Almost Optimal Sum of Squares Lower Bound for \textit{Planted Clique}

Pravesh K. Kothari
	
\textsc{UT Austin}

Based on joint works with Boaz Barak, Sam Hopkins, Jon Kelner, Ankur Moitra and Aaron Potechin.
Max-Clique

find the largest set of vertices all connected to each other
Max-Clique

find the largest set of vertices all connected to each other
Max-Clique

*find the largest set of vertices all connected to each other*

NP Hard - Karp’s original list
Max-Clique

find the largest set of vertices all connected to each other

NP Hard - Karp’s original list

NP Hard to approximate within $n^{1-\epsilon}$.

[Håstad’09, Zuckerman’06]
Max-Clique

find the largest set of vertices all connected to each other

Monotone Circuits [Razborov’88]
Max-Clique
find the largest set of vertices all connected to each other

Monotone Circuits [Razborov’88]

Strong SDP Hierarchies [Tulsiani’09]
Max-Clique: Average Case

find the largest set of vertices all connected to each other in a random graph $G \sim G(n, \frac{1}{2})$
Max-Clique: Average Case

find the largest set of vertices all connected to each other

in a random graph \( G \sim G(n, \frac{1}{2}) \)

each edge in \( G \) with prob. 0.5

w.h.p. max clique \( \sim 2 \log_2(n) \)
Max-Clique: Average Case

find the largest set of vertices all connected to each other

in a random graph \( G \sim G(n, \frac{1}{2}) \)

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w.h.p. max clique \( \sim 2 \log_2(n) \)

greedy finds cliques of size \( \sim \log_2(n) \)
Max-Clique: Average Case

find the largest set of vertices all connected to each other

in a random graph $G \sim G(n, \frac{1}{2})$

each edge in $G$ with prob. $0.5$

w.h.p. max clique $\sim 2 \log_2(n)$

greedy finds cliques of size $\sim \log_2(n)$

Challenge (Karp’76): efficiently find clique of size $\sim (1 + \epsilon) \log_2(n)$. 

find an $\omega$-clique added to a random graph $G \sim G(n, \frac{1}{2})$
Planted Clique

[Jerrum’92, Kucera’95]

find an \( \omega \)-clique added to a random graph \( G \sim G(n, \frac{1}{2}) \)

**spectral algorithm:** \( \omega > cn^{1/2} \)  

[AKS’98]
Planted Clique

[Jerrum’92, Kucera’95]

find an $\omega$-clique added to a random graph $G \sim G(n, \frac{1}{2})$

spectral algorithm: $\omega > cn^{1/2}$ [AKS’98]

no polynomial time algorithm when $\omega \ll \sqrt{n}$
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find an \( \omega \)-clique added to a random graph \( G \sim G(n, \frac{1}{2}) \)

central problem in average case complexity

Cryptography [Jue’02, ABW’10]

Motifs in Biological Networks [Milo et. al. Science’02, Lotem et. al. PNAS’14]

Certifying Restricted Isometry Property [KL’12]

Sparse PCA [BR’13]

Nash Equilibrium [HK’09]

Mathematical Finance [ABBG’09]
Planted Clique

[Jerrum’92, Kucera’95]

find an $\omega$-clique added to a random graph $G \sim G(n, \frac{1}{2})$

central problem in average case complexity

a testbed for algorithmic techniques for detecting "signal" (added clique) surrounded by "random noise" (random graph)
Sum of Squares SDP

[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

hierarchical strengthening of the basic SDP/spectral algo.
Sum of Squares SDP

[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

hierarchical strengthening of the basic SDP/spectral algo.

$d$: “degree” in the hierarchy.

$n^{O(d)}$: running time
Sum of Squares SDP

[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

highly successful convex relaxation for comb. optimization.

SPARSEST CUT [ARV04]

UNIQUE GAMES [ABS10, BRS12, GS12]
Sum of Squares SDP
[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

highly successful convex relaxation for comb. optimization.

SPARSEST CUT [ARV04]

UNIQUE GAMES [ABS10, BRS12, GS12]

….breaks known integrality gap instances for weaker hierarchies in polynomial time for

UNIQUE GAMES, BALANCED SEPARATOR, MAX CUT
[BBHKSZ12, OZ13, DMN13]
Sum of Squares SDP
[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

highly successful convex relaxation for comb. optimization.

SPARSEST CUT [ARV04]

UNIQUE GAMES [ABS10, BRS12, GS12]

...optimal amongst all poly size SDPs for a large class. [LRS15]
Sum of Squares SDP
[Grigoriev-Vorobjov99, Shor87, Nesterov99, Parillo00, Lasserre01]

highly successful convex relaxation for comb. optimization.

SPARSEST CUT [ARV04]

UNIQUE GAMES [ABS10, BRS12, GS12]

...successful for other average case problems. [BKS’14,15]

PLANTED SPARSE VECTOR, DICTIONARY LEARNING
SoS vs Planted Clique?
SoS vs Planted Clique?
posed by Meka and Wigderson (2013).
SoS vs Planted Clique?

“Degree $d$ SoS cannot detect planted cliques of size $\omega$”

$d = \Theta(1)$, $\tilde{O}$ ignores $n^{o(1)}$ factors.

<table>
<thead>
<tr>
<th>[Meka-Potechin-Wigderson’15]</th>
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Can deg 4 SoS detect \( n^{0.49} \) size planted cliques?
SoS vs Planted Clique?

\[ d = \Theta(1), \quad \tilde{O} \text{ ignores } n^{o(1)} \text{ factors.} \]

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SoS vs Planted Clique?

Poly time SoS can’t detect $\tilde{\omega}(\sqrt{n})$ size planted cliques.

[Barak-Hopkins-Kelner-K-Moitra-Potechin’16]
SoS vs Planted Clique?

Poly time SoS can’t detect \(\tilde{o}(\sqrt{n})\) size planted cliques.

almost optimal!

[Barak-Hopkins-Kelner-K-Moitra-Potechin’16]
SoS vs Planted Clique?

Previous SoS LBs

Certificates natural, same as ones fooling weaker algos.
SoS vs Planted Clique?

