Dependent Random Graphs and Multiparty Pointer Jumping

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1/21/16
(1) Dependent Random Graphs: Erdös-Rényi random graphs with edge dependencies

(2) Thorough understanding of clique number, chromatic number of dependent random graphs

(3) Application: new upper bound for NOF communication complexity of Multiparty Pointer Jumping
**Random Graphs**

\[ G \sim G(n,p) : \text{random graph on } n \text{ vertices } V = \{1, \ldots, n\} \]

each edge \((i,j) \in E\) independently with prob. \(p\)

\[ G(n,p) : \text{probability distribution} \]

\( G : \text{random variable} \)
Applications
Applications

Concentration of Measure:

- **clique number**: $\text{CL}(G) = \text{size of largest clique in } G$
- $\text{CL}(G) \sim \frac{2 \log(n)}{\log(1/p)}$ almost always
- **chromatic number**: $\chi(G) = \# \text{ colors needed to color } G$
- $\chi(G) \sim \frac{n \log(1/1-p)}{2 \log(n)}$ almost always
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Threshold functions:

- **Largest Component**:
  - $p < (1-c)/n$ then largest component is $O(\log n)$
  - $p = 1/n$ then largest component is $n^{2/3}$
  - $p > (1+c)/n$ then largest component is $> n/2$
- $\text{CL}(G) \geq 4$, Connectivity

Zero-One Laws, ...
Dependent Random Graphs

Definition: $G_d(n,p)$ is a distribution on $n$ vertex graphs, where

- each edge in graph w/prob $p$
- each edge depends on $\leq d$ other edges
Examples
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- Flip coin for each vertex
- add edge $(i,j)$ iff $X_i = X_j$
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Examples

- Flip coin for each vertex
- add edge \((i,j)\) iff \(X_i = X_j\)
- \(\text{CL}(G) \geq n/2\) w/prob 1

- Flip coin for each vertex
- add edge \((i,j)\) iff \(X_i \neq X_j\)
- \(\text{CL}(G) \leq 2\) w/prob 1
Examples

- Flip coin for each vertex
- Add edge \((i, j)\) iff \(X_i = X_j\)

- Flip coin for each vertex
- Add edge \((i, j)\) iff \(X_i \neq X_j\)

Insight: if \(d \ll n\), then bounds on properties of \(G_d(n, p)\) possible
Our Results
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Let $G \sim G_d(n,p)$. The following hold almost always:
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**Clique Number:**

- If $d/p \ll \sqrt{n}$ then $\text{clique}(G) \geq \log(n)/\log(1/p)$
- if $d < n/\log^2(n)$ then $\text{clique}(G) = O(d \log(n))$
Our Results

Let $G \sim G_d(n,p)$. The following hold almost always:

**Clique Number:**

- If $d/p << \sqrt{n}$ then $\text{clique}(G) \gtrapprox \log(n)/\log(1/p)$
- If $d < n/\log^2(n)$ then $\text{clique}(G) = O(d \log(n))$

**Chromatic Number:**

- If $d << n^{o(1)}$ then $\chi(G) < n \log(1/1-p) / \log(n)$
- If $d < n/\log^2(n)$ then $\chi(G) = \Omega(n/d\log(n))$
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Witnesses:
- $G_d(n,p)$’s with large clique, small $\chi(G)$
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**Witnesses:**

- $G_d(n,p)$’s with large clique, small $\chi(G)$
Key Intuition

Most k-cliques look independent.

Definition: $S \subseteq V$ is uncorrelated if $G|S$ is independent.
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Most k-cliques look independent.

Definition: \( S \subseteq V \) is uncorrelated if \( G|_S \) is independent.

