

**Proof.** (Sketch) The proof proceeds along the lines of the case for closed choice plus an additional invocation of Fubini’s Theorem. □

**Corollary**

\( \text{PC}_R \) is closed under composition for \( R \in \{\mathbb{N}, \mathbb{N}^\mathbb{N}, \mathbb{N} \times 2^\mathbb{N}, 2^\mathbb{N} \} \).

**Corollary**

WWKL is closed under composition.

**Corollary**

Las Vegas computable functions are closed under composition.
Independent Choice Theorem

Theorem (B., Gherardi and Hölzl 2015)

\[ PC_R \times PC_S \leq_W PC_{R \times S} \text{ for } R, S \subseteq \mathbb{N}^\mathbb{N} \text{ with } \sigma-\text{finite Borel measures and their product measure.} \]

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**Corollary**

\[ PC_R \text{ is closed under composition for } R \in \{\mathbb{N}, 2^\mathbb{N}, \mathbb{N} \times 2^\mathbb{N}, \mathbb{N}^\mathbb{N}\}. \]

**Corollary**

\[ \text{WWKL is closed under composition.} \]

**Corollary**

\[ \text{Las Vegas computable functions are closed under composition.} \]
Theorem (B., Gherardi and Hölzl 2015)

$$PC_R \times PC_S \leq_W PC_{R \times S} \text{ for } R, S \subseteq \mathbb{N}^\mathbb{N} \text{ with } \sigma- \text{finite Borel measures and their product measure.}$$

**Proof.** (Sketch) The proof proceeds along the lines of the case for closed choice plus an additional invocation of Fubini’s Theorem. □

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$$PC_R \text{ is closed under composition for } R \in \{\mathbb{N}, 2^{\mathbb{N}}, \mathbb{N} \times 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}\}.$$
Theorem (B., Gherardi and Hölzl 2015)

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\[ \text{PC}_R \text{ is closed under composition for } R \in \{ \mathbb{N}, 2^\mathbb{N}, \mathbb{N} \times 2^\mathbb{N}, \mathbb{N}^\mathbb{N} \}. \]

**Corollary**

\[ \text{WWKL is closed under composition.} \]

**Corollary**

\[ \text{Las Vegas computable functions are closed under composition.} \]
Theorem (B., de Brecht and Pauly 2012)

Let $X$ be a computable Polish space. Then

- $C_X \leq_{SW} C_{\mathbb{N}^\mathbb{N}}$, 
- $C_X \leq_{SW} C_{2^\mathbb{N}}$ if $X$ is computably compact, 
- $C_{2^\mathbb{N}} \leq_{SW} C_X$ if $X$ is perfect, 
- $C_X \leq_{SW} C_{\mathbb{N} \times 2^\mathbb{N}}$ if $X$ is a computable $K_\sigma$–space, 
- $C_X \equiv_{SW} C_{\mathbb{N}^\mathbb{N}}$ with respect to some oracle, if $X$ is not $K_\sigma$.

Corollary

For all $n \geq 1$:

- $C_{[0,1]^n} \equiv_{SW} C_{2^\mathbb{N}}$ 
- $C_{\mathbb{R}^n} \equiv_{SW} C_{\mathbb{N} \times 2^\mathbb{N}} \equiv_{SW} C_{\mathbb{N}} \times C_{2^\mathbb{N}} \equiv_{SW} C_{\mathbb{N}} \ast C_{2^\mathbb{N}}$ 
- $C_{C[0,1]} \equiv_{SW} C_{\ell_2} \equiv_{SW} C_{\mathbb{N}^\mathbb{N}}$
Theorem (B., de Brecht and Pauly 2012)

Let $X$ be a computable Polish space. Then

- $C_X \leq_{SW} C_{\mathbb{N}^\mathbb{N}}$,
- $C_X \leq_{SW} C_{2^\mathbb{N}}$ if $X$ is computably compact,
- $C_{2^\mathbb{N}} \leq_{SW} C_X$ if $X$ is perfect,
- $C_X \leq_{SW} C_{\mathbb{N} \times 2^\mathbb{N}}$ if $X$ is a computable $K_\sigma$–space,
- $C_X \equiv_{SW} C_{\mathbb{N}^\mathbb{N}}$ with respect to some oracle, if $X$ is not $K_\sigma$.

Corollary

For all $n \geq 1$:

- $C_{[0,1]^n} \equiv_{SW} C_{2^\mathbb{N}}$
- $C_{\mathbb{R}^n} \equiv_{SW} C_{\mathbb{N} \times 2^\mathbb{N}} \equiv_{SW} C_{\mathbb{N}} \times C_{2^\mathbb{N}} \equiv_{SW} C_{\mathbb{N}} \ast C_{2^\mathbb{N}}$
- $C_{C[0,1]} \equiv_{SW} C_{\ell_2} \equiv_{SW} C_{\mathbb{N}^\mathbb{N}}$
Choice for Computable Polish Spaces

- \( C_{\mathbb{N}} \equiv_{sW} C_{\ell_\infty} \equiv_{sW} C_{[0,1]} \)
- \( C_{\mathbb{N} \times 2} \equiv_{sW} C_{\mathbb{R}^n} \equiv_{sW} C_{2^\mathbb{N} \times \mathbb{N}} \)
- \( C_{\mathbb{N}} \equiv_{sW} C_{[0,1]^n} \equiv_{sW} C_{[0,1]^\mathbb{N}} \)
- \( C_{\mathbb{N}} \equiv_{sW} C_{\mathbb{Z}} \equiv_{sW} C_{\mathbb{Q}} \)

- Perfect non locally compact
- Perfect locally compact
- Perfect compact
- Countable discrete
- Finite

- \( C_1 \)
- \( C_2 \)
- \( C_3 \)
The following result is reminiscent of certain conservation results.

**Theorem (B., de Brecht and Pauly 2012)**

\[ f \preceq_W C_{2^N} \ast g \implies f \preceq_W g \]

for single-valued \( f : \subseteq X \rightarrow Y \) on computable metric spaces \( X, Y \).

**Proof.** (Idea.) A non-deterministic computation that yields a unique result cannot really exploit the advice \( r \in 2^N \). The compact set of successful advices can be systematically searched in order to find a successful advice.

**Corollary**

\[ f \preceq_W C_{2^N} \implies f \text{ computable (for } f \text{ as above).} \]

**Corollary**

\[ C_N \not\preceq_W C_{2^N}. \]

\[ \lim_N \equiv_W C_N \text{ is single-valued and non-computable.} \]
The following result is reminiscent of certain conservation results.

**Theorem (B., de Brecht and Pauly 2012)**

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The following result is reminiscent of certain conservation results.

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**Corollary**

\[ f \leq W C_{2^\mathbb{N}} \implies f \text{ computable (for } f \text{ as above).} \]

**Corollary**

\[ C_{\mathbb{N}} \not\leq W C_{2^\mathbb{N}}. \]

\[ \lim_{N} \equiv_{SW} C_{\mathbb{N}} \] is single-valued and non-computable.
The following result is reminiscent of certain conservation results.

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\[ f \leq_w C_{2^N} \ast g \implies f \leq_w g \]

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**Corollary**

\[ C_N \not\leq_w C_{2^N}. \]

\( \lim_{N} \equiv_s W C_N \) is single-valued and non-computable.
Choice Elimination for Choice on Natural Numbers

- \( f \) is called a **fractal** if there is a \( F : \subseteq N^N \rightarrow N^N \) with \( F \equiv_W f \) and \( F|_U \equiv_W f \) for every open \( U \subseteq N^N \) with \( U \cap \text{dom}(F) \neq \emptyset \).
- \( f \) is called a **total fractal** if there is a total \( F \) as above.
- **Strong (total) fractals** are defined analogously with \( \equiv_{sW} \).

**Theorem (Le Roux and Pauly 2015)**

\[
f \leq_W C_N * g \implies f \leq_W g \text{ for total fractals } f.
\]

**Proof.** (Idea.) Replace \( f \) by a total fractal and apply the Baire Category Theorem to the sets \( A_n \) of inputs to \( F \) for which \( C_N \) yields the number \( n \) as a possible result. Then \( N^N = \bigcup_{n=0}^{\infty} A_n \) and one of the sets \( A_n \) is somewhere dense. The fractality condition yields the desired reduction. \( \square \)

**Corollary (B. and Gherardi 2011)**

\( \text{IVT} \not\leq_W C_N \) and hence \( \text{IVT}|_W C_N \).

It is clear that also \( \text{PC}_{2^N} \not\leq_W C_N \).
$f$ is called a **fractal** if there is a $F : \subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ with $F \equiv_W f$ and $F|_U \equiv_W f$ for every open $U \subseteq \mathbb{N}^\mathbb{N}$ with $U \cap \text{dom}(F) \neq \emptyset$.

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strong (total) fractals are defined analogously with $\equiv_{sW}$.

**Theorem (Le Roux and Pauly 2015)**

$f \leq_W C_\mathbb{N} \ast g \implies f \leq_W g$ for total fractals $f$.

**Proof.** (Idea.) Replace $f$ by a total fractal and apply the Baire Category Theorem to the sets $A_n$ of inputs to $F$ for which $C_\mathbb{N}$ yields the number $n$ as a possible result. Then $\mathbb{N}^\mathbb{N} = \bigcup_{n=0}^\infty A_n$ and one of the sets $A_n$ is somewhere dense. The fractality condition yields the desired reduction. □

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Choice Elimination for Choice on Natural Numbers

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**Theorem (Le Roux and Pauly 2015)**

\[ f \leq_{_{W}} \mathcal{C}_{\mathbb{N}} \ast g \implies f \leq_{_{W}} g \text{ for total fractals } f. \]

**Proof.** (Idea.) Replace $f$ by a total fractal and apply the Baire Category Theorem to the sets $A_{n}$ of inputs to $F$ for which $\mathcal{C}_{\mathbb{N}}$ yields the number $n$ as a possible result. Then $\mathbb{N}^\mathbb{N} = \bigcup_{n=0}^{\infty} A_{n}$ and one of the sets $A_{n}$ is somewhere dense. The fractality condition yields the desired reduction. \(\square\)

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\[ \text{IVT} \not\leq_{_{W}} \mathcal{C}_{\mathbb{N}} \text{ and hence } \text{IVT}|_{_{W}} \mathcal{C}_{\mathbb{N}}. \]

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Choice Elimination for Choice on Natural Numbers

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$\text{IVT} \nleq_W C_{\mathbb{N}}$ and hence $\text{IVT} |_{W C_{\mathbb{N}}}$. 

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Basic Complexity Classes

\[
C_{NN} \Downarrow
\]

\[
\lim \equiv_{SW} \hat{C}_N
\]

\[
C_R \equiv_{W} C_N \times C_{2N}
\]

\[
WKL \equiv_{SW} C_{2N} \equiv_{SW} \hat{C}_2
\]

\[
WWKL \equiv_{SW} PC_{2N}
\]

\[
K_N \equiv_{SW} C^*_N
\]

\[
\lim_N \equiv_{SW} C_N
\]

\[
LLPO \equiv_{SW} C_2
\]
Join Irreducibility

For $g_n : \subseteq X \Rightarrow Y$ we define
$$\bigsqcup_{n=0}^{\infty} g_n : \subseteq \mathbb{N} \times X \Rightarrow Y, (n, x) \mapsto g_n(x).$$

Definition

$f$ is called **join irreducible**, if one of the following equivalent conditions hold:

- $f \equiv_W \bigsqcup_{n=0}^{\infty} g_n \Rightarrow (\exists n) f \equiv_W g_n.$
- $f \leq_W \bigsqcup_{n=0}^{\infty} g_n \Rightarrow (\exists n) f \leq_W g_n.$

Equivalence follows since the Weihrauch lattice is distributive.

Proposition (B., de Brecht and Pauly 2012)

*Every fractal $f$ is join irreducible.*

Corollary

$$C_N \sqcup C_{2^N} <_W C_N \times C_{2^N}.$$  

$$C_N \times C_{2^N} \equiv_W C_\mathbb{R}$$ is a fractal.
Join Irreducibility

For $g_n : \subseteq X \Rightarrow Y$ we define
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**Definition**

$f$ is called **join irreducible**, if one of the following equivalent conditions hold:

1. $f \equiv_W \bigsqcup_{n=0}^{\infty} g_n \Rightarrow (\exists n) f \equiv_W g_n$,
2. $f \leq_W \bigsqcup_{n=0}^{\infty} g_n \Rightarrow (\exists n) f \leq_W g_n$.

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$C_N \sqcup C_{2^N} \leq_W C_N \times C_{2^N}$.

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Join Irreducibility

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$$C_N \sqcup C_{2^N} \leq_W C_N \times C_{2^N}.$$

$$C_N \times C_{2^N} \equiv_W C_{\mathbb{R}}$$ is a fractal.
We consider:

- \( \min : \mathbb{N}^\mathbb{N} \to \mathbb{N} \), \( p \mapsto \min\{p(n) : n \in \mathbb{N}\} \),

- \( \max : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N} \), \( p \mapsto \max\{p(n) : n \in \mathbb{N}\} \).

Finding a minimum is simpler because the first element in the sequence is already an upper bound on the result and hence the search space is finite.

**Proposition**

\[ \max \equiv_{sW} C_N \quad \text{and} \quad \min \equiv_{sW} K_N \equiv_{sW} C_2^*. \]

This suggests the following correspondence:

- \( B\Sigma_1^0 \) (\( = \) boundedness for \( \Sigma_1^0 \) formulas) corresponds to \( K_N \),

- \( I\Sigma_1^0 \) (\( = \) induction for \( \Sigma_1^0 \) formulas) corresponds to \( C_N \).
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\( \max \equiv_{sW} C_\mathbb{N} \) and \( \min \equiv_{sW} K_\mathbb{N} \equiv_{sW} C^*_2 \).