Previous SoS LBs

Natural certificate for planted clique ([MW13, FK02]) yields suboptimal LB.

[Kelner’15]

Deg 4 MW witness NOT valid for $\omega \gg n^{\frac{1}{3}}$.
SoS vs Planted Clique?

Previous SoS LBs

Natural certificate for planted clique ([MW13, FK02]) yields suboptimal LB.

“Fix” the MW certificate for $d = 4$.

Proof via fine grained spectral analysis of random matrices.
SoS vs Planted Clique?

certificates natural, same as ones fooling weaker algos. Natural certificate for planted clique ([MW13, FK02]) yields suboptimal LB.

“Fix” the MW certificate for $d = 4$. Proof via fine grained spectral analysis of random matrices.

Moment matching method and canonical “Bayesian” certificate
SoS vs Planted Clique?

Previous SoS LBs

Natural certificate for planted clique ([MW13, FK02]) yields suboptimal LB.

“Fix” the MW certificate for $d = 4$.

Proof via fine grained spectral analysis of random matrices.

Moment matching method and canonical “Bayesian” certificate generalizes to other problems!
Talk Plan

1. Sum of Squares Proofs and Pseudodistributions
2. Meka-Wigderson Pseudodistribution
3. MW witness is not PSD
4. The Bayesian Pseudodistribution
Talk Plan

1. Sum of Squares Proofs and Pseudodistributions

2. Meka-Wigderson Pseudodistribution

3. MW witness is not PSD

4. The Bayesian Pseudodistribution
Is there an $\omega$ clique in $G$?
Planted Clique

[Jerrum’92, Kucera’95]

Is there an ω clique in G?

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \not\sim j \text{ in } G \]

\[ \sum_i x_i = \omega \]

\[ x_i^2 = x_i \forall i \]
Sum of Squares Proofs

Is the set of polynomials feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

wants to convince Brian that there’s a big clique in G

wants to refute Stewie’s evidence
Sum of Squares Proofs

Is the set of polynomials feasible?

\[ x_i x_j = 0 \ \forall \ i \not\sim j \ \text{in} \ G \]
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wants to convince Brian that there’s a big clique in \( G \)

wants to refute Stewie’s evidence

what kind of evidence?
Sum of Squares Proofs

Is the set of poly-equations feasible?

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all the vertices in the clique

wants to refute Stewie’s evidence
Sum of Squares Proofs

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not= j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]

Sum \[ \sum_i x_i = \omega \]

all the vertices in the clique

wants to convince Brian that there’s a big clique in \( G \)

must fully solve the problem!

wants to refute Stewie’s evidence
Sum of Squares Proofs

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wants to convince Brian that there’s a big clique in G

no evidence!

wants to refute Stewie’s evidence
Sum of Squares Proofs

Is the set of polynomial equations feasible?

\[
\begin{align*}
    x_i x_j &= 0 \quad \forall \, i \neq j \text{ in } G \\
    x_i^2 &= x_i \quad \forall \, i
\end{align*}
\]

\[\sum_i x_i = \omega\]

no evidence!

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Sum of Squares Proofs

Is the set of poly-equations feasible?

\[ \sum x_i = \omega \]

\[ x_i x_j = 0 \forall i \not\sim j \text{ in } G \]

\[ x_i^2 = x_i \forall i \]

Moments* of \( \mu \).

wants to convince Brian that there’s a big clique in \( G \)

wants to refute Stewie’s evidence

Claims he has a prob. distribution \( \mu \) over solutions (cliques)

\[ \mathbb{E}[\prod_{i \in S} x_i] \text{ for } |S| \leq d \]
Sum of Squares Proofs

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not= j \text{ in } G \]
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Moments* of \( \mu \).

Can lie!

of degree at most \( d \)!

wants to convince Brian that there’s a big clique in \( G \)

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Claims he has a prob. distribution \( \mu \) over solutions (cliques)

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Sum of Squares Proofs

Is the set of polynomials feasible?

\[ x_i x_j = 0 \quad \forall \; i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \; i \]

\[ \sum_i x_i = \omega \]

can lie!

Moments* of \( \tilde{\mu} \).

wants to convince Brian that there's a big clique in \( G \)

wants to refute Stewie's evidence

Claims he has a prob. distribution \( \tilde{\mu} \) over solutions (cliques)

of degree at most \( d \)!

“pseudo-moments”

\[ \mathbb{E}[\prod_{i \in S} x_i] \text{ for } |S| \leq d \]
Catching Lies

Is the set of poly-equations feasible? 

can lie!

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]

\[ x_i^2 = x_i \quad \forall \ i \]

\[ \sum_i x_i = \omega \]

Moments* of \( \tilde{\mu} \).

of degree at most \( d \)!

“pseudo-moments”

1. Pseudo-moments must obey the constraints.

\[ \tilde{E}[qx_i x_j] = 0 \quad \tilde{E}[q(x_i^2 - x_i)] = 0 \quad \tilde{E}[q(\sum_i x_i - \omega)] = 0 \]
Catching Lies

Is the set of poly-equations feasible?

can lie!

Moments* of \( \tilde{\mu} \).

of degree at most \( d \)!

“pseudo-moments”

1. Pseudo-moments must obey the constraints.

\[ \tilde{\mathbb{E}}[qp] = 0 \quad \text{for constraint} \quad \{ p = 0 \} \quad \text{and} \quad q: \deg(qp) \leq d \]
Catching Lies

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \neq j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_{i} x_i = \omega \]

can lie!

Moments* of \( \tilde{\mu} \).

of degree at most d!

“pseudo-moments”

1. Pseudo-moments must obey the constraints.

2. Pseudo-expectation of squares must be non-negative.

\[ \text{deg}(q) \text{ at most } d/2 \quad \tilde{E}[q^2] \geq 0 \]
Catching Lies

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \ \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \ \forall \ i \]
\[ \sum_i x_i = \omega \]

\text{can lie!}

Moments* of \( \tilde{\mu} \).

of degree at most d!