Lemma: Let \( G \sim G_d(n,p) \) and \( S \) be uniform subset of \( k \) vertices. If \( dk^3 << n \), then

\[
\Pr[S \text{ uncorrelated}] = 1 - o(1)
\]
Martingale: $Y_0, ..., Y_N$ are a martingale w.r.t. random vars $X_1, ..., X_N$ if for all $k < N$

- $E[|Y_k|] < \infty$
- $E[Y_{k+1} | X_1, .., X_k] = Y_k$
Classic Clique LB

[Bollobás 88]

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Edge exposure martingale:

- $X_1, ..., X_N$: edges of $G(n,p)$
- $Y_k: f(G|X_1, ..., X_k)$
Martingale: $Y_0, ..., Y_N$ are a martingale w.r.t. random vars $X_1, ..., X_N$ if for all $k < N$

- $\mathbb{E}[|Y_k|] < \infty$
- $\mathbb{E}[Y_{k+1} | X_1, .., X_k] = Y_k$

Azuma’s Inequality: If $a_k \leq Y_k - Y_{k-1} \leq b_k$ for all $k > 0$ then

$$\Pr[Y_N < Y_0 - t] \leq \exp(-2t^2/\sum_{k} (b_k-a_k)^2)$$
martingale: \[ Y := \text{largest # edge-disjoint k-cliques} \]

Goal: show w.h.p \( Y > 0 \)

- Select each \( S \) indep w/prob \( \gamma \)
- Remove all pairs of intersecting k-cliques
- \( L \) := remaining k-cliques
**Classic Clique LB**

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**martingale:**

\[ Y := \text{largest # edge-disjoint } k\text{-cliques} \]

**Goal:** show w.h.p \( Y > 0 \)

- Select each \( S \) indep w/prob \( \gamma \)
- Remove all pairs of intersecting \( k\)-cliques
- \( L := \text{remaining } k\text{-cliques} \)

**Analysis:**

\[
E[Y] \geq E[|L|] \\
\geq \gamma E[\#\text{cliques}] - 2 \gamma^2 E[\#\text{intersecting pairs}] \\
\geq ... (\text{small pain}) ... \geq n^2 p / 18k^5
\]
martingale: $Y :=$ largest # edge-disjoint $k$-cliques

Goal: show w.h.p $Y > 0$

• Select each $S$ indep w/prob $\gamma$
• Remove all pairs of intersecting $k$-cliques
• $L :=$ remaining $k$-cliques

Analysis: $E[Y] \geq E[|L|]$

$\geq \gamma E[\#cliques] - 2 \gamma^2 E[\#intersecting pairs]$

$\geq \ldots$ (small pain) $\ldots \geq n^2 p / 18k^5$

Azuma’s Inequality: $\Pr[Y=0] \approx \exp(-E[Y]^2/n^2) = \text{small}$
Clique LB for dependent random graphs
Clique LB for dependent random Graphs

\[ Y := \text{largest \# edge-disjoint uncorrelated } k\text{-cliques} \]

Goal: show w.h.p \( Y > 0 \)

- Select each uncorrelated \( S \) w/prob \( \gamma \)
- Remove all pairs of intersecting \( k\)-cliques
- \( L := \text{remaining uncorrelated } k\text{-cliques} \)
Clique LB for dependent random Graphs

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**Goal:** show w.h.p \( Y > 0 \)

- Select each \textit{uncorrelated} \( S \) w/ prob \( \gamma \)
- Remove all pairs of intersecting \( k \)-cliques
- \( L := \) remaining \textit{uncorrelated} \( k \)-cliques

\[ \mathbb{E}[Y] \geq \mathbb{E}[|L|] \geq \gamma \mathbb{E}[\# \text{cliques}] - 2 \gamma^2 \mathbb{E}[\# \text{intersecting pairs}] \geq ??? \]
Calculating #Uncorrelated Intersecting K-cliques

Independent case: $\Pr[S,T \text{ intersect}] = p^\binom{k}{2} \cdot p^\binom{k}{2} - \binom{\ell}{2}$
Calculating #Uncorrelated Intersecting K-cliques

Independent case: \( \Pr[S,T \text{ intersect}] = p^k \cdot p^k - \binom{\ell}{2} \)

Dependent case: \( \Pr[S,T \text{ intersect}] < p^k \cdot 1 \)
Clique LB for dependent random graphs
**Clique LB for dependent random Graphs**

\( Y := \) largest # edge-disjoint *uncorrelated* 

**k-cliques**

**Goal:** show w.h.p \( Y > 0 \)

- Select each *uncorrelated* \( S \) w/prob \( \gamma \)
- Remove all pairs of intersecting k-cliques
- \( L := \) remaining *uncorrelated* k-cliques
Clique LB for dependent random graphs

Goal: show w.h.p. \( Y > 0 \)