This suggests the following correspondence:

- \( B\Sigma^0_1 \) (= boundedness for \( \Sigma^0_1 \) formulas) corresponds to \( K_\mathbb{N} \),
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We consider:

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This suggests the following correspondence:

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Basic Complexity Classes and Reverse Mathematics

- $C_{\mathbb{N}}$
- $\lim \equiv_{sW} \widehat{C_N}$
- $C_R \equiv_{sW} C_N \times C_{2^N}$
- $WKL \equiv_{sW} C_{2^N} \equiv_{sW} \widehat{C_2}$
- $\text{WWKL} \equiv_{sW} \text{PC}_{2^N}$
- $K_N \equiv_{sW} C_2^*$
- $C_1$

ATR$_0$

ACA$_0$

WKL$_0 + I\Sigma^0_1$

WKL$_0$

WWKL$_0$

I\Sigma^0_1

B\Sigma^0_1

RCA$_0$
The Classification of Theorems
Choice on Natural Numbers

Theorem (B. and Gherardi 2012)

The following problems and theorems are Weihrauch equivalent:

- The choice problem $C_N$ on natural numbers.
- The Baire Category Theorem $\text{BCT}_1$.
- The Banach Inverse Mapping Theorem $\text{IMT}$.
- The Open Mapping Theorem.
- The Closed Graph Theorem.
- The Uniform Boundedness Theorem.

All for infinite dimensional computable normed spaces (in case of $\text{BCT}_1$ even for all perfect computable metric spaces).

All members of the equivalence class share the following features:

- All members map computable inputs to (some) computable outputs.
- All members are not uniformly computable.
- All members are computable with finitely many mind changes.
- All members have parallelizations that are equivalent to the limit map and they are closed under composition.
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The Baire Category Theorem

Theorem (Baire Category Theorem)

*Every complete metric space* $X$ *cannot be written as a countable union* $X = \bigcup_{i=0}^{\infty} A_i$ *of nowhere dense closed sets* $A_i \subseteq X$.

For perfect computable complete metric space $X$ we define:

- $\text{BCT}_0 : \subseteq \mathcal{A}(X)^\mathbb{N} \Rightarrow \mathbb{N}$, $(A_i)_{i \in \mathbb{N}} \mapsto X \setminus \bigcup_{i=0}^{\infty} A_i$ with $\text{dom}(\text{BCT}_0) = \{(A_i)_{i \in \mathbb{N}} : A_i^o = \emptyset\}$.

- $\text{BCT}_1 : \subseteq \mathcal{A}(X)^\mathbb{N} \Rightarrow \mathbb{N}$, $(A_i)_{i \in \mathbb{N}} \mapsto \{n \in \mathbb{N} : A_n^o \neq \emptyset\}$ with $\text{dom}(\text{BCT}_1) = \{(A_i)_{i \in \mathbb{N}} : X = \bigcup_{i=0}^{\infty} A_i\}$.

The strong Weihrauch equivalence class does not depend on the underlying space, but on the logical form.

Theorem (B. and Gherardi 2011)

$\text{BCT}_1 \equiv_{sW} C_\mathbb{N}$ *and* $\text{BCT}_0 \equiv_W \text{id}$.
The Baire Category Theorem

**Theorem (Baire Category Theorem)**

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For perfect computable complete metric space $X$ we define:

- $\text{BCT}_0 : \subseteq \mathcal{A}_{-}(X)^{\mathbb{N}} \ni (A_i)_{i \in \mathbb{N}} \mapsto X \setminus \bigcup_{i=0}^{\infty} A_i$ with $\text{dom}(\text{BCT}_0) = \{(A_i)_{i \in \mathbb{N}} : A_i^\circ = \emptyset\}$.

- $\text{BCT}_1 : \subseteq \mathcal{A}_{-}(X)^{\mathbb{N}} \ni (A_i)_{i \in \mathbb{N}} \mapsto \{n \in \mathbb{N} : A_n^\circ \neq \emptyset\}$ with $\text{dom}(\text{BCT}_1) = \{(A_i)_{i \in \mathbb{N}} : X = \bigcup_{i=0}^{\infty} A_i\}$.

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For perfect computable complete metric space $X$ we define:

- $\text{BCT}_0 : \subseteq A_-(X)^\mathbb{N} \nrightarrow X, (A_i)_{i\in\mathbb{N}} \mapsto X \setminus \bigcup_{i=0}^{\infty} A_i$ with $\text{dom}(\text{BCT}_0) = \{(A_i)_{i\in\mathbb{N}} : A_i^o = \emptyset\}$.
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The strong Weihrauch equivalence class does not depend on the underlying space, but on the logical form.

Theorem (B. and Gherardi 2011)

$\text{BCT}_1 \equiv_{\text{sW}} C_\mathbb{N}$ and $\text{BCT}_0 \equiv_{\text{W}} \text{id}$.
Proof.

Proof idea for $\text{BCT}_1 \equiv_W C_N$.

“$\text{BCT}_1 \leq_W C_N$” Given $(A_i)$, the set

$$\{\langle k, n \rangle : \emptyset \neq B_k \subseteq A_n\}$$

is co-c.e. in all parameters. Hence one can find a number $\langle k, n \rangle$ in this set using $C_N$. In this case $n \in \text{BCT}_1(A_i)$.

“$C_N \leq_W \text{BCT}_1$” Given a sequence $(n_i)_{i \in \mathbb{N}}$ that enumerates a set of natural numbers, we compute the sequence $(A_i)$ of closed subsets $A_i \subseteq X$ with

$$A_i := \begin{cases} \emptyset & \text{if } \exists i \ n = n_i \\ X & \text{otherwise} \end{cases}$$

This sequence is computable in $(n_i)$ and each $n \in \text{BCT}_1(n_i)$ has the property that $n$ does not appear in $(n_i)$.
The Baire Category Theorem

Proof.

Proof idea for $\text{BCT}_1 \equiv_W \text{C}_N$.

“$\text{BCT}_1 \leq_W \text{C}_N$” Given $(A_i)$, the set

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This sequence is computable in $(n_i)$ and each $n \in \text{BCT}_1(n_i)$ has the property that $n$ does not appear in $(n_i)$. $\square$
Banach’s Inverse Mapping Theorem

**Theorem (Banach’s Inverse Mapping Theorem)**

Every bijective bounded linear operator $T : X \rightarrow Y$ on Banach spaces $X, Y$ has a bounded inverse $T^{-1} : Y \rightarrow X$.

For computable Banach spaces $X, Y$ we define

$\text{IMT} : C(X, Y) \rightarrow C(Y, X), T \mapsto T^{-1}$ with $\text{dom}(\text{IMT}) = \{T : T \text{ linear}\}$.

The strong Weihrauch equivalence depends on the underlying spaces.

**Theorem (B. and Gherardi 2011)**

$\text{IMT} \equiv_{sW} C_N$ for infinite dimensional computable Banach spaces.

**Corollary (B. 2009)**

Every bijective computable linear operator $T : X \rightarrow Y$ on computable Banach spaces $X, Y$ has a computable inverse $T^{-1}$.
Banach’s Inverse Mapping Theorem

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*Every bijective bounded linear operator* $T : X \to Y$ *on Banach spaces* $X, Y$ *has a bounded inverse* $T^{-1} : Y \to X$.

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$\text{IMT} \equiv_{sW} C_N$ *for infinite dimensional computable Banach spaces*.

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Banach’s Inverse Mapping Theorem

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*Every bijective bounded linear operator $T : X \to Y$ on Banach spaces $X, Y$ has a bounded inverse $T^{-1} : Y \to X$.*

For computable Banach spaces $X, Y$ we define

- $\text{IMT} : \subseteq \mathcal{C}(X, Y) \to \mathcal{C}(Y, X), T \mapsto T^{-1}$ with $\text{dom}(\text{IMT}) = \{ T : T \text{ linear} \}$.

The strong Weihrauch equivalence depends on the underlying spaces.

**Theorem (B. and Gherardi 2011)**

$\text{IMT} \equiv_{\text{sW}} \mathcal{C}_N$ for infinite dimensional computable Banach spaces.

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Theorem

The following problems and theorems are Weihrauch equivalent:

- The choice problem $C_{2^\mathbb{N}}$ on Cantor space $2^\mathbb{N}$.
- Weak König’s Lemma $\text{WKL}$.  
- The Heine-Borel Theorem $\text{HB}_1$.  
- The Separation Problem for $\Sigma^0_1$ sets. (Gherardi and Marcone 2009)  
- The Hahn-Banach Theorem $\text{HBT}$. (Gherardi and Marcone 2009)  
- The Brouwer-Fixed Point Theorem $\text{BFT}_n$ for dimension $n \geq 2$. (B., Le Roux, J.S. Miller and Pauly 2012)

All members of the equivalence class share the following features:

- All members map computable inputs to (some) low outputs.
- All members are neither uniformly nor non-uniformly computable.
- All members are non-deterministically computable.
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Two different logical formalizations:

- **HB\(_0\)**: \(\subseteq \mathcal{O}([0, 1])^\mathbb{N} \Rightarrow \mathbb{N}, (U_i)_i \mapsto \{n \in \mathbb{N} : [0, 1] \subseteq \bigcup_{i=0}^n U_i\}\),
  \[\text{dom}(HB_0) := \{(U_i)_i : [0, 1] \subseteq \bigcup_{i=0}^\infty U_i\}\].

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The set \(\mathcal{O}(X)\) of open subsets of \(X\) is represented as \(\mathcal{A}_-(X)\), using complements.

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\(HB_0 \equiv^W \text{id is computable} \ HB_1 \equiv^W \text{WKL} \equiv^W \text{C}_{2^\mathbb{N}}\).
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*Every continuous map* $f : [0, 1]^n \rightarrow [0, 1]^n$ *has a fixed point* $x \in [0, 1]^n$, *i.e.*, $f(x) = x$.

- **BFT**$_n : C([0, 1]^n, [0, 1]^n) \Rightarrow [0, 1]^n, f \mapsto \{x : f(x) = x\}$.

- **Connected Choice** $CC_X : \subseteq A_\perp(X) \Rightarrow X, A \mapsto A$ *is the restriction of closed choice* $C_X$ *to connected sets.*

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is computable and maps any non-empty closed \( A \subseteq [0, 1] \) to a connected non-empty closed \( A \subseteq [0, 1]^3 \).

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There are computable \( f_n : [0, 1] \to \mathbb{R} \) with \( f_n(0) \cdot f_n(1) < 0 \) and without computable \( x_n \in [0, 1] \) such that \( f_n(x_n) = 0 \) for all \( n \in \mathbb{N} \).

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The following problems and theorems are Weihrauch equivalent:

- The choice problem $C_R$ on Euclidean space $\mathbb{R}$.
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Unique Choice \( UC_X : \subseteq A_-(X) \Rightarrow X \) is the restriction of closed choice \( C_X \) to

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**Proposition (B., Gherardi and Marcone 2012)**

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All or Unique Choice and Robust Division

- **All-or-Unique Choice** \( \text{AUC}_X : \subseteq \mathcal{A}_-(X) \Rightarrow X, A \mapsto A \) is the restriction of closed choice \( \mathcal{C}_X \) to

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  \text{RDIV} : [0, 1] \times [0, 1] \Rightarrow [0, 1], (x, y) \mapsto \begin{cases} \{ \frac{x}{\max(x,y)} \} & \text{if } y \neq 0 \\ [0, 1] & \text{if } y = 0 \end{cases}
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- **All-or-Unique Choice** \( \text{AUC}_X \subseteq \mathcal{A}(X) \Rightarrow X, A \mapsto A \) is the restriction of closed choice \( C_X \) to

\[ \text{dom}(\text{AUC}_X) := \{ A \subseteq X : A = X \text{ or } |A| = 1 \} \]

- \( \text{AUC}_N \equiv_{\text{SW}} C_N \).

- **Robust Division** is the mathematical problem

\[ \text{RDIV} : [0, 1] \times [0, 1] \Rightarrow [0, 1], (x, y) \mapsto \begin{cases} \frac{x}{\max(x, y)} & \text{if } y \neq 0 \\ [0, 1] & \text{if } y = 0 \end{cases} \]

- Robust division \( \text{RDIV} \) can be used to solve linear equations in compact domains: \( ax = b \).

- Likewise \( \text{RDIV}^* \) can be used to solve linear equations in compact domain of arbitrary finite dimension.

**Proposition**

\( \text{RDIV} \equiv_{\text{SW}} \text{AUC}_{[0, 1]} \).
Nash Equilibria

- A bi-matrix game is a pair $A, B \in \mathbb{R}^{m \times n}$ of $m \times n$–matrices.
- A vector $s = (s_1, ..., s_m) \in \mathbb{R}^m$ with $s_i \geq 0$ for all $i = 1, ..., m$ and $\sum_{j=1}^{m} s_j = 1$ is called a mixed strategy.
- By $S^m$ we denote the set of mixed strategies of dimension $m$.
- A Nash equilibrium is a pair $(x, y) \in S^n \times S^m$ such that $(\forall w \in S^n) x^T Ay \geq w^T Ay$ and $(\forall z \in S^m) x^T Bz \geq x^T Bz$.

Theorem (Nash 1951)

Every bi-matrix game admits a Nash equilibrium.

- $NASH_{n,m} : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \Rightarrow \mathbb{R}^n \times \mathbb{R}^m$, where $(A, B) \mapsto \{(x, y) : (x, y) \text{ is a Nash equilibrium for } (A, B)\}$.
- $NASH := \bigcup_{n,m \in \mathbb{N}} NASH_{n,m}$.