“pseudo-moments”

1. Pseudo-moments must obey the constraints.

2. Pseudo-expectation of squares must be non-negative.

3. Pseudo-moments must be normalized. \( \tilde{E}[1] = 1 \)
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \ in \ G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

Moments* of \( \tilde{\mu} \).

of degree at most \( d \)!

“pseudo-moments”

must invent pseudomoments that fool the three tests.

Winning Strategies
Winning Strategies

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \forall i \neq j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

Moments* of \( \tilde{\mu} \).

of degree at most \( d \)!

“pseudo-moments”

must invent pseudomoments that fool the three tests.

all constraints captured by a semi-definite program.
Is the set of poly-equations feasible?

\[
x_i x_j = 0 \quad \forall \ i \neq j \ \text{in} \ G
\]
\[
x_i^2 = x_i \quad \forall \ i
\]
\[
\sum_i x_i = \omega
\]

Moments* of \( \tilde{\mu} \).

of degree at most \( d \)!

“pseudo-moments”

must invent pseudomoments that fool the three tests.

all constraints captured by a semi-definite program.

SoS Algorithm: Optimize over all valid pseudomoments.
Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \quad \text{in} \ G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

Winning Strategies

Show deg d polynomials \( f \) and \( g \)

1. \( f = 0 \) on solutions.
2. \( g \) is sum of squares.

and \( -1 = f + g \)

Moments* of \( \tilde{\mu} \) of degree at most d!

“pseudo-moments”

wins if can come up with a “SoS” proof of infeasibility.
Winning Strategies

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \neq j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

Show deg d polynomials f and g
1. \( f = 0 \) on solutions.
2. \( g \) is sum of squares.

and \(-1 = f + g\)

Moments* of \( \bar{\mu} \).

of degree at most d!

“pseudo-moments”

wins if can come up with a “SoS” proof of infeasibility.

for any operator: \( \tilde{E}[f + g] < 0 \)
Is the set of polynomials feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \quad \text{in} \ G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

Moments* of \( \tilde{\mu} \).

of degree at most \( d \)!

“pseudo-moments”

For \( d, \omega \), construct a pseudo-expectation satisfying all the constraints,

w.h.p. for \( G \sim G(n, \frac{1}{2}) \).
Proving Lower Bounds

Is the set of poly-equations feasible?

$$x_i x_j = 0 \quad \forall \ i \not= j \text{ in } G$$
$$x_i^2 = x_i \quad \forall \ i$$
$$\sum_i x_i = \omega$$

Moments* of $\tilde{\mu}$.

of degree at most $d$!

“pseudo-moments”

For $d$, $\omega$, construct a pseudo-expectation satisfying all the constraints,

w.h.p. for $G \sim G(n, \frac{1}{2})$.

SoS “thinks” there’s a large clique in a random graph!
1. Sum of Squares Proofs and Pseudodistributions

2. Meka-Wigderson Pseudodistribution

3. MW witness is not PSD

4. The Bayesian Pseudodistribution
Meka-Wigderson Certificate

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \forall i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_S] \] (pseudo) prob. of \( S \subseteq C \)
Meka-Wigderson Certificate

Is the set of polynomials feasible?

\[ x_i x_j = 0 \ \forall \ i \not\sim j \ \text{in } G \]
\[ x_i^2 = x_i \ \forall \ i \]

\[ \tilde{E}[\chi_S] \] (pseudo) prob. of \( S \subseteq C \)

“fake” planted clique
Meka-Wigderson Certificate

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \ \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \ \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_S] \]
(pseudo) prob. of \( S \subseteq C \)

Every vertex is equally likely to be in \( C \).

\[ \tilde{E}[x_i] \approx \left( \frac{\omega}{n} \right) \]

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \quad \text{in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_S] \]  
(pseudo) prob. of \( S \subseteq C \)

Every vertex is equally likely to be in \( C \).

\[ \tilde{E}[x_i] \approx \left( \frac{\omega}{n} \right) \]

Every edge is equally likely to be in \( C \).

\[ \tilde{E}[x_i x_j] \approx \begin{cases} 
(\frac{\omega}{n})^2 & \text{if } i \sim j \\
0 & \text{otherwise.} 
\end{cases} \]

⚠ sloppy calculations only.
Meka-Wigderson Certificate

Is the set of poly-equations feasible?

\[
\begin{align*}
& x_i x_j = 0 \forall i \not\sim j \text{ in } G \\
& x_i^2 = x_i \forall i \\
& \sum_i x_i = \omega
\end{align*}
\]

\[\tilde{\mathbb{E}}[x_S] \] (pseudo) prob. of \( S \subseteq C \)

\[\tilde{\mathbb{E}}[x_S] \approx \begin{cases} 
\left(\frac{\omega}{n}\right)^{|S|} & \text{if } S \text{ is a clique} \\
0 & \text{otherwise.}
\end{cases}\]

⚠️ sloppy calculations only.
Meka-Wigderson Certificate

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_S] \]  
(pseudo) prob. of \( S \subseteq C \)

\[ \tilde{E}[x_S] \approx \begin{cases} 
\left(\frac{\omega}{n}\right)^{|S|} & \text{if } S \text{ is a clique} \\
0 & \text{otherwise.} 
\end{cases} \]

what's wrong with \( \tilde{E} \)?

⚠️ sloppy calculations only.
Talk Plan

1. Sum of Squares Proofs and Pseudodistributions

2. Meka-Wigderson Pseudodistribution

3. MW witness is not PSD

4. The Bayesian Pseudodistribution
Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \nmid j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E} \]：“best guess” about the clique by the algorithm.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]

\[ \sum_i x_i = \omega \]

\( \hat{E} \) : “best guess” about the clique by the algorithm, using some information about the graph.
Going Bayesian

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \; i \not= j \quad \text{in} \; G \]
\[ x_i^2 = x_i \quad \forall \; i \]
\[ \sum_i x_i = \omega \]

\[ \hat{E} \] : “best guess” about the clique by the algorithm, using some information about the graph.

what kind of information can SoS base its guess on?
Is the set of poly-equations feasible?

\( x_ix_j = 0 \ \forall \ i \neq j \) in \( G \)

\( x_i^2 = x_i \ \forall \ i \)

\[ \sum_i x_i = \omega \]

\((H, C) \sim G(n, \frac{1}{2}, \omega)\): random graph + planted \( \omega \)-clique.

