Interpolation on Symmetric Spaces and Variational Discretization of Gauge Field Theories

Melvin Leok

Joint work with Evan Gawlik

Department of Mathematics
University of California, San Diego

IMS Workshop on State-of-the-Art Shape Research,
Singapore, July 18, 2016.

Supported by NSF DMS-0726263, DMS-100152, DMS-1010687 (CAREER), CMMI-1029445, DMS-1065972, CMMI-1334759, DMS-1411792, DMS-1345013.
Manifold-valued data and manifold-valued functions play an important role in a variety of applications:

- Mechanics
- Reduced-order modeling
- Numerical relativity

Source: http://www.ode.org/
A **gauge symmetry** is a continuous local transformation on the field variables that leaves the system physically indistinguishable.

A consequence of this is that the Euler–Lagrange equations are **underdetermined**, i.e., the evolution equations are insufficient to propagate all the fields.

The **kinematic fields** have no physical significance, but the **dynamic fields** and their conjugate momenta have physical significance.

The Euler–Lagrange equations are **overdetermined**, and the initial data on a Cauchy surface satisfies a constraint (usually elliptic).

These degenerate systems are naturally described using **multi-Dirac** mechanics and geometry.
Electromagnetism

- Let $\mathbf{E}$ and $\mathbf{B}$ be the electric and magnetic vector fields respectively.
- We can write Maxwell’s equations in terms of the scalar and vector potentials $\phi$ and $\mathbf{A}$ by,

\[
\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = 0,
\]
\[
\mathbf{B} = \nabla \times \mathbf{A}, \quad \Box \mathbf{A} + \nabla \left( \nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} \right) = 0.
\]

- The following transformation leaves the equations invariant,

\[
\phi \rightarrow \phi - \frac{\partial f}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A} + \nabla f.
\]

- The associated Cauchy initial data constraints are,

\[
\nabla \cdot \mathbf{B}^{(0)} = 0, \quad \nabla \cdot \mathbf{E}^{(0)} = 0.
\]
One often addresses the indeterminacy due to gauge freedom in a field theory through the choice of a **gauge condition**.

The **Lorenz gauge** is $\nabla \cdot A = -\frac{\partial \phi}{\partial t}$, which yields,

$$\Box \phi = 0, \quad \Box A = 0.$$  

The **Coulomb gauge** is $\nabla \cdot A = 0$, which yields,

$$\nabla^2 \phi = 0, \quad \Box A + \nabla \frac{\partial \phi}{\partial t} = 0.$$  

Given different initial and boundary conditions, some problems may be easier to solve in certain gauges than others. There is no systematic way of deciding which gauge to use for a given problem.
Noether’s Theorem

For every continuous symmetry of an action, there exists a quantity that is conserved in time.

- The simplest illustration of the principle comes from classical mechanics: a time-invariant action implies a conservation of the Hamiltonian, which is usually identified with energy.

- More precisely, if \( S = \int_{t_a}^{t_b} L(q, \dot{q}) dt \) is invariant under the transformation \( t \rightarrow t + \epsilon \), then

\[
\frac{d}{dt} \left( \dot{q} \frac{\partial L}{\partial \dot{q}} - L \right) = \frac{dH}{dt} = 0
\]
Noether’s Theorem for Gauge Field Theories

For every differentiable, local symmetry of an action, there exists a **Noether current** obeying a continuity equation. Integrating this current over a spacelike surface yields a conserved quantity called a **Noether charge**.

- The action principle for electromagnetism is

\[ S = \frac{1}{2} \int (B^2 - E^2) d^4x. \]

Applying Noether’s theorem to the gauge symmetry yields the following currents:

\[ j_0 = E \cdot \nabla f \]
\[ j = -E \frac{\partial f}{\partial t} + (B \times \nabla) f \]
Our long-term goal is to develop geometric structure-preserving numerical discretizations that systematically addresses the issue of gauge symmetries. Eventually, we wish to study discretizations of general relativity that address the issue of general covariance.

Towards this end, we will consider multi-Dirac mechanics based on a Hamilton–Pontryagin variational principle for field theories that is well adapted to degenerate field theories.

The issue of general covariance also leads us to avoid using a tensor product discretization that presupposes a slicing of spacetime, rather we will consider 4-simplicial complexes in spacetime.

More generally, we will need to study discretizations that are invariant to some discrete analogue of the gauge symmetry group.
Consider the **Pontryagin bundle** \( TQ \oplus T^*Q \), which has local coordinates \((q, v, p)\).