Theorem (Pauly 2010)

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A Las Vegas Algorithm for Robust Division

Proposition

Robust division $\text{RDIV}$ is Las Vegas computable.

1. Given $x, y \in [0, 1]$ and a random advice $r \in [0, 1]$, we aim to compute the fraction $z = \frac{x}{\max(x, y)}$.
2. We guess that $r$ is a correct solution, i.e., $r = z$ if $y > 0$, and we produce approximations of $r$ (rational intervals $(a, b) \ni r$).
3. Simultaneously, we try to find out whether $y > 0$, which we will eventually recognize, if this is correct.
4. If we find that $y > 0$, then we can compute the true result $z = \frac{x}{\max(x, y)}$ and produce approximations of it.
5. If at some stage we find that the best approximation $(a, b)$ of $r$ that was already produced as output is incompatible with $z$, i.e., if $z \not\in (a, b)$, then we indicate a failure.

Corollary

$\text{NASH} \equiv_w \text{RDIV}^* \leq_w \text{WWKL}$. 
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A Probabilistic Algorithm for Zero Finding

1. A continuous function $f : [0, 1] \rightarrow \mathbb{R}$ with $f(0) \cdot f(1) < 0$ is given as input.

2. Guess a binary sequence or, equivalently, a bit $b \in \{0, 1\}$ and a point $x \in [0, 1]$.

3. Interpret the guess $b = 1$ such that the zero set $f^{-1}\{0\}$ contains no open intervals and use the trisection method to compute a zero $z \in [0, 1]$ with $f(z) = 0$ in this case (disregarding $x$).

4. Interpret the guess $b = 0$ such that the zero set $f^{-1}\{0\}$ does contain an open interval and check whether $f(x) = 0$ in this case. Stop after finite time if this test fails and output $x$ otherwise.

Warning: This is not a Las Vegas algorithm! But it yields:

**Theorem**

$\text{IVT} \leq_w \text{WWKL}'$. 
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There is no Las Vegas Algorithm for Zero Finding

Theorem

$\text{IVT} \not\leq_W \text{WWKL}.$

Proof. (Idea) The proof is based on a finite extension construction: under the assumption that there is an algorithm for the reduction, one can create an instance (a function $f$) by finite extension that forces the reduction to translate this function into a tree that has measure zero.

Corollary

$\text{IVT} |_W \text{WWKL}.$

The inverse result $\text{WWKL} \not\leq_W \text{IVT}$ is easy to see: IVT maps computable inputs to computable outputs, WWKL does not.
There is no Las Vegas Algorithm for Zero Finding

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\[ \square \]

**Corollary**

\[ \text{IVT} \mid_W \text{WWKL}. \]

The inverse result \( \text{WWKL} \not\leq_W \text{IVT} \) is easy to see: \( \text{IVT} \) maps computable inputs to computable outputs, \( \text{WWKL} \) does not.
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Nash Equilibria and the Intermediate Value Theorem

\[ \text{lim} \equiv_{sW} \hat{C}_N \]

\[ C_R \equiv_{sW} C_N \times C_{2^N} \]

\[ \text{WKL} \equiv_{sW} C_{2^N} \quad \text{BCT}_1 \equiv_{sW} C_N \]

\[ \text{WWKL} \equiv_{sW} PC_{2^N} \]

\[ \text{IVT} \equiv_{sW} CC_{[0,1]} \quad \text{NASH} \equiv_{sW} AUC^*_{[0,1]} \]

\[ K_N \equiv_{sW} C_{2^*} \quad \text{RDIV} \equiv_{sW} AUC_{[0,1]} \]

\[ \text{LLPO} \equiv_{sW} C_2 \]

\[ \text{ACC}_N \]
All or Co-Unique Choice and Diagonal Non-Computability

- **All-or-Co-Unique Choice** $\text{ACC}_X : \subseteq A_\rightarrow(X) \Rightarrow X, A \mapsto A$ is the restriction of closed choice $C_X$ to

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**Proposition (Weihrauch 1992)**

$\text{ACC}_{n+1} <_W \text{ACC}_n$ for all $n \geq 2$.

- **Diagonally non-computable functions** for $X \subseteq \mathbb{N}$:

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**Theorem (Higuchi, Kihara 2014 and B., Hendtlass, Kreuzer 2015)**

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### Proposition (Weihrauch 1992)

\[\text{ACC}_{n+1} \prec_w \text{ACC}_n \text{ for all } n \geq 2.\]

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### Theorem (Higuchi, Kihara 2014 and B., Hendtlass, Kreuzer 2015)

\[\text{DNC}_n \equiv\) sW \(\widehat{\text{ACC}}_n \text{ for all } n \geq 2 \text{ and } \text{DNC}_\mathbb{N} \equiv\) sW \(\widehat{\text{ACC}}_\mathbb{N}\).\]

### Corollary (Jockusch 1989)

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$\text{DNC}_n \equiv_{\text{sW}} \widehat{\text{ACC}_n}$ for all $n \geq 2$ and $\text{DNC}_N \equiv_{\text{sW}} \widehat{\text{ACC}_N}$.

**Corollary (Jockusch 1989)**

$\text{DNC}_N \ll W \text{ DNC}_{n+1} \ll W \text{ DNC}_n$ for all $n \geq 2$. 
PA, Diagonal Non-Computability and WKL

- **PA**: $\mathcal{D} \ni \mathcal{D}, a \leftrightarrow \{b : b \gg a\}$ is the problem of Peano arithmetic.

**Corollary**

$\text{PA} \prec W \text{DNC}_n$ for all $n \geq 2$.

- **WKL**$_n$ $\subseteq \text{Tr}_n \ni n^\mathbb{N}, T \leftrightarrow [T]$ denotes Weak König’s Lemma for big $n$–ary trees.

- A tree $T \subseteq n^* = \{0, 1, \ldots, n - 1\}^*$ is called big, if it satisfies the following condition: if $w$ is a node of $T$ which is on an infinite path, then all but at most one successor nodes are on an infinite path of $T$ too.

**Theorem (B., Hendtlass and Kreuzer 2015)**

$\text{WKL}_n \equiv_{sW} \text{DNC}_n$ for all $n \geq 2$. 
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\[
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\[
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\text{PA} \rightarrow \text{DNC}_N \equiv_{sW} \widehat{\text{ACC}_N}
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Dense Realization and All or Co-Unique Choice

- $f : \subseteq X \Rightarrow Y$ is called densely realized, if $f^r(p)$ is dense in $\text{dom}(\delta_Y)$ for every $p \in \text{dom}(f \delta_X)$.
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- The set $\mathcal{D}$ of Turing degrees with its standard representation $\delta_D : \mathbb{N}^\mathbb{N} \rightarrow \mathcal{D}, p \mapsto [p]$ is densely realized.
- In particular, every $\Pi_2$ statement that claims the existence of a Turing degree translates into a densely realized problem.
- $\text{PA} : \mathcal{D} \Rightarrow \mathcal{D}, a \mapsto \{ b : b \gg a \}$ is densely realized.
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Proposition (B., Hendtlass and Kreuzer 2015)

If $f$ is densely realized, then $\text{ACC}_N \not\leq_W f$.

- $\text{ACC}_N$ is the weakest choice principles studied so far.
- All typical theorems from analysis are above $\text{ACC}_N$. 

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Jumps
Basic Complexity Classes and Reverse Mathematics

\[ \text{lim} \equiv_{sW} \hat{C}_N \]

\[ C_R \equiv_{sW} C_N \times C_2^N \]

\[ \text{WKL} \equiv_{sW} C_{2^N} \equiv_{sW} C_2 \]

\[ \text{WWKL} \equiv_{sW} PC_{2^N} \]

\[ \text{lim}_N \equiv_{sW} C_N \]

\[ \text{K}_N \equiv_{sW} C_2^* \]

\[ C_1 \]

\[ \text{ATR}_0 \]

\[ \text{ACA}_0 \]

\[ \text{WKL}_0 + \text{I}_1^0 \]

\[ \text{WKL}_0 \]

\[ \text{WWKL}_0 \]

\[ \text{I}_1^0 \]

\[ \text{B}_1^0 \]

\[ \text{RCA}_0 \]
Limits and LPO

- $\lim_X : \subseteq X^\mathbb{N} \to X, (x_n)_n \mapsto \lim_{n \to \infty} x_n$ denotes the limit operation of a Hausdorff space $X$.

- $\lim : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}, \langle p_0, p_1, p_2, ... \rangle \mapsto \lim_{n \to \infty} p_n$ denotes the limit operation of Baire space $\mathbb{N}^\mathbb{N}$ with encoded input.

Proposition (B. 2005)

$\lim \equiv_{sW} \lim_X$ for all perfect computable metric spaces $X$.

- LPO : $\mathbb{N}^\mathbb{N} \to \mathbb{N}, p \mapsto \begin{cases} 1 & \text{if } (\forall n) p(n) = 0 \\ 0 & \text{otherwise} \end{cases}$

  denotes the limited principle of omniscience.

- $C_2 \equiv_{sW} LLPO \leq_{W} \text{RDIV} \leq_{W} \text{LPO} \leq_{W} C_\mathbb{N}$.

Proposition (B. and Gherardi 2011)

$\hat{\text{LPO}} \equiv_{sW} \hat{C}_\mathbb{N} \equiv_{sW} \hat{\lim}$. 
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\[ \widehat{\text{LPO}} \equiv_{sW} \widehat{C_\mathbb{N}} \equiv_{sW} \text{lim}. \]
The following problems and theorems are Weihrauch equivalent:

- The parallelization $\widehat{C_N}$ of the choice problem on natural numbers.
- The limit problem $\lim : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, \langle p_0, p_1, p_2, \ldots \rangle \mapsto \lim_{n \to \infty} p_n$.
- The differentiability problem $d : \subseteq C[0, 1] \to C[0, 1], f \mapsto f'$ (von Stein 1989).
- The Monotone Convergence Theorem MCT.
- The Fréchet-Riesz Theorem for Hilbert spaces. (follows from B. and Yoshikawa 2006)
- The Radon-Nikodym Theorem. (Hoyrup, Rojas, Weihrauch 2012)

All members of the equivalence class share the following features:

- All members map computable inputs to (some) limit computable outputs.
- All members are neither uniformly nor non-uniformly computable, but limit computable.
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Parallelized Choice on Natural Numbers

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A Dichotomy for Linear Operators

**Theorem (B. 1999)**

Let $X$, $Y$ be computable Banach spaces and $T : \subseteq X \rightarrow Y$ a densely defined linear operator with a c.e. closed graph. Then:

- $T \leq \text{id}$ $\iff$ $T$ computable $\iff$ $T$ bounded.
- $\lim \leq \text{id}$ $\iff$ $T$ unbounded.

**Corollary (von Stein 1992)**

$d \equiv \text{id}$, where $d : \subseteq C[0, 1] \rightarrow C[0, 1], f \mapsto f'$

**Corollary (First Main Theorem of Pour-El and Richards 1989)**

An unbounded $T : \subseteq X \rightarrow Y$ as above admits a computable $x \in \text{dom}(T)$ such that $T(x)$ is not computable.

**Corollary (Myhill 1971)**

There exists a computable and continuously differentiable $f : [0, 1] \rightarrow \mathbb{R}$ such that $f'$ is not computable.
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For instance $\text{id}' \equiv_{sW} \lim, \text{id}'' \equiv_{sW} \lim \circ \lim$, etc.

Proposition (B., Gherardi and Marcone 2011)

$f \leq_{sW} g \implies f' \leq_{sW} g'$ and $f \leq_{sW} f'$.

$f <_{W} f'$ does not hold in general: $f \equiv_{sW} f'$ for a constant $f$.

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Jumps

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- For instance \( \text{id}' \equiv_{\text{sw}} \lim, \text{id}'' \equiv_{\text{sw}} \lim \circ \lim \), etc.

Proposition (B., Gherardi and Marcone 2011)

\[ f \leq_{\text{sw}} g \implies f' \leq_{\text{sw}} g' \text{ and } f \leq_{\text{sw}} f'. \]

- \( f <_{\text{sw}} f' \) does not hold in general: \( f \equiv_{\text{sw}} f' \) for a constant \( f \).
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Proposition (B., Gherardi and Marcone 2011)

- \( f' \equiv_{\text{w}} f' \times \lim \equiv_{\text{w}} f \ast \lim \), if \( f \) is a cylinder.
- \( f \) is a cylinder \( \implies \) \( f' \) is a cylinder.
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Jumps and the Algebraic Structure

Proposition (B., Gherardi and Marcone 2011)

- \( (f \circ g)' = f \circ g' \)
- \( (f \times g)' \equiv_{SW} f' \times g' \)
- \( \hat{f}' \equiv_{SW} \hat{f}' \)
- \( (f \sqcap g)' \equiv_{SW} f' \sqcap g' \)
- \( (f \sqcup g)' \leq_{SW} f' \sqcup g' \)
- \( f^* \leq_{SW} f'^* \)

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- \( f \) strongly idempotent \( \implies f' \) strongly idempotent,
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In particular, not every \( f \) with \( \lim \leq_{W} f \) is a jump.
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The Weihrauch Lattice refines the Borel Hierarchy

- \( f^{(0)} := f \) and \( f^{(n+1)} := (f^{(n)})' \) for all \( n \in \mathbb{N} \).

Theorem (B. 2005)

\[
f \leq_W \text{id}^{(n)} \iff f \text{ is effectively } \Sigma^0_{n+1} \text{-measurable for all } n \in \mathbb{N}.
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The Cluster Point Problem

- \( \text{CL}_X : \subseteq X^\mathbb{N} \Rightarrow X, (x_n)_n \mapsto \{x : x \text{ is a cluster point of } (x_n)_n\} \) is called the **cluster point problem** of a topological space \( X \).