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \not\sim j \text{ in } G \]
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\((H, C) \sim G(n, \frac{1}{2}, \omega)\) : random graph + planted \(\omega\)-clique.

“prior” prob of \(i\) in \(C\) \(\approx \frac{\omega}{n}\)

⚠️ sloppy calculations only.
Is the set of polynomial equations feasible?

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\]
\[
x_i^2 = x_i \quad \forall \ i
\]
\[
\sum_i x_i = \omega
\]

\((H, C) \sim G(n, \frac{1}{2}, \omega)\): random graph + planted \(\omega\)-clique.

"prior" prob of \(i\) in \(C\) \(\approx \frac{\omega}{n}\)

Suppose you now see \(\deg(i) = \frac{n}{2} + \Delta, \Delta \pm \Theta(\sqrt{n})\)

\(\text{sloppy calculations only.}\)
Is the set of poly-equations feasible?

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\((H, C) \sim G(n, \frac{1}{2}, \omega)\): random graph + planted \(\omega\)-clique.

“prior” prob of \(i\) in \(C\) \(\approx \frac{\omega}{n}\)

Suppose you now see \(\text{deg}(i) = \frac{n}{2} + \Delta, \Delta \pm \Theta(\sqrt{n})\)

best guess about the clique based on \(\text{deg}(i)\)?

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

$$x_i x_j = 0 \forall i \not= j \text{ in } G$$
$$x_i^2 = x_i \forall i$$
$$\sum_i x_i = \omega$$

$$(H, C) \sim G(n, \frac{1}{2}, \omega)$$: random graph + planted $\omega$-clique.

“prior” prob of $i$ in $C \approx \frac{\omega}{n}$

Suppose you now see $\deg(i) = \frac{n}{2} + \Delta, \Delta \pm \Theta(\sqrt{n})$

Given $\deg(i)$, posterior “probability” of $i$ in $C$?

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
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\[(H, C) \sim G(n, \frac{1}{2}, \omega) : \text{random graph + planted } \omega\text{-clique.}\]

“prior” prob of \(i\) in \(C\) \(\approx \frac{\omega}{n}\)

Suppose you now see \(\text{deg}(i) = \frac{n}{2} + \Delta, \Delta \pm \Theta(\sqrt{n})\)

Given \(\text{deg}(i)\), posterior “probability” of \(i\) in \(C\)?

\textbf{Bayes’ Rule!}

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

$x_i x_j = 0 \forall i \not\sim j$ in $G$
$x_i^2 = x_i \forall i$

If $i \notin C$, 
$\deg(i) \sim \mathcal{N}(\frac{n}{2}, \frac{n}{4})$

$\sum_i x_i = \omega$

⚠️ sloppy calculations only.
Going Bayesian

Is the set of polynomials feasible?

\begin{align*}
  x_i x_j &= 0 \forall i \not\approx j \text{ in } G \\
  x_i^2 &= x_i \forall i \\
  \sum_i x_i &= \omega
\end{align*}

\text{If } i \notin C, \quad \deg(i) \sim \mathcal{N}\left(\frac{n}{2}, \frac{n}{4}\right)

\text{If } i \in C, \quad \deg(i) \sim \mathcal{N}\left(\frac{n}{2} + \omega, \frac{n}{4}\right)

⚠️ sloppy calculations only.
Going Bayesian

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_{i} x_i = \omega \]

\[ \text{If } i \notin C, \quad \deg(i) \sim N \left( \frac{n}{2}, \frac{n}{4} \right) \]
\[ \text{If } i \in C, \quad \deg(i) \sim N \left( \frac{n}{2} + \omega, \frac{n}{4} \right) \]

Given \( \deg(i) \), posterior “probability” of \( i \) is in \( C \)?

Bayes’ Rule!

⚠️ sloppy calculations only.
Is the set of polynomial equations feasible?

\[ x_ix_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

If \( i \not\in C \), \( \text{deg}(i) \sim \mathcal{N}(\frac{n}{2}, \frac{n}{4}) \)

If \( i \in C \), \( \text{deg}(i) \sim \mathcal{N}(\frac{n}{2} + \omega, \frac{n}{4}) \)

Given \( \text{deg}(i) \), posterior “probability” of \( i \) is in \( C \)?

\[ \text{deg}(i) = \frac{n}{2} + \Delta \]
\[ \Delta \pm \Theta(\sqrt{n}) \]

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \, i \neq j \quad \text{in } G \]
\[ x_i^2 = x_i \quad \forall \, i \]
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If \( i \notin C \), \( \text{deg}(i) \sim \mathcal{N}\left(\frac{n}{2}, \frac{n}{4}\right) \)

If \( i \in C \), \( \text{deg}(i) \sim \mathcal{N}\left(\frac{n}{2} + \omega, \frac{n}{4}\right) \)

Given \( \text{deg}(i) \), posterior “probability” of \( i \) is in \( C \)?

\[ \text{deg}(i) = \frac{n}{2} + \Delta \]
\[ \Delta \pm \Theta(\sqrt{n}) \]

Multiplicative update by

\[
\frac{\exp\left(\frac{2(\Delta-\omega)^2}{n}\right)}{\exp\left(\frac{2\Delta^2}{n}\right)} \approx \exp\left(\frac{2\Delta \omega}{n}\right) \\
\approx \left(1 + \frac{2\Delta \omega}{n}\right)
\]

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \text{deg}(i) = \frac{n}{2} + \Delta \]

Bayesian dictates:
\[ \tilde{E}[x_i] \approx \frac{\omega}{n} \left(1 + \frac{\Delta \omega}{n}\right) \]

⚠️ sloppy calculations only.
Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]

\[ \deg(i) = \frac{n}{2} + \Delta \]

Bayesian dictates:

\[ \tilde{E}[x_i] \approx \frac{\omega}{n} \left( 1 + \frac{\Delta \omega}{n} \right) \]

\[ \sum x_i = \omega \]

large when \( \omega \), approaches \( \sqrt{n} \).

\[ \tilde{E}[x_i] \approx \frac{\omega}{n} \left( 1 + \frac{\Delta \omega}{n} \right) \]

sloppy calculations only.
Going Bayesian

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \; \forall \; i \not= j \; \text{in} \; G \]
\[ x_i^2 = x_i \; \forall \; i \]
\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning based on \textit{deg}.
Bayesian Reasoning

Is the set of poly-equations feasible?