- Select each uncorrelated \( S \) w/prob \( \gamma \)
- Remove all pairs of intersecting \( k \)-cliques
- \( L := \) remaining uncorrelated \( k \)-cliques

Analysis: \[ E[Y] \geq E[|L|] \]
\[ \geq \gamma E[\#cliques] - 2 \gamma^2 E[\#intersecting pairs] \]
\[ \geq \frac{n^2p}{19k^5} \]
Clique LB for dependent random Graphs

\[ Y := \text{largest } \# \text{ edge-disjoint uncorrelated } k\text{-cliques} \]

**Goal:** show w.h.p \( Y > 0 \)

- Select each uncorrelated \( S \) w/prob \( \gamma \)
- Remove all pairs of intersecting \( k\)-cliques
- \( L := \text{remaining uncorrelated } k\text{-cliques} \)

**Analysis:**

\[ E[Y] \geq E[|L|] \geq \gamma E[\#\text{cliques}] - 2 \gamma^2 E[\#\text{intersecting pairs}] \geq n^2p/19k^5 \]

Azuma’s Inequality:

\[ \Pr[Y=0] \approx \exp(-E[Y]^2/n^2d^2) = \text{still small} \]
Number-On-The-Forehead Communication

Input: $x = (x_1, x_2, ..., x_k)$ is split between $k$ players

Goal: minimize communication needed to compute $f(x)$

Communication Model:

- **NOF communication**: Player $i$ sees all input except $x_i$
- **One-way communication**: each player speaks once, in order
- **Simultaneous Messages**: each player sends msg to Referee, Referee outputs answer.
- **Blackboard**: all players see every message sent
Motivation/Applications

• Current *frontier* in communication complexity
• Lower bounds *harder*
• Surprising *upper bounds* possible
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• Current *frontier* in communication complexity
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**Open Problem:** Give polynomial $\Omega(n^{0.001})$ communication lower bound for any $k = \text{polylog}(n)$ player NOF problem
Motivation/Applications

• Current *frontier* in communication complexity
• Lower bounds *harder*
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Open Problem: Give polynomial $\Omega(n^{0.001})$ communication lower bound for any $k = \text{polylog}(n)$ player NOF problem

Fact: If $f \in \text{ACC}^0$ then $f$ has NOF protocol with $\text{poly}(\log n)$ communication and $k = \text{poly}(\log n)$ players

[Y90], [HG91], [BT94]
MPJ₃: Multiparty Pointer Jumping

\[ i \in \{1, \ldots, n\} \quad f : \{0,1\}^n \rightarrow \{0,1\}^n \quad x \in \{0,1\}^n \]

Output: \[ x[f(i)] \]
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Output: \[ x[f(i)] \]
Permutation Pointer Jumping

\[ i \in \{1, \ldots, n\} \quad \Pi \in S_n \quad x \in \{0,1\}^n \]

Output: \[ x[\Pi(i)] \]
Previous Results, Our Result

Lower Bounds:

- \( \text{MPJ}_3: \Omega(n^{1/2}) \)  
- \( \text{MPJ}_k: \Omega(n^{1/(k-1)}) \)

Also: strong lower bounds for restricted protocols

Upper Bounds:

- \( \text{MPJ}_3^{\text{perm}}: O(n (\log \log n)/\log(n)) \)  
- \( \text{MPJ}_3: O(n \sqrt{(\log \log n)/\log n}) \)  
- \( \text{MPJ}_3: O(n (\log \log n)/\log n) \)

[W97] [VW 07] [PRS 97] [BC 08] [this work]
Previous Results, Our Result

Lower Bounds:

- $\text{MPJ}_3: \Omega(n^{1/2})$  
  \cite{W97}
- $\text{MPJ}_k: \Omega(n^{1/(k-1)})$  
  \cite{VW07}

Also: strong lower bounds for \textit{restricted protocols}  
\cite{DJS98, G06, C07, BC08}

Upper Bounds:

- $\text{MPJ}_3^{\text{perm}}: O(n \cdot \log\log n / \log(n))$  
  \cite{PRS97}
- $\text{MPJ}_3: O(n \sqrt{\log\log n / \log n})$  
  \cite{BC08}
- $\text{MPJ}_3: O(n \cdot \log\log n / \log n)$  
  (this work)
The PRS Protocol

[Prášek-Rödl-Sgall 97]

The PRS gameplan:

(1) For each bipartite graph $H$ define protocol $P_H$.