**Theorem (B., Gherardi and Marcone 2011)**

\[ \text{CL}_X \equiv_{\text{sW}} C'_X \text{ for every computable metric space } X. \]

**Proof.** (Idea) This can be proved by showing that the jump of \( \psi_- \) is equivalent to the cluster point representation of \( \mathcal{A}_-(X) \). One direction follows since

\[ X^\mathbb{N} \to \mathcal{A}_-(X), (x_n)_n \mapsto \{x : x \text{ is a cluster point of } (x_n)_n\} \]

is limit computable. The other direction is more involved. \( \square \)

**Example**

- \( C'_2 \equiv_{\text{sW}} \text{CL}_2 \) is the infinite pigeonhole principle,
- \( C'_{2^\mathbb{N}} \equiv_{\text{sW}} \text{CL}_{2^\mathbb{N}} \) is the Bolzano-Weierstraß Theorem of \( 2^\mathbb{N} \),
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The Jump of Choice on Cantor Space

- \( \text{BWT}_X : \subseteq X^\mathbb{N} \Rightarrow X, (x_n)_n \mapsto \{x : x \text{ is a cluster point of } (x_n)_n\} \) is \( \text{CL}_X \) rest. to \( \text{dom}(\text{BWT}_X) := \{(x_n)_n : \{x_n : n \in \mathbb{N}\} \text{ is compact}\} \).

The following problems and theorems are strongly Weihrauch equivalent:

- The jump \( C_{2^\mathbb{N}}' \) of choice on Cantor space \( 2^\mathbb{N} \).
- The jump of Weak König’s Lemma \( \text{WKL}' \).
- König’s Lemma \( \text{KL} \). (B. and Rakotoniaina 2015)
- The Bolzano-Weierstraß Theorem \( \text{BWT}_\mathbb{R} \) on \( \mathbb{R} \). (B., Gherardi, Marcone 2011)

All members of the equivalence class share the following features:

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The following problems and theorems are strongly Weihrauch equivalent:

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The Bolzano-Weierstraß Theorem

Proposition (B., Gherardi, Marcone 2011)

- \( \text{WKL} \,'
\( \equiv \text{sw} \) \( \text{BWT}_X \) for perfect computable metric spaces \( X \).
- \( K'_N \equiv \text{sw} \) \( \text{BWT}_N \).

Proposition (B. and Rakotoniaina 2015)

\[
K_{N}^{(n)} \leq_{\text{SW}} C_{N}^{(n)} \leq_{\text{SW}} K_{N}^{(n+1)} \quad \text{for all } n \in \mathbb{N}.
\]

Proof. (Idea) This follows from

\[
K_N \leq_{\text{SW}} C_N \equiv_{\text{SW}} \lim_N \leq_{\text{SW}} \text{BWT}_N \equiv_{\text{SW}} K'_N.
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\( C_{2}^{(n)} \) is \( \Sigma_{n+2}^{0} \)-measurable but not \( \Sigma_{n+1}^{0} \)-measurable for all \( n \in \mathbb{N} \).
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Higher Complexity Classes

\[ \text{ATR}_0 \quad \text{\( \rightarrow \)} \quad C_{NN} \]

\[ \text{\( \Sigma_4^0 \)} \quad \text{\( \rightarrow \)} \quad \text{lim''} \]

\[ \text{\( \Sigma_3^0 \)} \quad \text{\( \rightarrow \)} \quad \text{\( \text{\( \text{C}_{\text{IR}}' \))} \rightarrow \text{\( \text{C}_{\text{IR}}'' \))} \rightarrow \text{\( \text{WKL''} \equiv_{\text{sw}} C_{2N}'' \))} \rightarrow \text{\( \text{K''} \))} \rightarrow \text{\( \text{LPO''} \))} \rightarrow \text{\( \text{I\Sigma_3^0} \))} \]

\[ \text{\( \Sigma_3^0 \)} \quad \text{\( \rightarrow \)} \quad \text{lim'} \]

\[ \text{\( \Sigma_2^0 \)} \quad \text{\( \rightarrow \)} \quad \text{lim} \]

\[ \text{\( \rightarrow \)} \quad \text{WKL' \equiv_{\text{sw}} C_{2N'} \) \rightarrow \text{\( \text{K'} \) \rightarrow \text{\( \text{LPO'} \))} \rightarrow \text{\( \text{I\Sigma_2^0} \))} \]

\[ \text{\( \rightarrow \)} \quad \text{ATR_0 \equiv_{\text{sw}} C_{2N} \) \rightarrow \text{\( \text{K} \) \rightarrow \text{\( \text{LPO} \))} \rightarrow \text{\( \text{I\Sigma_1^0} \))} \]
The Cluster Point Problem in the Role of Induction

- We recall that $\text{DNC}_{n+1} <_W \text{DNC}_n$ for all $n \geq 2$.
- R. Friedberg proved that non-uniformly the corresponding Turing degrees coincide.
- Dorais, Hirst and Shafer (2015) refined this construction and analyzed it in reverse mathematics.

**Proposition (B., Hendtlass, Kreuzer 2015)**

$\text{DNC}_2 \leq_W \text{DNC}_n \ast C'_N$ for all $n \geq 2$.

- The proof is a uniform version of the construction of Dorais, Hirst and Shafer (2015).

**Question**

*How can $(\text{DNC}_{n+1} \rightarrow \text{DNC}_n)$ be characterized?*

The result above only implies $(\text{DNC}_{n+1} \rightarrow \text{DNC}_n) \leq_W C'_N$. 
We recall that $\text{DNC}_{n+1} \prec_W \text{DNC}_n$ for all $n \geq 2$.

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The proof is a uniform version of the construction of Dorais, Hirst and Shafer (2015).

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*How can $(\text{DNC}_{n+1} \rightarrow \text{DNC}_n)$ be characterized?*

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- We recall that \( \text{DNC}_{n+1} <_W \text{DNC}_n \) for all \( n \geq 2 \).
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We define the cardinality $\#f$ as the supremum of all cardinalities $|M|$ of sets $M \subseteq \text{dom}(f)$ such that the sets $f(x)$ with $x \in M$ are pairwise disjoint.

**Proposition (B., Gherardi and Hölzl 2015)**

$f \leq_{SW} g \implies \#f \leq \#g$.

**Proposition**

If $f : \subseteq X \Rightarrow \mathbb{N}$ is a strong fractal and $\text{range}(g)$ compact, then $f \leq_W g \implies f \leq_{SW} g$.

**Corollary (B., Gherardi and Marcone 2012)**

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Cohesiveness and the Bolzano-Weierstraß Theorem

- \( \text{WBWT}_X \subseteq X^\mathbb{N} \Rightarrow X', (x_n)_n \mapsto \text{BWT}_X \) is called the Weak Bolzano Weierstraß Theorem of \( X \).

- \( \text{COH} : (2^\mathbb{N})^\mathbb{N} \Rightarrow 2^\mathbb{N} \) where \( \text{COH}(R_i) \) contains all infinite \( X \subseteq \mathbb{N} \) such that for all \( i \in \mathbb{N} \) one of the sets \( X \cap R_i \) or \( X \cap (\mathbb{N} \setminus R_i) \) is finite is called the Cohesiveness Problem.

**Theorem (Kreuzer 2011)**

\[ \text{COH} \equiv_W \text{WBWT}_\mathbb{R}. \]

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Recall: \( (\lim \rightarrow \text{BWT}_X) = \min\{h : \text{BWT}_X \leq_W \lim \ast h\} \).

**Corollary**

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**Theorem (B., Hendtlass and Kreuzer 2015)**

$$WKL' \equiv_W \lim^* \text{COH}.$$  

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$$\text{COH} \equiv_W \widehat{\text{WBWT}}_2.$$  

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### On the Combinatorial “Core” of Problems

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Ramsey’s Theorem
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Theorem (Ramsey 1930)

Every coloring \( c : [\mathbb{N}]^n \rightarrow k \) admits an infinite homogeneous set \( M \subseteq \mathbb{N} \).

- Here \([M]^n\) denotes the set of \( n\)–element subsets of \( M \subseteq \mathbb{N} \).
- We identify \( k \) with \( \{0, 1, \ldots, k - 1\} \) for all \( k \in \mathbb{N} \).
- A set \( M \subseteq \mathbb{N} \) is called homogeneous for the coloring \( c \), if there is some \( i \in k \) such that \( c(A) = i \) for all \( A \in [M]^n \).
- By \( C_{n,k} \) we denote the set of colorings \( c : [\mathbb{N}]^n \rightarrow k \).
- By \( RT_k^n : C_{n,k} \Rightarrow 2^\mathbb{N} \) we denote the corresponding multi-valued function, where \( RT_k^n(c) \) contains exactly all infinite homogeneous sets \( M \subseteq \mathbb{N} \) for \( c \).
- We also consider the case \( k = \mathbb{N} \), which corresponds to an unspecified but finite number of colors.
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*Every coloring* \( c : [\mathbb{N}]^n \rightarrow k \) *admits an infinite homogeneous set* \( M \subseteq \mathbb{N} \).

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- By \( \text{RT}_{k}^{n} : C_{n,k} \Rightarrow 2^{\mathbb{N}} \) we denote the corresponding multi-valued function, where \( \text{RT}_{k}^{n}(c) \) contains exactly all infinite homogeneous sets \( M \subseteq \mathbb{N} \) for \( c \).
- We also consider the case \( k = \mathbb{N} \), which corresponds to an unspecified but finite number of colors.
Proposition (B. and Rakotoniaina 2015)

\[ C_2^{(n)} \leq_W RT^n_2 \text{ for all } n \geq 1. \]

**Proof.** (Idea.) We note that \( C_2^{(n)} \equiv_{sW} BWT_2 \circ \lim_{2^N}^{[n-1]} \). Let \( p \in \text{dom}(BWT_2 \circ \lim_{2^N}^{[n-1]}) \) and \( q := \lim_{2^N}^{[n-1]}(p) \). Then

\[
q(i_0) = \lim_{i_1 \to \infty} \lim_{i_2 \to \infty} \ldots \lim_{i_{n-1} \to \infty} p(i_{n-1}, \ldots, i_0)
\]

for all \( i_0 \in \mathbb{N} \). We compute the coloring \( c : [\mathbb{N}]^n \to 2 \) with

\[
c\{i_0 < i_1 < \ldots < i_{n-1}\} := p(i_{n-1}, i_{n-2}, \ldots, i_1, i_0).
\]

For \( M \in RT^n_2 \) we obtain \( c(M) \in BWT_2(q) \).

**Corollary**

\[ WKL^{(n)} \leq_W \widehat{RT}^n_k \text{ for all } n \geq 1, k \geq 2. \]
Proposition (B. and Rakotoniaina 2015)

\[ C_2(n) \leq WRT^n \text{ for all } n \geq 1. \]

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\[ WKL^{(n)} \leq W \hat{RT}^n_k \text{ for all } n \geq 1, k \geq 2. \]
Theorem (B. and Rakotoniaina 2015)

\[ \text{RT}_N^n \times \text{RT}_k^{n+1} \leq_{\text{SW}} \text{RT}_{k+1}^{n+1} \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) Given a coloring \( c_1 : [N]^n \to N \) with finite range and a coloring \( c_2 : [N]^{n+1} \to k \) we construct a coloring \( c^+ : [N]^{n+1} \to k+1 \) as follows:

\[
c^+(A) := \begin{cases} 
c_2(A) & \text{if } A \text{ is homogeneous for } c_1 \\
k & \text{otherwise}
\end{cases}
\]

for all \( A \in [N]^{n+1} \). Then \( \text{RT}_2^{n+1}(c^+) \subseteq \text{RT}_N^n(c_1) \cap \text{RT}_k^{n+1}(c_2) \) and hence the desired reduction follows. \( \square \)

**Corollary**

\( (\text{RT}_k^n)^* \leq_{\text{W}} \text{RT}_2^{n+1} \text{ for all } n, k \geq 1. \)
Products and Parallellization of Ramsey

**Theorem (B. and Rakotoniaina 2015)**

\[ \text{RT}_N^n \times \text{RT}_{k}^{n+1} \leq_{sW} \text{RT}_{k+1}^{n+1} \quad \text{for all } n, k \geq 1. \]

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\( (\text{RT}_k^n)^* \leq_{W} \text{RT}_2^{n+1} \quad \text{for all } n, k \geq 1. \)
Parallelization of Ramsey

Theorem (B. and Rakotoniaina 2015)

\[ \hat{\text{RT}}_k^n \leq_{SW} \text{RT}_{2}^{n+2} \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) Given a sequence \((c_i)_i\) of colorings \(c_i : [\mathbb{N}]^n \to k\), we compute a sequence \((d_m)_m\) of colorings \(d_m \in C_{n,k^m}\) that capture the products \((\text{RT}_k^n)^m\) and a sequence \((d_m^+)_m\) of colorings \(d_m^+ : [\mathbb{N}]^{n+1} \to 2\) by

\[
d_m^+(A) := \begin{cases} 
0 & \text{if } A \text{ is homogeneous for } d_m \\
1 & \text{otherwise}
\end{cases}
\]

for all \(A \in [\mathbb{N}]^{n+1}\). Now, in a final step we compute a coloring \(c : [\mathbb{N}]^{n+2} \to 2\) with

\[
c(\{m\} \cup A) := d_m^+(A)
\]

for all \(A \in [\mathbb{N}]^{n+1}\) and \(m < \min(A)\). Given an infinite homogeneous set \(M \in \text{RT}_{2}^{n+2}(c)\) we determine a sequence \((M_i)_i\) as follows: for each fixed \(i \in \mathbb{N}\) we first search for a number \(m > i\) in \(M\) and then we let \(M_i := \{x \in M : x > m\}\).
Corollary