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\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning based on \textit{deg}.

Can catch it?
Bayesian Reasoning

Is the set of poly-equations feasible?

\[
x_ix_j = 0 \ \forall \ i \neq j \text{ in } G \\
x_i^2 = x_i \ \forall \ i \\
\sum_i x_i = \omega
\]

MW violates Bayesian reasoning based on deg. 

low-degree polynomial that notices the discrepancy?
Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \ \forall \ i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \ \forall \ i \]
\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning based on \textbf{deg}. 

\[
 r_i(j) = \begin{cases} 
 +1 & \text{if } i \sim j \\
 0 & \text{if } j = i \\
 -1 & \text{if } i \not\sim j 
\end{cases}
\]
Bayesian Reasoning

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
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\[ r_i(j) = \begin{cases} 
+1 & \text{if } i \sim j \\
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-1 & \text{if } i \not\sim j 
\end{cases} \]

\[ G(n, \frac{1}{2}, \omega) \] random graph + planted $\omega$-clique.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \quad \text{in} \ G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning based on \textit{deg}.

Our candidate polynomial

\[ r_i(j) = \begin{cases} 
+1 & \text{if } i \sim j \\
0 & \text{if } j = i \\
-1 & \text{if } i \not\sim j 
\end{cases} \]

\[ P(x) = \sum_{i \leq n} \left( \sum_{j \leq n} r_i(j) x_j \right)^4 \]

Claim: MW gives it a value too tiny.
Is the set of poly-equations feasible?

MW violates Bayesian reasoning based on \textbf{deg}.

Our candidate polynomial

\[ P(x) = \sum_{i \leq n} \left( \sum_{j \leq n} r_i(j)x_j \right)^4 \]

Suppose \((H, C) \sim G(n, \frac{1}{2}, \omega)\) value:

\[ \sum_{i \leq n} \langle r_i, 1(C) \rangle^4 \]
Bayesian Reasoning

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
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MW violates Bayesian reasoning.

Our candidate polynomial

\[ P(x) = \sum_{i \leq n} \left( \sum_{j \leq n} r_i(j) x_j \right)^4 \]

Suppose \((H, C) \sim G(n, \frac{1}{2}, \omega)\)

value: \[ \sum_{i \leq n} \langle r_i, 1(C) \rangle^4 \]

value: \[ \mathbb{E} \left[ \sum_{i \leq n} \langle r_i, x \rangle^4 \right] \]

\[ r_i(j) = \begin{cases} +1 & \text{if } i \sim j \\ 0 & \text{if } j = i \\ -1 & \text{if } i \not\sim j \end{cases} \]
Bayesian Reasoning

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \; i \sim j \quad \text{in} \quad G \]
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MW violates Bayesian reasoning.

Our candidate polynomial

\[ P(x) = \sum_{i \leq n} \left( \sum_{j \leq n} r_i(j) x_j \right)^4 \]

Suppose \( (H, C) \sim G(n, \frac{1}{2}, \omega) \)
\[ \sum_{i \leq n} \langle r_i, 1(C) \rangle^4 \approx \sum_{i \leq n} \frac{\omega}{n} \omega^4 \approx \omega^5 \]

⚠️ sloppy calculations only.
Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \forall i \sim j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
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Our candidate polynomial

\[ P(x) = \sum_{i \leq n} \left( \sum_{j \leq n} r_i(j)x_j \right)^4 \]

Suppose \( G \sim G(n, \frac{1}{2}) \)

\[ \sum_{i \leq n} \tilde{E}[\langle r_i, x \rangle^4] \]

⚠️ sloppy calculations only.
Bayesian Reasoning

Is the set of polynomials feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]
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\[ P(x) = \sum_{i \leq n} \left( \sum_{j \leq n} r_i(j)x_j \right)^4 \]

Suppose \( G \sim G(n, \frac{1}{2}) \)

\[ \sum_{i \leq n} \tilde{E}[\langle r_i, x \rangle^4] \approx \sum_{i \leq n} \tilde{E}[\|x\|^4] \approx n\omega^2 \]

⚠️ sloppy calculations only.
Bayesian Reasoning

Is the set of polynomial equations feasible?

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MW violates Bayesian reasoning.

Our candidate polynomial

\[ P(x) = \left( \sum_{j \leq n} r_i(j) x_j \right)^4 \]

Suppose \((H, C) \sim G(n, \frac{1}{2}, \omega)\)

\[ \sum_{i \leq n} \langle r_i, 1(C) \rangle^4 \approx \sum_{i \leq n} \frac{\omega}{n} \omega^4 \approx \omega^5 \]

Suppose \(G \sim G(n, \frac{1}{2})\)

\[ \sum_{i \leq n} \tilde{E}[\langle r_i, x \rangle^4] \approx \sum_{i \leq n} \tilde{E}[\|x\|^4] \approx n\omega^2 \]

\[ n\omega^2 \ll \omega^5 \quad \text{when} \quad \omega \gg n^{\frac{1}{3}} \]

⚠️ sloppy calculations only.
Is the set of polynomial equations feasible?

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MW violates Bayesian reasoning.

Our candidate polynomial

\[ P(x) = \left( \sum_{j \leq n} r_i(j) x_j \right)^4 \]

Suppose \( (H, C) \sim G(n, \frac{1}{2}, \omega) \)
\[ \sum_{i \leq n} \langle r_i, 1(C) \rangle^4 \approx \sum_{i \leq n} \frac{\omega}{n} \omega^4 \approx \omega^5 \]

Suppose \( G \sim G(n, \frac{1}{2}) \)
\[ \sum_{i \leq n} \tilde{E}[\langle r_i, x \rangle^4] \approx \sum_{i \leq n} \tilde{E}[\|x\|^4] \approx n\omega^2 \]

\[ n\omega^2 \ll \omega^5 \quad \text{when} \quad \omega \gg n^{\frac{1}{3}} \]

⚠️ sloppy calculations only.
Bayesian Reasoning

Is the set of poly-equations feasible?