(2) Show each $P_H$ computes $MPJ_3^{\text{perm}}$.

(3) Probabilistic Method: $\exists H$ with $\text{cost}(P_H) = O(n \log \log(n) / \log(n))$
Protocol $P_H$

(1) Alice sees $H, \prod, x$, creates graph $G_{\prod H}$
Protocol $P_H$

(1) Alice sees $H, \Pi, x$, creates graph $G_{\Pi H}$
Protocol $P_H$

(1) Alice sees $H$, $\Pi$, $x$, creates graph $G_{\Pi H}$
Protocol $P_H$

(1) Alice sees $H$, $\prod$, $x$, creates graph $G_{\prod H}$

Add edge $(a,b) \in G_{\prod H}$ if:

- $(\prod^{-1}(a), b)$ edge in $H$
- $(\prod^{-1}(b), a)$ edge in $H$
**Protocol $P_{H}$**

1. Alice sees $H$, $\mathcal{P}$, $x$, creates graph $G_{\mathcal{P}H}$

Add edge $(a,b) \in G_{\mathcal{P}H}$ if:

- $(\mathcal{P}^{-1}(a), b)$ edge in $H$
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Protocol $P_H$

(1) Alice sees $H$, $\mathcal{P}$, $x$, creates graph $G_{\mathcal{P}H}$

Add edge $(a,b) \in G_{\mathcal{P}H}$ if:
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$C = \{C_1, ..., C_m\}$: clique cover of $G_{\mathcal{P}H}$
Protocol $P_H$

(1) Alice sees $H$, $\Pi$, $x$, creates graph $G_{\Pi H}$

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Add edge $(a,b) \in G_{\mathcal{P}H}$ if:
- $(\mathcal{P}^{-1}(a), b)$ edge in $H$
- $(\mathcal{P}^{-1}(b), a)$ edge in $H$

$\mathcal{C} = \{C_1, \ldots, C_m\}$:
clique cover of $G_{\mathcal{P}H}$

Alice sends $\text{parity bit}$ of each $C \in \mathcal{C}$

$x_1 \oplus x_2, \quad x_3 \oplus x_4 \oplus x_5$
Protocol $P_H$
**Protocol $P_H$**

1. Alice sees $H$, $\Pi$, $x$, creates $G_{\Pi H}$
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   - $C^*$: clique containing $\Pi(i)$
**Protocol \( P_H \)**

1. Alice sees \( H, \Pi, \mathbf{x} \), creates \( G^{\Pi H} \)
   - sends parity bit for each \( C \in \mathcal{C} \)
   - \( C^* \): clique containing \( \Pi(i) \)

2. Bob sees \( H, i, \mathbf{x} \)
   - sends \( x_j \) for each \( (i,j) \in H \)
Protocol $P_H$

1. Alice sees $H, \prod, x$, creates $G_{\prod H}$
   - sends parity bit for each $C \in \mathcal{C}$
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   - sends $x_j$ for each $(i,j) \in H$
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(1) Alice sees $H$, $\prod$, $x$, creates $G_{\prod H}$
- sends parity bit for each $C \in \mathcal{C}$
- $C^*$: clique containing $\prod(i)$

(2) Bob sees $H$, $i$, $x$
- sends $x_j$ for each $(i,j) \in H$

Fact: Bob sends $x_j$ for each $j \in C^*$ ($j \neq \prod(i)$)
**Protocol \( P_H \)**

1. Alice sees \( H, \pi, x \), creates \( G_{\pi H} \)
   - sends parity bit for each \( C \in \mathcal{C} \)
   - \( C^* \): clique containing \( \pi(i) \)

2. Bob sees \( H, i, x \)
   - sends \( x_j \) for each \( (i,j) \in H \)

3. Carol sees \( H, i, \pi \), creates \( G_{\pi H} \)
   - takes parity bit from \( C^* \) Alice’s msg.
   - XORs out \( x_j \) for all \( j \) but \( \pi(i) \)

**Fact:** Bob sends \( x_j \) for each \( j \in C^* \) (\( j \neq \pi(i) \))
Protocol \textbf{PH}: Creating $G^\Pi_H$

\[ i \in \{1, \ldots, n\} \quad \Pi \in S_n \quad x \in \{0,1\}^n \]