For all $n \geq 2$ we obtain:

- $\lim\equiv^N W SRT^1_N$
- $\lim \leq^W SRT^3_2$
- $WKL' \leq^W RT^3_2$ (Hirschfeldt and Jockusch 2015)
- $WKL^{(n)} \leq^W SRT^{n+2}_2$

- A coloring $c : [\mathbb{N}]^n \to k$ is called stable, if $\lim_{i \to \infty} c(A \cup \{i\})$ exists for all $A \in [\mathbb{N}]^{n-1}$.
- $SRT^*_k$ is the restriction of $RT^*_k$ to stable colorings.
Upper Bounds

Theorem (Cholak, Jockusch, Slaman 2009)

\[
\text{RT}_k^n \leq_{W} \text{SRT}_k^n \ast \text{COH} \text{ for all } n, k \geq 1.
\]

Theorem

\[
\text{SRT}_k^{n+1} \leq_{W} \text{RT}_k^n \ast \lim \text{ for all } n, k \geq 1.
\]

Proof. (Idea.) In fact, we even proved \( \text{SRT}_k^{n+1} \equiv_{W} (\text{CRT}_k^n)' \). \( \square \)

Corollary

\[
\widehat{\text{RT}}_k^n \leq_{W} \text{RT}_k^n \ast \text{WKL}' \text{ for all } n, k \geq 1.
\]

Proof. (Idea.) We use \( \text{WKL}' \equiv_{W} \text{lim} \ast \text{COH} \). \( \square \)

Corollary

\[
\widehat{\text{RT}}_k^n =_{W} \text{WKL}^{(n)} \text{ for all } n \geq 1, k \geq 2.
\]

Corollary

\( \text{RT}_k^n \) is effectively \( \Sigma^0_{n+2} \)-, but not \( \Sigma^0_{n+1} \)-measurable for \( n, k \geq 2 \).
Upper Bounds

Theorem (Cholak, Jockusch, Slaman 2009)

\[ \text{RT}_k^n \leq \text{WSRT}_k^n \ast \text{COH} \text{ for all } n, k \geq 1. \]

Theorem

\[ \text{SRT}_{k}^{n+1} \leq \text{WRT}_k^n \ast \text{lim} \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) In fact, we even proved \( \text{SRT}_{k}^{n+1} \equiv_{\text{W}} \text{(CRT}_{k}^{n})' \). □

**Corollary**

\[ \text{RT}_{k}^{n+1} \leq \text{WRT}_k^n \ast \text{WKL}' \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) We use \( \text{WKL}' \equiv_{\text{W}} \text{lim} \ast \text{COH}. \) □

**Corollary**

\[ \text{RT}_{k}^{n} =_{\text{W}} \text{WKL}^{(n)} \text{ for all } n \geq 1, k \geq 2. \]

**Corollary**

\( \text{RT}_k^n \) is effectively \( \Sigma_{n+2}^0 \)-, but not \( \Sigma_{n+1}^0 \)-measurable for \( n, k \geq 2. \)
## Upper Bounds

**Theorem (Cholak, Jockusch, Slaman 2009)**

\[
RT^n_k \leq_W SRT^n_k \ast \text{COH} \quad \text{for all } n, k \geq 1.
\]

\[
\text{Theorem}
\]

\[
SRT^{n+1}_k \leq_W RT^n_k \ast \lim \quad \text{for all } n, k \geq 1.
\]

**Proof.** (Idea.) In fact, we even proved \( SRT^{n+1}_k \equiv_W (CRT^n_k)' \). □

**Corollary**

\[
RT^{n+1}_k \leq_W RT^n_k \ast \text{WKL'} \quad \text{for all } n, k \geq 1.
\]

**Proof.** (Idea.) We use \( \text{WKL'} \equiv_W \lim \ast \text{COH} \). □

**Corollary**

\[
\widetilde{RT}^n_k \equiv_W \text{WKL}^{(n)} \quad \text{for all } n \geq 1, k \geq 2.
\]

**Corollary**

\( RT^n_k \) is effectively \( \Sigma^0_{n+2} \)-, but not \( \Sigma^0_{n+1} \)-measurable for \( n, k \geq 2 \).
Theorem (Cholak, Jockusch, Slaman 2009)

\[ \text{RT}_k^n \leq_w \text{SRT}_k^n \ast \text{COH} \text{ for all } n, k \geq 1. \]

Theorem

\[ \text{SRT}_{k}^{n+1} \leq_w \text{RT}_k^n \ast \lim \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) In fact, we even proved \( \text{SRT}_{k}^{n+1} \equiv_w \text{CRT}_k^n \). \( \square \)

Corollary

\[ \text{RT}_k^{n+1} \leq_w \text{RT}_k^n \ast \text{WKL}' \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) We use \( \text{WKL}' \equiv_w \lim \ast \text{COH}. \) \( \square \)

Corollary

\[ \hat{\text{RT}}_k^n \equiv_w \text{WKL}^{(n)} \text{ for all } n \geq 1, k \geq 2. \]

Corollary

\( \text{RT}_k^n \) is effectively \( \Sigma_{n+2}^0 \)-, but not \( \Sigma_{n+1}^0 \)-measurable for \( n, k \geq 2. \)
Upper Bounds

Theorem (Cholak, Jockusch, Slaman 2009)

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Theorem

\[ \text{SRT}_k^{n+1} \leq_{W} \text{RT}_k^n \ast \lim \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) In fact, we even proved \( \text{SRT}_k^{n+1} \equiv_{W} (\text{CRT}_k^n)' \).

\[ \square \]

Corollary

\[ \text{RT}_k^{n+1} \leq_{W} \text{RT}_k^n \ast \text{WKL}' \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) We use \( \text{WKL}' \equiv_{W} \lim \ast \text{COH} \).

\[ \square \]

Corollary

\[ \widehat{\text{RT}}_k^n \equiv_{W} \text{WKL}^{(n)} \text{ for all } n \geq 1, k \geq 2. \]

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\( \text{RT}_k^n \) is effectively \( \Sigma^0_{n+2} \)-, but not \( \Sigma^0_{n+1} \)-measurable for \( n, k \geq 2 \).
Upper Bounds

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\[ \mathsf{RT}_k^{n+1} \leq_W \mathsf{RT}_k^n \ast \mathsf{WKL}' \text{ for all } n, k \geq 1. \]

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\[ \widehat{\mathsf{RT}}_k^n \equiv_W \mathsf{WKL}^{(n)} \text{ for all } n \geq 1, k \geq 2. \]

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\[ \text{SRT}_k^{n+1} \leq \text{W RT}_k^n \ast \text{lim} \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) In fact, we even proved \( \text{SRT}_k^{n+1} \equiv \text{W (CRT}_k^n)' \). □

Corollary

\[ \text{RT}_k^{n+1} \leq \text{W RT}_k^n \ast \text{WKL}' \text{ for all } n, k \geq 1. \]

**Proof.** (Idea.) We use \( \text{WKL}' \equiv \text{W lim} \ast \text{COH}. \) □

Corollary

\[ \widehat{\text{RT}}_k^n \equiv \text{W WKL}^{(n)} \text{ for all } n \geq 1, k \geq 2. \]

Corollary

\( \text{RT}_k^n \) is effectively \( \Sigma^0_{n+2} \)-, but not \( \Sigma^0_{n+1} \)-measurable for \( n, k \geq 2. \)
Ramsey’s Theorem and Cohesiveness

\[
\Sigma_4^0 \rightarrow \lim'' \rightarrow WKL'' \\
\rightarrow (RT_2^1)' \times \lim \\
\rightarrow SRT_2^2 \times COH \rightarrow \rightarrow SRT_2^2 \times COH \rightarrow \rightarrow SRT_2^2 \sqcup COH \rightarrow \rightarrow SRT_2^2 \rightarrow \rightarrow (CRT_2^1)'
\]

\[
\Sigma_3^0 \rightarrow \lim' \rightarrow WKL' \equiv_{sW} KL \\
\rightarrow C_2' \equiv_{sW} BWT_2 \rightarrow RT_2^1 = D_2^1
\]

\[
\Sigma_2^0 \rightarrow \lim \rightarrow COH \\
\rightarrow C_N \equiv_{sW} \lim N \rightarrow C_F
\]
The Squashing Theorem

**Definition**

\[ f : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \] is called **finitely tolerant** if there is a computable
\[ T : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \] such that for all \( p, q \in \text{dom}(f) \), \( r \in \mathbb{N}^\mathbb{N} \), \( k \in \mathbb{N} \):

\[
((\forall n \geq k)(p(n) = q(n)) \ \text{and} \ \ r \in f(q)) \implies T\langle r, k \rangle \in f(p).
\]

- \( f \) finitely tolerant \( \implies \) \( f \) fractal.
- \( \lim, BWT_n, BWT_N, BWT_{2^N}, RT^n_k, RT^n_N \) are finitely tolerant.

**Theorem (Dorais, Dzhafarov, Hirst, Milet and Shafer 2016)**

Let \( f, g : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \) and let \( f \) be finitely tolerant and total. Then
\[ g \times f \leq_W f \implies \widehat{g} \leq_W f. \]

**Note.** \( BWT_N \) is not total.

**Corollary**

Under the same assumptions on \( f \) it holds that
\[ f \text{ idempotent} \implies f \text{ parallelizable}. \]
The Squashing Theorem

Definition

\( f : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N} \) is called \textbf{finitely tolerant} if there is a computable \( T : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \) such that for all \( p, q \in \text{dom}(f), \ r \in \mathbb{N}^\mathbb{N}, \ k \in \mathbb{N}: ((\forall n \geq k)(p(n) = q(n)) \text{ and } r \in f(q)) \implies T\langle r, k \rangle \in f(p) \).

\[
\begin{align*}
\text{\( \triangleright \) } f \text{ finitely tolerant } \implies f \text{ fractal.} \\
\text{\( \triangleright \) } \text{lim, BWT}_n, \text{BWT}_n, \text{BWT}_2^n, \text{RT}_k^n, \text{RT}_N^n \text{ are finitely tolerant.}
\end{align*}
\]

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Note. \( \text{BWT}_N \) is not total.

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Under the same assumptions on \( f \) it holds that \( f \) idempotent \( \implies f \) parallelizable.
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\[ \left( (\forall n \geq k)(p(n) = q(n)) \text{ and } r \in f(q) \right) \Longrightarrow T\langle r, k \rangle \in f(p). \]

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Under the same assumptions on \( f \) it holds that

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Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)
\[ RT^n_k <_{sW} RT^n_{k+1} \text{ for all } n, k \geq 1. \]

Theorem (B. & Rakotoniaina, Hirschfeldt & Jockusch, Patey 2015)
\[ RT^n_k <_W RT^n_{k+1} \text{ for all } n, k \geq 1. \]

Proof.

\[ RT^n_2 \times RT^{n+1}_k \leq_W RT^{n+1}_k \text{ by the Product Theorem.} \]

\[ RT^n_2 \times RT^{n+1}_k \leq_W RT^{n+1}_k \text{ implies } \widehat{RT}^{n}_2 \leq_W RT^{n+1}_k \text{ by the Squashing Theorem which leads to a contradiction:} \]
\[ \lim^{(n-1)} \leq_W WKL^{(n)} \equiv_W \widehat{RT}^{n}_2 \leq_W RT^{n+1}_k \]

\[ RT^n_2 \times RT^{n+1}_k \not\leq_W RT^{n+1}_k \text{ for all } n, k \geq 1 \text{ follows.} \]

\[ RT^{n+1}_k <_W RT^{n+1}_{k+1} \text{ for all } n, k \geq 1 \text{ follows.} \]
Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)
\[ \text{RT}_k^n <_{\text{sW}} \text{RT}_{k+1}^n \quad \text{for all } n, k \geq 1. \]

Theorem (B. & Rakotoniaina, Hirschfeldt & Jockusch, Patey 2015)
\[ \text{RT}_k^n <_{\text{W}} \text{RT}_{k+1}^n \quad \text{for all } n, k \geq 1. \]

Proof.