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\begin{align*}
  x_i x_j &= 0 \quad \forall \ i \not\sim j \text{ in } G \\
  x_i^2 &= x_i \quad \forall \ i \\
  \sum_i x_i &= \omega
\end{align*}
\]

MW violates Bayesian reasoning.

\[
 r_i(j) = \begin{cases} 
  +1 & \text{if } i \sim j \\
  0 & \text{if } j = i \\
  -1 & \text{if } i \not\sim j 
\end{cases}
\]

[Kelner]

for \( \omega \gg n^{1/3} \) and some constant \( c \),

w.h.p.

\[
\mathbb{E}\left[ (\langle r_i, x \rangle^2 - cx_i)^2 \right] < 0
\]
Bayesian Reasoning

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
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MW violates Bayesian reasoning.

\[ r_i(j) = \begin{cases} 
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\end{cases} \]

[Kelner]

for \( \omega \gg n^{1/3} \) and some constant \( c \),

w.h.p. \( \tilde{\mathbb{E}}[(\langle r_i, x \rangle^2 - cx_i)^2] < 0 \)

generalizes to higher \( d \).
Bayesian Reasoning

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]

\[ x_i^2 = x_i \quad \forall \ i \]

\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning.

\[ r_i(j) = \begin{cases} 
+1 & \text{if } i \sim j \\
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\end{cases} \]

[Kelner]

for \( \omega \gg n^{1/3} \) and some constant \( c \),

w.h.p. \[ \tilde{E}[(r_i, x)^2 - cx_i)^2] < 0 \]

MW Pseudomoments work for \( \omega \ll n^{\frac{1}{d+1}} \).
Fixing MW at degree 4

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_S] \approx \begin{cases} 
  \left( \frac{\omega}{n} \right)^{|S|} & \text{if } S \text{ is a clique} \\
  0 & \text{otherwise.} 
\end{cases} \]
Is the set of poly-equations feasible?

\[ x_ix_j = 0 \quad \forall \ i \neq j \quad \text{in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
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\[ \tilde{E}[x_S] \approx \begin{cases} 
\left( \frac{\omega}{n} \right)^{|S|} & \text{if } S \text{ is a clique} \\
0 & \text{otherwise.} 
\end{cases} \]

\[ \tilde{E}[x_S] \approx \left( \frac{\omega}{n} \right)^{|S|} \left( 1 + r_S \frac{\omega}{n} \right) \]

where \[ r_S = \sum_i \prod_{j \in S} r_i(j) \]
Fixing MW at degree 4

Is the set of polynomials feasible?

\[ \begin{align*}
    x_i x_j &= 0 \quad \forall \ i \neq j \text{ in } G \\
    x_i^2 &= x_i \quad \forall \ i \\
    \sum_i x_i &= \omega
\end{align*} \]

\[ \mathbb{E}[x_S] \approx \left( \frac{\omega}{n} \right)^{|S|} \left( 1 + r_S \frac{\omega}{n} \right) \]

where \( r_S = \sum_i \Pi_{j \in S} r_i(j) \)

[HKP’15] ([RS’15])

P is the only “bad” polynomial for MW.

fixing P doesn’t cause new problems.
Is the set of poly-equations feasible?

\[ x_ix_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_S] \approx \left( \frac{\omega}{n} \right)^{|S|} \left( 1 + r_S \frac{\omega}{n} \right) \]

where \( r_S = \sum_i \Pi_{j \in S} r_i(j) \)

[HKP’15] ([RS’15])

P is the only “bad” polynomial for MW.
fixing P doesn’t cause new problems.

approximate eigen-split of MW Mom. matrix
spectral norm upper bounds for random matrices
Fixing MW at degree 4

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \forall i \not= j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

\[ \bar{F}[x_S] \approx \left( \frac{\omega}{n} \right)^{|S|} \left( 1 + r_S \frac{\omega}{n} \right) \]

where

\[ r_S = \sum_i \prod_{j \in S} r_i(j) \]

[HKP’15] (RS’15)

P is the only “bad” polynomial for MW. Fixing P doesn’t cause new problems.

doesn’t even generalize to \( d = 6 \).
Talk Plan

1. Sum of Squares Proofs and Pseudodistributions

2. Meka-Wigderson Pseudodistribution

3. MW witness is not PSD

4. The Bayesian Pseudodistribution
Bayesian Reasoning

Is the set of poly-equations feasible?

\[
\begin{align*}
x_i x_j &= 0 \forall i \neq j \text{ in } G \\
x_i^2 &= x_i \forall i \\
\sum_i x_i &= \omega
\end{align*}
\]

SoS seems to reason about \( \text{deg}(i) \).

What other statistics can SoS reason about?
The Bayesian Pseudomoments

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

build an operator that satisfies all “simple” Bayesian constraints?
The Bayesian Pseudomoments

Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

build an operator that satisfies all “simple” Bayesian constraints?

No “simple” low-deg polynomial notices the discrepancy.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \not\approx j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

build an operator that satisfies all "simple" Bayesian constraints?

No "simple" low-deg polynomial notices the discrepancy.

at least in expectation?
The Bayesian Pseudomoments

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \not= j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

build an operator that satisfies all “simple” Bayesian constraints?

No “simple” low-deg polynomial notices the discrepancy.

at least in expectation?

For every “simple” map \( G \rightarrow p_G(x) \)

\[ \mathbb{E}_{G \sim G(n, \frac{1}{2})} \tilde{\mathbb{E}}[p_G(x)] \approx \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)} [p_G(1(C))] \]
Is the set of polynomial equations feasible?

\[ x_i x_j = 0 \forall i \neq j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

build an operator that satisfies all "simple" Bayesian constraints?

No "simple" low-deg polynomial notices the discrepancy.

at least in expectation?

No choice in the operator!