Output: $x[\Pi(i)]$
Claim: There exists $H$ such that

$$\text{cost}(P_H) = O(n\log\log(n)/\log(n))$$
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Proof: consider random $H$, each edge w/prob $p$
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$$\text{cost}(P_H) = O(n \log \log(n) / \log(n))$$

**Proof:** consider random $H$, each edge w/prob $p$

- $G_{\overline{H}} \sim G(n, p^2)$
- $\text{clique-cover}(G) = \chi\left( \overline{G} \right) = \chi(G(n, 1-p^2))$
Claim: There exists $H$ such that 

$$\text{cost}(P_H) = O(n \log \log(n) / \log(n))$$

Proof: consider random $H$, each edge w/prob $p$

- $G_{\prod H} \sim G(n, p^2)$
- $\text{clique-cover}(G) = \chi(\overline{G}) = \chi(G(n, 1-p^2))$
- $\text{BAD}_i$: event that $i$ has $> 2pn$ neighbors
- $\text{BAD}_{\prod H}$: event that $G_{\prod H}$ has $> 2n \log(1/p)/\log(n)$ cliques
- Choose $p := \log \log(n) / \log(n)$
Final Analysis

Claim: There exists $H$ such that

$$\text{cost}(P_H) = O(n\log\log(n)/\log(n))$$

Proof: consider random $H$, each edge w/prob $p$

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- Choose $p := \log\log(n)/\log(n)$
- $\Pr[\text{BAD}] \leq n \Pr[\text{BAD}_i] + n! \Pr[\text{BAD}_{\prod H}] < 1$
Generalizing to MPJ$^3$
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PRS protocol works only for permutations.
Generalizing to $\text{MPJ}_3$

PRS protocol works only for permutations.

Our Protocol:

- Create graph $G_{f,H}$ on domain of $H$.
- Add edge $(a,b) \in G_{f,H}$ if $(a,f(b))$ and $(b,f(a)) \in H$
- Edges in $G_{f,H}$ no longer independent
**Generalizing to MPJ$_3$**

PRS protocol works only for permutations.

**Our Protocol:**

- Create graph $G_{fH}$ on *domain* of $H$.
- Add edge $(a,b) \in G_{fH}$ if $(a,f(b))$ and $(b,f(a)) \in H$
- Edges in $G_{fH}$ no longer independent
- **Dependent Random Graphs:** each depends on a few other edges.
- dependency $\approx$ largest preimage of $f$
- handle $|f^{-1}(a)| > \log(n)$ separately
Generalizing to $\text{MPJ}_3$

PRS protocol works only for permutations.

Our Protocol:
- Create graph $G_{fH}$ on **domain** of $H$.
- Add edge $(a,b) \in G_{fH}$ if $(a,f(b))$ and $(b,f(a)) \in H$
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**Dependent Random Graphs**: each depends on a few other edges.

- dependency $\approx$ largest preimage of $f$
- handle $|f^{-1}(a)| > \log(n)$ separately

- lower bound on **clique #**, upper bound on **chromatic number** still (asymptotically) the same.
Conclusions/Open Problems

Current state of MPJ₃:

• best u.b.’s *deterministic, SM*, work for general *f*
• best l.b.’s *randomized, one-way*
• known lower bound techniques *can’t get better than* $\sqrt{n}$.
• MPJ₃ l.b.’s should generalize to MPJₖ.
Conclusions/Open Problems

Current state of MPJ3:

• best u.b.’s **deterministic, SM**, work for general $f$

• best l.b.’s **randomized, one-way**

- Open Problem: Prove $\text{CC}(\text{MPJ}_3) \geq 10\sqrt{n}$.  

• MPJ3 l.b.’s should generalize to MPJ$_k$. 
Conclusions/Open Problems

Current state of MPJ₃:
• best u.b.’s **deterministic, SM**, work for general f
• best l.b.’s **randomized, one-way**
• known lower bound techniques can’t get better than $\sqrt{n}$.
• **MPJ₃** l.b.’s should generalize to **MPJₖ**.

Open Problems for Dependent RG’s:
• tighten analysis for **clique/chromatic #**
• other properties: **connectivity, diameter, ...**
• [B-Sanchez16]: **jumbledness**
Thanks!