\begin{itemize}
  \item \[ \text{RT}_2^n \times \text{RT}_{k}^{n+1} \leq_{\text{W}} \text{RT}_{k+1}^{n+1} \] by the Product Theorem.
  \item \[ \text{RT}_2^n \times \text{RT}_{k}^{n+1} \leq_{\text{W}} \text{RT}_{k}^{n+1} \] implies \( \hat{\text{RT}}_2^n \leq_{\text{W}} \text{RT}_{k}^{n+1} \) by the Squashing Theorem which leads to a contradiction:
  \[ \lim_{(n-1)}^{(n-1)} \leq_{\text{W}} \text{WKL}^{(n)} \equiv_{\text{W}} \hat{\text{RT}}_2^n \leq_{\text{W}} \text{RT}_{k}^{n+1} \]
  \item \[ \text{RT}_2^n \times \text{RT}_{k}^{n+1} \not<_{\text{W}} \text{RT}_{k}^{n+1} \] for all \( n, k \geq 1 \) follows.
  \item \[ \text{RT}_{k}^{n+1} <_{\text{W}} \text{RT}_{k+1}^{n+1} \] for all \( n, k \geq 1 \) follows.
\end{itemize}
Ramsey’s Theorem in the Weihrauch Lattice

\[
\begin{align*}
\Sigma^0_6 \quad \lim^{(4)} \quad \downarrow \\
\text{WKL}^{(4)} \equiv_{sW} \hat{C}^{(4)}_2 \quad \rightarrow \quad \text{RT}^4_N \quad \rightarrow \quad \ldots \quad \rightarrow \quad \text{RT}^4_N \quad \rightarrow \quad \text{RT}^4_3 \quad \rightarrow \quad \text{RT}^4_2 \quad \rightarrow \quad \hat{C}^{(4)}_2
\\
\Sigma^0_5 \quad \lim^{(3)} \quad \downarrow \\
\text{WKL}^{(3)} \equiv_{sW} \hat{C}^{(3)}_2 \quad \rightarrow \quad \text{RT}^3_N \quad \rightarrow \quad \ldots \quad \rightarrow \quad \text{RT}^3_4 \quad \rightarrow \quad \text{RT}^3_3 \quad \rightarrow \quad \text{RT}^3_2 \quad \rightarrow \quad \hat{C}^{(3)}_2
\\
\Sigma^0_4 \quad \lim'' \quad \downarrow \\
\text{WKL}'' \equiv_{sW} \hat{C}''_2 \quad \rightarrow \quad \text{RT}^2_N \quad \rightarrow \quad \ldots \quad \rightarrow \quad \text{RT}^2_4 \quad \rightarrow \quad \text{RT}^2_3 \quad \rightarrow \quad \text{RT}^2_2 \quad \rightarrow \quad \hat{C}''_2
\\
\Sigma^0_3 \quad \lim' \quad \downarrow \\
\text{WKL}' \equiv_{sW} \hat{C}'_2 \quad \rightarrow \quad \text{RT}^1_N \quad \rightarrow \quad \ldots \quad \rightarrow \quad \text{RT}^1_4 \quad \rightarrow \quad \text{RT}^1_3 \quad \rightarrow \quad \text{RT}^1_2 \quad \rightarrow \quad \hat{C}'_2
\\
\Sigma^0_2 \quad \lim \equiv_{sW} \hat{C}_N \quad \downarrow \quad \downarrow \\
\text{WKL} \equiv_{sW} \hat{C}_2 \quad \rightarrow \quad K_N \equiv_{sW} \hat{C}^*_2 \quad \rightarrow \quad \ldots \quad \rightarrow \quad \hat{C}_4 \quad \rightarrow \quad \hat{C}_3 \quad \rightarrow \quad \hat{C}_2
\end{align*}
\]
Corollary (Jump of compact choice)

\[ K'_N \equiv^W \text{RT}^1_N, \ K'_N \preceq^W \text{SRT}^2_N, \ K'_N \preceq^W \text{SRT}^2_2 \ast \text{SRT}^2_2 \quad \text{and} \quad K^{(n)}_N \preceq^W \text{SRT}^n_N \quad \text{for} \ n \geq 2. \]

- Case \( n = 2 \) can be seen as a uniform version of the fact that \( \text{SRT}^2_{<\infty} \) proves \( \text{B} \Sigma^0_3 \) over \( \text{RCA}_0 \) (Cholak, Jockusch, Slaman).
- \( \text{RT}^1_{<\infty} \) is equivalent to \( \text{B} \Sigma^0_2 \) over \( \text{RCA}_0 \) (Hirst)
- \( \text{SRT}^2_2 \) proves \( \text{RT}^1_{<\infty} \) over \( \text{RCA}_0 \) (Cholak, Jockusch, Slaman)
Corollary (Jump of compact choice)

\[ K'_N \equiv_W RT^1_N, \ K'_N \not\leq_W SRT^2_N, \ K'_N \leq_W SRT^2_N \ast SRT^2_N \quad \text{and} \quad K^{(n)}_N \leq_W SRT^n_N \quad \text{for} \ n \geq 2. \]

- Case \( n = 2 \) can be seen as a uniform version of the fact that \( SRT^2_{<\infty} \) proves \( B\Sigma^0_3 \) over \( RCA_0 \) (Cholak, Jockusch, Slaman).
- \( RT^1_{<\infty} \) is equivalent to \( B\Sigma^0_2 \) over \( RCA_0 \) (Hirst)
- \( SRT^2_2 \) proves \( RT^1_{<\infty} \) over \( RCA_0 \) (Cholak, Jockusch, Slaman) in contrast to the statement above!
Corollary (Jump of compact choice)

\[ K'_N \equiv W \ RT^1_N, \ K'_N \not\leq W \ SRT^2_N, \ K'_N \leq W \ SRT^2_N \ast SRT^2_N \text{ and } \]
\[ K^{(n)}_N \leq W \ SRT^*_N \text{ for } n \geq 2. \]

- Case \( n = 2 \) can be seen as a uniform version of the fact that \( SRT^2_{< \infty} \) proves \( B\Sigma^0_3 \) over \( RCA_0 \) (Cholak, Jockusch, Slaman).
- \( RT^1_{< \infty} \) is equivalent to \( B\Sigma^0_2 \) over \( RCA_0 \) (Hirst)
- \( SRT^2_2 \) proves \( RT^1_{< \infty} \) over \( RCA_0 \) (Cholak, Jockusch, Slaman) in contrast to the statement above!
Lowness

\[ L = J^{-1} \circ \text{lim} \]
The Uniform Low Basis Theorem

- $J : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, p \mapsto p'$ denotes the Turing jump.
- $J \equiv_{sw} \lim$ and $J^{-1} : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable.
- $L := J^{-1} \circ \lim$ is the low map.
- $q \in \mathbb{N}^\mathbb{N}$ is low : $\iff q' \leq_T \emptyset'$ $\iff (\exists p \text{ comp.}) L(p) = q$.

Definition (B., de Brecht and Pauly 2011)

- $f$ is low : $\iff f \leq_{sw} L.$

- $L$ is not a cylinder, hence $\leq_{sw}$ cannot be replaced by $\leq_w$.
- $L$ is also not idempotent.

Theorem (B., de Brecht and Pauly 2011)

- $C_R \leq_{sw} L$, that is $C_R$ is low.

This is a uniform version of the Low Basis Theorem.

Corollary

- $WKL \equiv_{sw} C_{2^\mathbb{N}}$ and $BCT_1 \equiv_{sw} C_\mathbb{N}$ are low.
The Uniform Low Basis Theorem

- $J : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, p \mapsto p'$ denotes the Turing jump.
- $J \equiv_{SW} \text{lim}$ and $J^{-1} : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable.
- $L := J^{-1} \circ \text{lim}$ is the low map.
- $q \in \mathbb{N}^\mathbb{N}$ is low $\iff q' \leq_T \emptyset' \iff (\exists p \text{ comp.}) L(p) = q$.

**Definition (B., de Brecht and Pauly 2011)**

$f$ is low $\iff f \leq_{SW} L$.

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**Theorem (B., de Brecht and Pauly 2011)**

$C_\mathbb{R} \leq_{SW} L$, that is $C_\mathbb{R}$ is low.

This is a uniform version of the Low Basis Theorem.

**Corollary**

$WKL \equiv_{SW} C_2^\mathbb{N}$ and $BCT_1 \equiv_{SW} C_\mathbb{N}$ are low.
The Uniform Low Basis Theorem

- $J : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, $p \mapsto p'$ denotes the Turing jump.
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- $L := J^{-1} \circ \text{lim}$ is the low map.
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- \( J : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}, p \mapsto p' \) denotes the Turing jump.
- \( J \equiv_{sW} \text{lim} \) and \( J^{-1} : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \) is computable.
- \( \text{L} := J^{-1} \circ \text{lim} \) is the low map.
- \( q \in \mathbb{N}^\mathbb{N} \) is low : \( \iff q' \leq_T \emptyset' \iff (\exists p \text{comp.}) \text{L}(p) = q. \)

**Definition (B., de Brecht and Pauly 2011)**

\( f \) is low : \( \iff f \leq_{sW} \text{L}. \)

- \( \text{L} \) is not a cylinder, hence \( \leq_{sW} \) cannot be replaced by \( \leq_{W} \).
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\( C_R \leq_{sW} \text{L}, \) that is \( C_R \) is low.

This is a uniform version of the Low Basis Theorem.

**Corollary**

\( \text{WKL} \equiv_{sW} C_{2^\mathbb{N}} \) and \( \text{BCT}_1 \equiv_{sW} C_{\mathbb{N}} \) are low.
The Uniform Low Basis Theorem

- \( J : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}, p \mapsto p' \) denotes the Turing jump.
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The Low Basis Theorem

- $\text{LBT} : \subseteq \text{Tr} \Rightarrow 2^\mathbb{N}$, $T \mapsto \{ p \in [T] : p' \leq_T T' \}$ denotes the Low Basis Theorem with $\text{dom}(\text{LBT})$ as the set of all infinite binary trees.

Theorem (B., Hendtlass and Kreuzer 2015)

$\text{WKL} \prec_w \text{LBT} \prec_w \text{L}$ and $\text{LBT} \nmid_w \text{CR}$.

Proof. (Idea) It is clear that $\text{WKL} \leq_w \text{LBT} \leq_w \text{L}$ and $\text{LBT} \not\leq_w \text{CR}$ follows from the Hyperimmune Free Basis Theorem. $\text{CR} \not\leq_w \text{LBT}$ follows from the following proposition. □

Proposition

$\text{LPO} \not\leq_w \text{LBT}$.

The proof exploits the fact that $\text{LBT}$ restricted to computable inputs is parallelizable.
The Low Basis Theorem

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$\text{WKL} <_W \text{LBT} <_W \text{L}$ and $\text{LBT} \not<_W \text{CR}$.

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**Proposition**

$LPO \not<_W \text{LBT}$.

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- $\text{LBT} \subseteq T \rightarrow 2^\mathbb{N}$, $T \mapsto \{ p \in [T] : p' \leq_T T' \}$ denotes the Low Basis Theorem with $\text{dom}(\text{LBT})$ as the set of all infinite binary trees.

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$\text{WKL} \lesssim_W \text{LBT} \lesssim_W \text{L}$ and $\text{LBT} \not\lesssim_W \text{C}_R$.

**Proof.** (Idea) It is clear that $\text{WKL} \leq_W \text{LBT} \leq_W \text{L}$ and $\text{LBT} \not\leq_W \text{C}_R$ follows from the Hyperimmune Free Basis Theorem. $\text{C}_R \not\leq_W \text{LBT}$ follows from the following proposition. □

**Proposition**

$LPO \not\leq_W \text{LBT}$.

The proof exploits the fact that LBT restricted to computable inputs is parallelizable.
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- \( \text{LBT} : \subseteq \text{Tr} \Rightarrow 2^\mathbb{N}, T \mapsto \{ p \in [T] : p' \leq_T T' \} \) denotes the Low Basis Theorem with \( \text{dom(LBT)} \) as the set of all infinite binary trees.

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\[ \text{WKL} \preceq \text{LBT} \preceq \text{L} \text{ and } \text{LBT} \npreceq \text{W} \text{C}_\text{R}. \]

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Lowness in the Weihrauch Lattice

\[ C_{NN} \]

\[ J \equiv_{sW} \lim \equiv_{sW} \hat{C}_N \]

\[ L : = J^{-1} \circ \lim \]

\[ COH \]

\[ LBT \]

\[ C_R \equiv_{sW} C_N \times C_{2^N} \]

\[ WKL \equiv_{sW} C_{2^N} \]

\[ BCT_1 \equiv_{sW} C_N \]

\[ K_N \equiv_{sW} C_{2^*} \]

\[ LPO \]

\[ LLPO \equiv_{sW} C_2 \]

\[ ACC_N \]
A Characterization and Application of Lowness

- \( f \ast_s g := \sup \{ f_0 \circ g_0 : f_0 \leq_{sW} f \text{ and } g_0 \leq_{sW} g \} \).
- \( \lim \ast_s g \) always exists as a maximum (and is realized by \( J \circ g^r \)).
- \( L_2 := J^{-1} \circ J^{-1} \circ \lim \circ \lim \) characterizes low\(_2\) similarly as \( L \) characterizes lowness.
- \( f \text{ low}_2 : \iff f \leq_{sW} L_2 \).

Theorem (B., Gherardi, Marcone 2012)

- \( f \text{ low} \iff f \leq_{sW} L \iff \lim \ast_s f \leq_{W} \lim \).
- \( f \text{ low}_2 \iff f \leq_{sW} L_2 \iff \lim' \ast_s f \leq_{W} \lim' \).

Theorem (B., Hendtlass and Kreuzer 2015)

COH and WBWT\(_\mathbb{R}\) are low\(_2\) but not low.

The proof uses WKL' \( \equiv_{W} \lim \ast \text{COH} \) and the fact that WKL is low.
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Genericity
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**Proposition (B., de Brecht and Pauly 2011)**

- $p \in \mathbb{N}^\mathbb{N}$ 1–generic and limit computable $\implies p$ limit computable in the jump.
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**Theorem (B., de Brecht and Pauly 2011)**

$\text{DNC}_\mathbb{N} \not\leq_W \lim_J$ and $\text{C}_\mathbb{N} \equiv_s W \lim_N <_W \lim_J <_W L$.

Surprisingly, $\lim_J \equiv_s W L$ with respect to some oracle.
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Genericity

- **1-GEN**: $2^\mathbb{N} \Rightarrow 2^\mathbb{N}$, $p \mapsto \{q : q \text{ is } 1\text{–generic in } p\}$.

**Proposition (B., Hendtlass and Kreuzer 2015)**

$f \leq_w \text{lim}_J$ if $f$ has a limit computable realizer with only 1–generic points in its range.

**Theorem (B., Hendtlass and Kreuzer 2015)**

$\text{BCT}_0 <_w 1\text{-WGEN} <_w 1\text{-GEN} <_w \text{BCT}_0' \equiv_{sw} \Pi^0_1 G <_w \text{lim}_J$.