For every "simple" map \( G \rightarrow p_G(x) \)

\[ \mathbb{E}_{G \sim G(n, \frac{1}{2})} \tilde{\mathbb{E}}[p_G(x)] \approx \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[p_G(1(C))] \]
The Bayesian Pseudomoments

For every "simple" map \( G \rightarrow p_G(x) \)

\[
\mathbb{E}_{G \sim G(n, \frac{1}{2})} \left[ \tilde{\mathbb{E}}[p_G(x)] \right] = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[p_G(1(C))]
\]

\( \tilde{\mathbb{E}}[x_S] : \{0, 1\}^{n \choose 2} \rightarrow \mathbb{R} \quad \sum_{T \subseteq {n \choose 2}} \tilde{\mathbb{E}}[x_S](T) \cdot \chi_T \)
The Bayesian Pseudomoments

For every "simple" map $G \rightarrow p_G(x)$

$$
\mathbb{E}_{G \sim G(n, \frac{1}{2})}[\tilde{\mathbb{E}}[p_G(x)]] = \mathbb{E}_{(H, C) \sim G(n, \frac{1}{2}, \omega)}[p_G(1(C))]$$

$$
\tilde{\mathbb{E}}[x_S] : \{0, 1\}^{(n \choose 2)} \rightarrow \mathbb{R} \quad \sum_{T \subseteq \{0, 1\}^{(n \choose 2)}} \tilde{\mathbb{E}}[x_S](T) \cdot \chi_T
$$

$$
\tilde{\mathbb{E}}[x_S](T) = \mathbb{E}_{G \sim G(n, \frac{1}{2})}[\tilde{\mathbb{E}}[x_S] \chi_T(G)]
$$
The Bayesian Pseudomoments

For every $G \rightarrow p_{G}(x)$

$$\mathbb{E}_{G \sim G(n, \frac{1}{2})}[\tilde{\mathbb{E}}[p_{G}(x)]] = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[p_{G}(1(C))]$$

$$\tilde{\mathbb{E}}[x_{S}] : \{0, 1\}^{\binom{n}{2}} \rightarrow \mathbb{R}$$

$$\sum_{T \subseteq \binom{n}{2}} \tilde{\mathbb{E}}[x_{S}](T) \cdot \chi_{T}$$

$$\tilde{\mathbb{E}}[x_{S}](T) = \mathbb{E}_{G \sim G(n, \frac{1}{2})}[\tilde{\mathbb{E}}[x_{S}]]$$
The Bayesian Pseudomoments

\[ \tilde{E}[x_S] : \{0, 1\}^{(n/2)} \rightarrow \mathbb{R} \sum_{T \subseteq (n/2)} \tilde{E}[x_S](T) \cdot \chi_T \]

\[ \tilde{E}[x_S](T) = \mathbb{E}_{G \sim G(n, \frac{1}{2})} [\tilde{E}[x_S] \chi_T(G)] \]

\[ = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)} [1(S \subseteq C) \chi_T(H)] \]

match “moments” (Fourier coeffs) with planted dist.
The Bayesian Pseudomoments

\[ \mathbb{E}_{G \sim G(n, \frac{1}{2})}[\tilde{\mathbb{E}}[p_G(x)]] = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[p_G(1(C))] \]

\[ \tilde{\mathbb{E}}[x_S](T) = \mathbb{E}_{G \sim G(n, \frac{1}{2})}[\tilde{\mathbb{E}}[x_S]\chi_T(G)] \]

\[ = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[1(S \subseteq C)\chi_T(H)] \]

completely determined by the “planted” distribution!
The Bayesian Pseudomoments

Is the set of poly-equations feasible?

\[ x_ix_j = 0 \ \forall \ i \not\equiv j \ \text{in} \ G \]
\[ x_i^2 = x_i \ \forall \ i \]
\[ \sum_{i} x_i = \omega \]

build an operator that satisfies all Bayesian constraints of low-instance degree.

“instance degree” of \[ \sum_{S} p_S x_S = \deg \text{ of } p_S \] edge indicators
The Bayesian Pseudomoments

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

build an operator that satisfies all Bayesian constraints of low-instance degree.

For every “simple” map \( G \rightarrow p_G(x) \)

\[ \mathbb{E}_{G \sim G(n, \frac{1}{2})} [\tilde{\mathbb{E}}[p_G(x)]] = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)} [p_G(1(C))] \]
Is the set of polyequations feasible?

\[\begin{align*}
  x_ix_j &= 0 \quad \forall \ i \neq j \text{ in } G \\
  x_i^2 &= x_i \quad \forall \ i
\end{align*}\]

\[\sum_i x_i = \omega\]

build an operator that satisfies

of \textit{low-instance degree}

No choice in the operator if want to satisfy all Bayesian constraints!

For every \textit{“simple”} map \( G \rightarrow p_G(x) \)

\[E_{G \sim G(n, \frac{1}{2})}[\hat{E}[p_G(x)]] = E_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[p_G(1(C))]\]
Is the set of poly-equations feasible?

\[
x_ix_j = 0 \forall i \not\sim j \text{ in } G
\]
\[
x_i^2 = x_i \forall i
\]
\[
\sum_i x_i = \omega
\]

build an operator that satisfies all Bayesian constraints of low-instance degree.

For every “simple” map \( G \rightarrow p_G(x) \)

\[
\mathbb{E}_{G \sim G(n, \frac{1}{2})} \left[ \tilde{\mathbb{E}}[p_G(x)] \right] = \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)}[p_G(1(C))]
\]
The Bayesian Pseudomoments

\[ \tilde{E}[x_s] : \{0, 1\}^{\binom{n}{2}} \to \mathbb{R} \sum_{T \subseteq \binom{[n]}{2} : |V(T)| \leq \tau} \tilde{E}[x_s](T)\chi_T \]

use a truncated Fourier polynomial as the pseudomoment.
Is the set of poly-equations feasible?

\[
x_i x_j = 0 \quad \forall \ i \neq j \quad \text{in } G
\]
\[
x_i^2 = x_i \quad \forall \ i
\]
\[
\sum_i x_i = \omega
\]

build an operator that satisfies \textit{all} Bayesian constraints of \textit{low-instance degree}.