- **1-WGEN** denotes the problem of weakly 1–generics (defined similarly as above).
- **$\Pi^0_1 G$** denotes the so called $\Pi^0_1$–genericity problem studied in reverse mathematics (interpreted in the straightforward sense).
- **$\text{BCT}_0'$** is densely realized and parallelizable.

**Corollary**

$\text{ACC}_N \not\leq_w \text{BCT}_0'$.
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\[ \text{BCT}_0 <_W 1\text{-WGEN} <_W 1\text{-GEN} <_W \text{BCT}'_0 \equiv_s W \Pi^0_1 G <_W \text{lim}_J. \]

- **1-WGEN** denotes the problem of weakly 1–generics (defined similarly as above).
- **\(\Pi^0_1 G\)** denotes the so called \(\Pi^0_1\)–genericity problem studied in reverse mathematics (interpreted in the straightforward sense).
- **\(\text{BCT}'_0\)** is densely realized and parallelizable.

**Corollary**

\[ \text{ACC}_N \not\leq_W \text{BCT}'_0. \]
Genericity in the Weihrauch Lattice

\[ C_{\text{NN}} \]

\[ J \equiv_{sW} \lim \equiv_{sW} \hat{C}_N \]

\[ L \equiv_{sW} (J^{-1})' \]

\[ C_{\mathbb{R}} \equiv_{\mathbb{W}} C_N \times C_{2_N} \]

\[ \text{DNC}_N \]

\[ \text{LLPO} \equiv_{sW} C_2 \]

\[ \text{ACC}_N \]

\[ \text{NON} \]

\[ \text{WKL} \equiv_{sW} C_{2_N} \]

\[ \text{BCT}_1 \equiv_{sW} C_N \]

\[ \text{K}_N \equiv_{sW} C_2^* \]

\[ \text{LPO} \]

\[ \text{HYP} \]

\[ \text{BCT}_0 \equiv_{sW} \Pi_{1}^0 G \]

\[ \text{1-GEN} \]

\[ \text{1-WGEN} \]

\[ \text{BCT}_0 \]

\[ J^{-1} \]
Randomness
▶ **MLR** : \(2^\mathbb{N} \Rightarrow 2^\mathbb{N}\), the problem of **Martin-Löf randomness** is defined by
\[
\text{MLR}(p) := \{ q \in 2^\mathbb{N} : q \text{ is Martin-Löf random relative to } p \}.
\]

▶ **q** is called **Martin-Löf random relative to** **p**, if for every sequence \((U_i)_i\) of open sets \(U_i \subseteq 2^\mathbb{N}\) that is computable relative to **p** with \(\mu(U_i) < 2^{-i}\), we obtain \(p \notin \bigcap_{i=0}^{\infty} U_i\).

▶ **MLR** is densely realized, hence \(C_2 \not\lesssim_W \text{MLR}\).

▶ **MLR** is parallelizable and hence idempotent.

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**Proposition (B., Gherardi and Hölzl 2015)**

\[
\text{MLR} \ast \text{MLR} \leq_W \text{MLR}.
\]

Follows from van Lambalgen's Theorem.
MLR : $2^\mathbb{N} \xrightarrow{\equiv} 2^\mathbb{N}$, the problem of Martin-Löf randomness is defined by
\[ \text{MLR}(p) := \{ q \in 2^\mathbb{N} : q \text{ is Martin-Löf random relative to } p \}. \]

$q$ is called Martin-Löf random relative to $p$, if for every sequence $(U_i)_i$ of open sets $U_i \subseteq 2^\mathbb{N}$ that is computable relative to $p$ with $\mu(U_i) < 2^{-i}$, we obtain $p \notin \bigcap_{i=0}^{\infty} U_i$.

MLR is densely realized, hence $C_2 \not\leq W \text{MLR}$.

MLR is parallelizable and hence idempotent.

**Proposition (B., Gherardi and Hölzl 2015)**

\[ \text{MLR} * \text{MLR} \leq W \text{ MLR}. \]

Follows from van Lambalgen's Theorem.
Martin-Löf Randomness

- $\text{MLR} : 2^\mathbb{N} \rightarrow 2^\mathbb{N}$, the problem of Martin-Löf randomness is defined by 
  $$\text{MLR}(p) := \{ q \in 2^\mathbb{N} : q \text{ is Martin-Löf random relative to } p \}.$$ 
- $q$ is called Martin-Löf random relative to $p$, if for every sequence $(U_i)_i$ of open sets $U_i \subseteq 2^\mathbb{N}$ that is computable relative to $p$ with $\mu(U_i) < 2^{-i}$, we obtain $p \not\in \bigcap_{i=0}^{\infty} U_i$. 
- $\text{MLR}$ is densely realized, hence $C_2 \not\leq_W \text{MLR}$. 
- $\text{MLR}$ is parallelizable and hence idempotent.

Proposition (B., Gherardi and Hölzl 2015)

$$\text{MLR} \ast \text{MLR} \leq_W \text{MLR}.$$ 

Follows from van Lambalgen’s Theorem.
MLR : \(2^\mathbb{N} \Rightarrow 2^\mathbb{N}\), the problem of Martin-Löf randomness is defined by
\[
\text{MLR}(p) := \{ q \in 2^\mathbb{N} : q \text{ is Martin-Löf random relative to } p \}.
\]

- \(q\) is called Martin-Löf random relative to \(p\), if for every sequence \((U_i)_i\) of open sets \(U_i \subseteq 2^\mathbb{N}\) that is computable relative to \(p\) with \(\mu(U_i) < 2^{-i}\), we obtain \(p \not\in \bigcap_{i=0}^{\infty} U_i\).

- MLR is densely realized, hence \(C_2 \not\leq W \text{ MLR}\).
- MLR is parallelizable and hence idempotent.

**Proposition (B., Gherardi and Hölzl 2015)**

\[
\text{MLR} \ast \text{MLR} \leq W \text{ MLR}.
\]

Follows from van Lambalgen’s Theorem.
MLR : $2^\mathbb{N} \nrightarrow 2^\mathbb{N}$, the problem of Martin-Löf randomness is defined by
\[ \text{MLR}(p) := \{ q \in 2^\mathbb{N} : q \text{ is Martin-Löf random relative to } p \}. \]

$q$ is called Martin-Löf random relative to $p$, if for every sequence $(U_i)_i$ of open sets $U_i \subseteq 2^\mathbb{N}$ that is computable relative to $p$ with $\mu(U_i) < 2^{-i}$, we obtain $p \notin \bigcap_{i=0}^{\infty} U_i$.

MLR is densely realized, hence $C_2 \not\leq_W \text{MLR}$. 

MLR is parallelizable and hence idempotent.

Proposition (B., Gherardi and Hölzl 2015)

\[ \text{MLR} \ast \text{MLR} \leq_W \text{MLR}. \]

Follows from van Lambalgen’s Theorem.
The problem of Martin-Löf randomness is defined by

$$\text{MLR}(p) := \{ q \in 2^\mathbb{N} : q \text{ is Martin-Löf random relative to } p \}.$$ 

$q$ is called Martin-Löf random relative to $p$, if for every sequence $(U_i)_i$ of open sets $U_i \subseteq 2^\mathbb{N}$ that is computable relative to $p$ with $\mu(U_i) < 2^{-i}$, we obtain $p \notin \bigcap_{i=0}^{\infty} U_i$.

MLR is densely realized, hence $C_2 \not\leq W \text{MLR}$.

MLR is parallelizable and hence idempotent.

**Proposition (B., Gherardi and Hölzl 2015)**

$$\text{MLR} \ast \text{MLR} \leq W \text{MLR}.$$ 

Follows from van Lambalgen’s Theorem.
Characterization of Martin-Löf Randomness

Theorem (B. and Pauly 2013)

\[ \text{MLR} \equiv_W (C_N \rightarrow \text{WWKL}). \]

**Proof.** (Sketch.) \((C_N \rightarrow \text{WWKL}) \leq_W \text{MLR}\): It suffices to prove \(\text{WWKL} \leq_W C_N \ast \text{MLR}\). By Kučera’s Lemma, every Martin-Löf random real \(p\) is a path in every infinite binary tree \(T\) of positive measure up to some finite prefix. Using \(C_N\) we can cut away longer and longer prefixes of \(p\) until we find a path in \(T\).

\(\text{MLR} \leq_W (C_N \rightarrow \text{WWKL})\): Given some \(h\) with \(\text{WWKL} \leq_W C_N \ast h\) we need to prove that \(\text{MLR} \leq_W h\). Given some universal Martin-Löf test \((U_i)_i\), the complement \(A_0 := 2^{\mathbb{N}} \setminus U_0\) is a closed set of positive measure and given the corresponding tree \(T\) with \(A = [T]\) the function \(h\) will deliver some sequence \(q\) that can be converted into a Martin-Löf random real by a finite mind change computation. This computation can be converted into a regular computation that yields a Martin-Löf random real. □
Theorem (B. and Pauly 2013)

\[ \text{MLR} \equiv_{W} (C_N \rightarrow \text{WWKL}). \]

**Proof.** (Sketch.) \((C_N \rightarrow \text{WWKL}) \leq_{W} \text{MLR}:\) It suffices to prove \(\text{WWKL} \leq_{W} C_N \ast \text{MLR}.\) By Kučera’s Lemma, every Martin-Löf random real \(p\) is a path in every infinite binary tree \(T\) of positive measure up to some finite prefix. Using \(C_N\) we can cut away longer and longer prefixes of \(p\) until we find a path in \(T\).

\(\text{MLR} \leq_{W} (C_N \rightarrow \text{WWKL}):\) Given some \(h\) with \(\text{WWKL} \leq_{W} C_N \ast h\) we need to prove that \(\text{MLR} \leq_{W} h.\) Given some universal Martin-Löf test \((U_i)_i\), the complement \(A_0 := 2^\mathbb{N} \setminus U_0\) is a closed set of positive measure and given the corresponding tree \(T\) with \(A = [T]\) the function \(h\) will deliver some sequence \(q\) that can be converted into a Martin-Löf random real by a finite mind change computation. This computation can be converted into a regular computation that yields a Martin-Löf random real. \(\square\)
Characterization of Martin-Löf Randomness

Theorem (B. and Pauly 2013)

\[ \text{MLR} \equiv_{W} (C_{N} \rightarrow \text{WWKL}). \]

Proof. (Sketch.) \( (C_{N} \rightarrow \text{WWKL}) \leq_{W} \text{MLR} \): It suffices to prove \( \text{WWKL} \leq_{W} C_{N} * \text{MLR} \). By Kučera’s Lemma, every Martin-Löf random real \( p \) is a path in every infinite binary tree \( T \) of positive measure up to some finite prefix. Using \( C_{N} \) we can cut away longer and longer prefixes of \( p \) until we find a path in \( T \).

\( \text{MLR} \leq_{W} (C_{N} \rightarrow \text{WWKL}) \): Given some \( h \) with \( \text{WWKL} \leq_{W} C_{N} * h \) we need to prove that \( \text{MLR} \leq_{W} h \). Given some universal Martin-Löf test \( (U_{i})_{i} \), the complement \( A_{0} := 2^{N} \setminus U_{0} \) is a closed set of positive measure and given the corresponding tree \( T \) with \( A = [T] \) the function \( h \) will deliver some sequence \( q \) that can be converted into a Martin-Löf random real by a finite mind change computation. This computation can be converted into a regular computation that yields a Martin-Löf random real. \( \Box \)
Quantitative Versions of WWKL

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

By \( \varepsilon\text{-WWKL} \subseteq \text{Tr} \Rightarrow 2^\mathbb{N} \) we denote the restriction of WKL to \( \text{dom}(\varepsilon\text{-WWKL}) := \{ T : \mu([T]) > \varepsilon \} \) for \( \varepsilon \in \mathbb{R} \).

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016 and B., Gherardi and Hölzl 2015)

\( \varepsilon\text{-WWKL} \leq_w \delta\text{-WWKL} \iff \varepsilon \geq \delta \) for all \( \varepsilon, \delta \in [0, 1] \).

Proof. (Idea) “\( \Rightarrow \)” Assume \( \varepsilon < \delta \). Then there are positive integers \( a, b \) with \( \varepsilon < \frac{a}{b} \leq \delta \). We consider

- \( C_{a,b} \) which is \( C_b \) restricted to sets \( A \subseteq \{0, ..., b - 1\} \) with \( |A| \geq a \).

Then \( C_{a,b} \leq_w \varepsilon\text{-WWKL} \) and \( C_{a,b} \nleq_w \delta\text{-WWKL} \). Hence \( \varepsilon\text{-WWKL} \nleq_w \delta\text{-WWKL} \) \( \square \)

Proposition (B., Hendtlass and Kreuzer 2015)

\( \varepsilon\text{-WWKL} \) is not parallelizable for \( \varepsilon \in [0, 1) \).
Quantitative Versions of WWKL

**Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)**

By $\varepsilon$-WWKL : $\subseteq \text{Tr} \Rightarrow 2^\mathbb{N}$ we denote the restriction of WKL to
$\text{dom}(\varepsilon\text{-WWKL}) := \{T : \mu([T]) > \varepsilon\}$ for $\varepsilon \in \mathbb{R}$.

**Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016 and B., Gherardi and Hölzl 2015)**

$\varepsilon$-WWKL $\leq_w \delta$-WWKL $\iff \varepsilon \geq \delta$ for all $\varepsilon, \delta \in [0, 1]$.

**Proof.** (Idea) “$\implies$” Assume $\varepsilon < \delta$. Then there are positive integers $a, b$ with $\varepsilon < \frac{a}{b} \leq \delta$. We consider
- $C_{a,b}$ which is $C_b$ restricted to sets $A \subseteq \{0, ..., b - 1\}$ with $|A| \geq a$.

Then $C_{a,b} \leq_w \varepsilon$-WWKL and $C_{a,b} \not\leq_w \delta$-WWKL. Hence $\varepsilon$-WWKL $\not\leq_w \delta$-WWKL $\square$

**Proposition (B., Hendtlass and Kreuzer 2015)**

$\varepsilon$-WWKL is not parallelizable for $\varepsilon \in [0, 1)$.
Quantitative Versions of WWKL

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By $\varepsilon$-WWKL $:\subseteq \text{Tr} \Rightarrow 2^\mathbb{N}$ we denote the restriction of WKL to 
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Proof. (Idea) “$\implies$” Assume $\varepsilon < \delta$. Then there are positive integers $a, b$ with $\varepsilon < \frac{a}{b} \leq \delta$. We consider

- $C_{a,b}$ which is $C_b$ restricted to sets $A \subseteq \{0, \ldots, b-1\}$ with $|A| \geq a$.

Then $C_{a,b} \leq_W \varepsilon$-WWKL and $C_{a,b} \not\leq_W \delta$-WWKL. Hence $\varepsilon$-WWKL $\not\leq_W \delta$-WWKL.

Proposition (B., Hendtlass and Kreuzer 2015)

$\varepsilon$-WWKL is not parallelizable for $\varepsilon \in [0, 1)$.
Quantitative Versions of WKKL

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

By \( \varepsilon\text{-WWKL} \subseteq \text{Tr} \Rightarrow 2^\mathbb{N} \) we denote the restriction of \( \text{WKL} \) to
\[
\text{dom}(\varepsilon\text{-WWKL}) := \{ T : \mu([T]) > \varepsilon \} \text{ for } \varepsilon \in \mathbb{R}.
\]

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016 and B., Gherardi and Hölzl 2015)

\( \varepsilon\text{-WWKL} \leq_W \delta\text{-WWKL} \iff \varepsilon \geq \delta \) for all \( \varepsilon, \delta \in [0, 1] \).

\textbf{Proof.} (Idea) "\( \Rightarrow \)" Assume \( \varepsilon < \delta \). Then there are positive integers \( a, b \) with \( \varepsilon < \frac{a}{b} \leq \delta \). We consider

- \( C_{a,b} \) which is \( C_b \) restricted to sets \( A \subseteq \{0, \ldots, b - 1\} \) with \( |A| \geq a \).

Then \( C_{a,b} \leq_W \varepsilon\text{-WWKL} \) and \( C_{a,b} \not\leq_W \delta\text{-WWKL} \). Hence \( \varepsilon\text{-WWKL} \not\leq_W \delta\text{-WWKL} \). \( \square \)

Proposition (B., Hendtlass and Kreuzer 2015)

\( \varepsilon\text{-WWKL} \text{ is not parallelizable for } \varepsilon \in [0, 1] \).
Quantitative Versions of WWKL

\[(1−\ast)-\text{WWKL} \subseteq \text{Tr}^\mathbb{N} \Rightarrow 2^\mathbb{N}, (T_i); \mapsto \bigcup_{i=0}^{\infty} (1−2^{-i})-\text{WWKL}(T_i)\]

Theorem (B., Hendtlass and Kreuzer 2015)

\[(1−\ast)-\text{WWKL} \text{ is parallelizable.}\]

Proposition (B., Hendtlass and Kreuzer 2015)

\[\text{ACC}_\mathbb{N} \leq_W (1−\ast)-\text{WWKL}.\]

Corollary

\[\text{DNC}_\mathbb{N} \leq_W (1−\ast)-\text{WWKL}.\]

Proposition (B., Hendtlass and Kreuzer 2015)

\[\text{DNC}_\mathbb{N} \mid_W \text{MLR}.\]
Quantitative Versions of WWKL

\( (1-\ast)\text{-WWKL} : \subseteq \text{Tr}^\mathbb{N} \Rightarrow 2^\mathbb{N}, (T_i) \mapsto \bigcup_{i=0}^{\infty} (1-2^{-i})\text{-WWKL}(T_i) \)

Theorem (B., Hendtlass and Kreuzer 2015)

\( (1-\ast)\text{-WWKL} \) is parallelizable.

Proposition (B., Hendtlass and Kreuzer 2015)

\( \text{ACC}_\mathbb{N} \leq_W (1-\ast)\text{-WWKL} \).

Corollary

\( \text{DNC}_\mathbb{N} \leq_W (1-\ast)\text{-WWKL} \).

Proposition (B., Hendtlass and Kreuzer 2015)

\( \text{DNC}_\mathbb{N} |_W \text{MLR} \).
Quantitative Versions of WWKL

\[(1-*\)-WWKL \subseteq \text{Tr}^\mathbb{N} \Rightarrow 2^\mathbb{N}, (T_i); \mapsto \bigcup_{i=0}^{\infty} (1-2^{-i})\text{-WWKL}(T_i)\]

**Theorem (B., Hendtlass and Kreuzer 2015)**

\[(1-*\)-WWKL is parallelizable.\]

**Proposition (B., Hendtlass and Kreuzer 2015)**

\[\text{ACC}_\mathbb{N} \leq_W (1-*)\text{-WWKL}.\]

**Corollary**

\[\text{DNC}_\mathbb{N} \leq_W (1-*)\text{-WWKL}.\]

**Proposition (B., Hendtlass and Kreuzer 2015)**

\[\text{DNC}_\mathbb{N} \mid_W \text{MLR}.\]
Quantitative Versions of WWKL

\[
(1−\ast)\text{-WWKL} : \subseteq Tr^\mathbb{N} \implies 2^\mathbb{N}, (T_i)_i \mapsto \bigcup_{i=0}^{\infty} (1−2^{-i})\text{-WWKL}(T_i)
\]

Theorem (B., Hendtlass and Kreuzer 2015)

\( (1−\ast)\text{-WWKL} \) is parallelizable.

Proposition (B., Hendtlass and Kreuzer 2015)

\[ ACC_\mathbb{N} \leq_W (1−\ast)\text{-WWKL}. \]

Corollary

\[ DNC_\mathbb{N} \leq_W (1−\ast)\text{-WWKL}. \]

Proposition (B., Hendtlass and Kreuzer 2015)

\[ DNC_\mathbb{N} \mid_W MLR. \]
Quantitative Versions of WWKL

\[(1 - *)\text{-WWKL} : \subseteq \text{Tr}^\mathbb{N} \Rightarrow 2^\mathbb{N}, (T_i)_i \mapsto \bigsqcup_{i=0}^{\infty} (1 - 2^{-i})\text{-WWKL}(T_i)\]

**Theorem (B., Hendtlass and Kreuzer 2015)**

\((1 - *)\text{-WWKL} is parallelizable.\)

**Proposition (B., Hendtlass and Kreuzer 2015)**

\(\text{ACC}_\mathbb{N} \leq_W (1 - *)\text{-WWKL}.\)

**Corollary**

\(\text{DNC}_\mathbb{N} \leq_W (1 - *)\text{-WWKL}.\)

**Proposition (B., Hendtlass and Kreuzer 2015)**

\(\text{DNC}_\mathbb{N} \mid_W \text{MLR}.\)
Theorem (B., Gherardi and Hölzl 2015)

\[ \text{MLR} \leq_W (1 - *)\text{-WWKL}. \]

**Proof.** (Sketch) We use a universal Martin-Löf test, which is a computable sequence \((U_i)_i\) of c.e. open sets \(U_i \subseteq 2^\mathbb{N}\) such that 
\[ \mu(U_i) < 2^{-n} \] and 
\[ \bigcap_{i=0}^{\infty} U_i \] is exactly the set of all sequences which are not Martin-Löf random. Hence, \(A_i := 2^\mathbb{N} \setminus U_i\) is a co-c.e. closed set with 
\[ \mu(A_i) > 1 - 2^{-n} \] and each \(A_i\) only contains Martin-Löf random sequences. Hence, we can compute a corresponding sequence \((T_i)_i\) of infinite binary trees with 
\[ [T_i] = A_i. \] Upon input of this sequence \((1 - *)\text{-WWKL}\) yields a Martin-Löf random sequence. The entire argument can be relativized, i.e., it also works in presence of some oracle \(p \in 2^\mathbb{N}\). This yields the reduction \(\text{MLR} \leq_W (1 - *)\text{-WWKL}\). In order to see that the reduction is strict, one has to take into account that \(\text{MLR}\) is densely realized. \(\square\)
Theorem of Kurtz. Every 2–random computes a 1–generic.

Theorem (B., Hendtlass and Kreutzer 2015)

$$1\text{-GEN} \prec_W (1 - *)\text{-WWKL}'$$.

Proof. (Idea) We apply the “fireworks technique” of Rumyantsev and Shen to get a uniform reduction. □

Theorem (B., Hendtlass and Kreutzer 2015)

$$\text{BCT}'_0 \not\leq_W \text{WWKL}^{(n)}$$ for all $$n \in \mathbb{N}$$.

Proof. (Idea) There exists a co-c.e. comeager set $$A \subseteq 2^\mathbb{N}$$ such that no point of $$A$$ is low for $$\Omega$$. $$\text{WWKL}^{(n)}$$ has a realizer that maps computable inputs to outputs that are low for $$\Omega$$ for $$n \geq 1$$. □

Corollary

$$\text{BCT}'_0 \not\leq_W 1\text{-GEN}$$.
Theorem of Kurtz. Every 2–random computes a 1–generic.

Theorem (B., Hendtlass and Kreutzer 2015)

1-GEN $\leq^w W(1 - \ast)$-WWKL$'$.

Proof. (Idea) We apply the “fireworks technique” of Rumyantsev and Shen to get a uniform reduction. □

Theorem (B., Hendtlass and Kreutzer 2015)

$BCT'_0 \not\leq_W WWKL^{(n)}$ for all $n \in \mathbb{N}$.

Proof. (Idea) There exists a co-c.e. comeager set $A \subseteq 2^\mathbb{N}$ such that no point of $A$ is low for $\Omega$. $WWKL^{(n)}$ has a realizer that maps computable inputs to outputs that are low for $\Omega$ for $n \geq 1$. □

Corollary

$BCT'_0 \not\leq_W 1$-GEN.
Theorem of Kurtz. Every 2–random computes a 1–generic.

**Theorem (B., Hendtlass and Kreutzer 2015)**

\[ 1\text{-GEN} \prec_W (1 - *)\text{-WWKL}' \]

**Proof.** (Idea) We apply the “fireworks technique” of Rumyantsev and Shen to get a uniform reduction.

**Theorem (B., Hendtlass and Kreutzer 2015)**

\[ \text{BCT}_0' \nleq_W \text{WWKL}^{(n)} \text{ for all } n \in \mathbb{N}. \]

**Proof.** (Idea) There exists a co-c.e. comeager set \( A \subseteq 2^\mathbb{N} \) such that no point of \( A \) is low for \( \Omega \). \( \text{WWKL}^{(n)} \) has a realizer that maps computable inputs to outputs that are low for \( \Omega \) for \( n \geq 1 \).

**Corollary**

\[ \text{BCT}_0' \nleq_W 1\text{-GEN}. \]
Theorem of Kurtz. Every 2–random computes a 1–generic.

**Theorem (B., Hendtlass and Kreutzer 2015)**

\[ 1 \text{-GEN} <_W (1 - *) \text{-WWKL}' \]

**Proof.** (Idea) We apply the “fireworks technique” of Rumyantsev and Shen to get a uniform reduction.

**Theorem (B., Hendtlass and Kreutzer 2015)**

\[ \text{BCT}'_0 \not<_W \text{WWKL}^{(n)} \text{ for all } n \in \mathbb{N} \]

**Proof.** (Idea) There exists a co-c.e. comeager set \( A \subseteq 2^{\mathbb{N}} \) such that no point of \( A \) is low for \( \Omega \). \( \text{WWKL}^{(n)} \) has a realizer that maps computable inputs to outputs that are low for \( \Omega \) for \( n \geq 1 \).

**Corollary**

\[ \text{BCT}'_0 \not<_W 1 \text{-GEN} \]
Theorem of Kurtz. Every 2–random computes a 1–generic.

Theorem (B., Hendtlass and Kreutzer 2015)

\[ 1\text{-GEN} \prec_W (1 - \ast)\text{-WWKL}' . \]

**Proof.** (Idea) We apply the “fireworks technique” of Rumyantsev and Shen to get a uniform reduction.  

Theorem (B., Hendtlass and Kreutzer 2015)

\[ \text{BCT}'_0 \nless_W \text{WWKL}^{(n)} \text{ for all } n \in \mathbb{N} . \]

**Proof.** (Idea) There exists a co-c.e. comeager set \( A \subseteq 2^\mathbb{N} \) such that no point of \( A \) is low for \( \Omega \). \( \text{WWKL}^{(n)} \) has a realizer that maps computable inputs to outputs that are low for \( \Omega \) for \( n \geq 1 \).  

Corollary

\[ \text{BCT}'_0 \nless_W 1\text{-GEN} . \]
Summary on Weihrauch Complexity

- Weihrauch complexity is a **uniform and resource sensitive** computable version of reverse mathematics.
- It measures the amount of resources needed to compute certain realizers of theorems.
- Positive and negative results are directly constructed without any need for further models.
- Results have immediate interpretations in computable analysis.
- Many results from reverse mathematics are fully uniform with only one usage of the resource.
- Sometimes proofs can be transferred, sometimes completely new methods have to be developed.
- The Weihrauch lattice can be seen as a refinement of the Borel hierarchy for functions and hence methods of descriptive set theory and topology can be applied directly.
- Many complexity classes have direct computational interpretations.