[Barak-Hopkins-Kelner-K-Moitra-Potechin’16]

For every \( d, \varepsilon \), Bayesian \( \tilde{E} \) passes all the tests whenever \( \omega \ll n^{0.5-\varepsilon} \).
The Bayesian Pseudomoments

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \not= j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
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[Barak-Hopkins-Kelner-K-Moitra-Potechin’16]

For every \( d, \epsilon \), Bayesian \( \tilde{E} \) passes all the tests whenever \( \omega \ll n^{0.5 - \epsilon} \).

More details in the next talk!
The Bayesian Pseudomoments

A general procedure to construct SoS LB witnesses.
  
  unify SoS LBs, prove LBs for planted problems, 2->4 norm...

SoS can simulate “computationally bounded” Bayesian reasoning.
  
  Applications to upper bounds?
Thank you for your attention.

Acknowledgment
Some illustrations based from Boaz and Aaron’s slides.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \forall i \neq j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

\[ \tilde{E}[x_i] \approx \left( \frac{\omega}{n} \right) \quad \text{Vertex } i : \quad \text{deg}(i) = \frac{n}{2} + \Delta, \Delta \pm \Theta(\sqrt{n}) \]

If \( i \notin C \), \( \text{deg}(i) \sim \mathcal{N} \left( \frac{n}{2}, \frac{n}{4} \right) \)

If \( i \in C \), \( \text{deg}(i) \sim \mathcal{N} \left( \frac{n}{2} + \omega, \frac{n}{4} \right) \)

Given \( \text{deg}(i) \), posterior “probability” of \( i \) is in \( C \)?

\[
\text{Pr}[i \in C] \cdot \frac{\text{Pr}[\text{deg}(i) = q \mid i \in C]}{\text{Pr}[i \in C] \cdot \text{Pr}[\text{deg}(i) = q \mid i \in C] + \text{Pr}[i \notin C] \cdot \text{Pr}[\text{deg}(i) = q \mid i \notin C]}
\]
Is the set of polyequations feasible?

\[
x_{ij} = 0 \quad \forall \ i \not\sim j \text{ in } G
\]
\[
x_i^2 = x_i \quad \forall \ i
\]
\[
\sum_i x_i = \omega
\]

MW violates Bayesian reasoning.

\[
r_i(j) = \begin{cases} 
+1 & \text{if } i \sim j \\
0 & \text{if } j = i \\
-1 & \text{if } i \not\sim j 
\end{cases}
\]

\[
\mathbb{E}_{G \sim G(n, \frac{1}{2})} \mathbb{E}[\langle r_i, x \rangle^4] \approx \|x\|^4
\]
\[
\approx \omega^2
\]

\[r_i\] is treated as a random vector.

⚠️ sloppy calculations only.
Is the set of polynomials feasible?

\[ x_i x_j = 0 \forall i \not\sim j \text{ in } G \]
\[ x_i^2 = x_i \forall i \]
\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning.

\[ r_i(j) = \begin{cases} 
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0 & \text{if } j = i \\
-1 & \text{if } i \not\sim j 
\end{cases} \]

\[
\mathbb{E}_{G \sim G(n, \frac{1}{2})} \mathbb{E} [ \langle r_i, x \rangle^4 ] \approx \| x \|^4 \approx \omega^2 \\
\mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)} \langle r_i, 1(C) \rangle^4 \approx \frac{\omega}{n} \omega^4 \approx \frac{\omega^5}{n} 
\]

⚠️ sloppy calculations only.
Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \not \sim j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

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\[ r_i(j) = \begin{cases} 
+1 & \text{if } i \sim j \\
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-1 & \text{if } i \not \sim j 
\end{cases} \]

\[ \mathbb{E}_{G \sim G(n, \frac{1}{2})} \mathbb{E}[\langle r_i, x \rangle^4] \approx \| x \|^4 \]
\[ \approx \omega^2 \]

\[ \mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)} \langle r_i, 1(C) \rangle^4] \approx \frac{\omega}{n} \omega^4 \]
\[ \approx \frac{\omega^5}{n} \]

\( \text{sloppy calculations only.} \)
Bayesian Reasoning

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning.

\[ r_i(j) = \begin{cases} 
+1 & \text{if } i \sim j \\
0 & \text{if } j = i \\
-1 & \text{if } i \not\sim j 
\end{cases} \]

\[
\mathbb{E}_{G \sim G(n, \frac{1}{2})} \mathbb{E}[\langle r_i, x \rangle^4] \approx \|x\|^4 \\
\approx \omega^2 \\
\mathbb{E}_{(H,C) \sim G(n, \frac{1}{2}, \omega)} \langle r_i, 1(C) \rangle^4 \approx \frac{\omega}{n} \omega^4 \\
\approx \frac{\omega^5}{n}
\]

If \( \omega \gg n^{\frac{1}{3}} \) pseudo expectation is too small.

⚠️ sloppy calculations only.
Bayesian Reasoning

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \ i \neq j \text{ in } G \]
\[ x_i^2 = x_i \quad \forall \ i \]
\[ \sum_i x_i = \omega \]

MW violates Bayesian reasoning.

SoS captures “low-instance degree” Bayesian reasoning?
Bayesian Reasoning

Is the set of poly-equations feasible?

\[ x_i x_j = 0 \quad \forall \quad i \neq j \quad \text{in} \quad G \]
\[ x_i^2 = x_i \quad \forall \quad i \]
\[ \sum_{i} x_i = \omega \]

MW violates Bayesian reasoning.

“instance degree” of \[ \sum_{S} p_S x_S = \deg \text{ of } p_S \quad \text{edge indicators} \]

SoS captures “low-instance degree” Bayesian reasoning?
Bayesian Reasoning

Is the set of poly-equations feasible?

$$x_i x_j = 0 \ orall \ i \neq j \text{ in } G$$

$$x_i^2 = x_i \ \forall \ i$$

$$\sum_i x_i = \omega$$

MW violates Bayesian reasoning.

“instance degree” of

$$\sum_S p_S x_S = \text{deg of } p_S \text{ edge indicators}$$

SoS captures “low-instance degree” Bayesian reasoning.

degree of vertices, number of triangles, etc.