The **Hamilton–Pontryagin principle** is given by

\[
\delta \int [L(q, v) - p(v - \dot{q})] = 0,
\]

where we impose the second-order curve condition, \( v = \dot{q} \) using Lagrange multipliers \( p \).
Taking variations in $q$, $v$, and $p$ yield

$$\delta \int [L(q, v) - p(v - \dot{q})] dt$$

\[= \int \left[ \frac{\partial L}{\partial q} \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v - (v - \dot{q}) \delta p + p \delta \dot{q} \right] dt \]

\[= \int \left[ \left( \frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left( \frac{\partial L}{\partial v} - p \right) \delta v - (v - \dot{q}) \delta p \right] dt, \]

where we used integration by parts, and the fact that the variation $\delta q$ vanishes at the endpoints.

This recovers the **implicit Euler–Lagrange equations**,\n
$$\dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}, \quad v = \dot{q}.$$
Multisymplectic Geometry

- **Base space** $\mathcal{X}$. $(n + 1)$-spacetime.
- **Configuration bundle**. Given by $\pi : Y \to \mathcal{X}$, with the fields as the fiber.
- **Configuration** $q : \mathcal{X} \to Y$. Gives the field variables over each spacetime point.
- **First jet** $J^1 Y$. The first partials of the fields with respect to spacetime.
- **Lagrangian density** $L : J^1 Y \to \Omega^{n+1}(\mathcal{X})$.
- **Action integral** given by, $S(q) = \int_{\mathcal{X}} L(j^1 q)$.
- **Hamilton’s principle** states, $\delta S = 0$. 
In coordinates, the Hamilton–Pontryagin principle for fields is
\[
S(y^A, y^A_\mu, p^\mu_A) = \int_U \left[ p^\mu_A \left( \frac{\partial y^A}{\partial x^\mu} - v^A_\mu \right) + L(x^\mu, y^A, v^A_\mu) \right] d^{n+1}x.
\]

By taking variations with respect to \(y^A\), \(v^A_\mu\) and \(p^\mu_A\) (where \(\delta y^A\) vanishes on \(\partial U\)) we obtain the implicit Euler–Lagrange equations,
\[
\frac{\partial p^\mu_A}{\partial x^\mu} = \frac{\partial L}{\partial y^A}, \quad p^\mu_A = \frac{\partial L}{\partial v^A_\mu}, \quad \text{and} \quad \frac{\partial y^A}{\partial x^\mu} = v^A_\mu.
\]

The **covariant Legendre transform** involves both the energy and momentum,
\[
p^\mu_A = \frac{\partial L}{\partial v^A_\mu}, \quad p = L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu.
\]
**Discrete Lagrangian Variational Principle**

- **Discrete Lagrangian**

\[
L_d(q_0, q_1) \approx L_d^{\text{exact}}(q_0, q_1) \equiv \int_0^h L(q_{0,1}(t), \dot{q}_{0,1}(t)) \, dt,
\]

where \(q_{0,1}(t)\) satisfies the Euler–Lagrange equations for \(L\) and the boundary conditions \(q_{0,1}(0) = q_0, q_{0,1}(h) = q_1\).

- This is related to **Jacobi's solution** of the **Hamilton–Jacobi equation**.
Discrete Lagrangian Variational Principle

- **Discrete Hamilton’s principle**

\[ \delta S_d = \delta \sum L_d(q_k, q_{k+1}) = 0, \]

where \(q_0, q_N\) are fixed.

- **Discrete Euler-Lagrange equation**

\[ D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0. \]

The associated discrete flow \((q_{k-1}, q_k) \mapsto (q_k, q_{k+1})\) is automatically symplectic, since it is equivalent to,

\[ p_k = -D_1 L_d(q_k, q_{k+1}), \quad p_{k+1} = D_2 L_d(q_k, q_{k+1}), \]

which is the characterization of a symplectic map in terms of a **Type I generating function** (discrete Lagrangian).
Main Advantages of Variational Integrators

- **Discrete Noether’s Theorem**
  If the discrete Lagrangian $L_d$ is (infinitesimally) $G$-invariant under the diagonal group action on $Q \times Q$,

  \[ L_d(gq_0, gq_1) = L_d(q_0, q_1) \]

  then the **discrete momentum map** $J_d : Q \times Q \to g^*$,

  \[ \langle J_d(q_k, q_{k+1}), \xi \rangle \equiv \langle D_1 L_d(q_k, q_{k+1}), \xi Q(q_k) \rangle \]

  is preserved by the discrete flow.
**Main Advantages of Variational Integrators**

- **Variational Error Analysis**
  Since the exact discrete Lagrangian generates the exact solution of the Euler–Lagrange equation, the exact discrete flow map is *formally* expressible in the setting of variational integrators.

- This is analogous to the situation for B-series methods, where the exact flow can be expressed formally as a B-series.

- If a computable discrete Lagrangian $L_d$ is of order $r$, i.e.,

$$L_d(q_0, q_1) = L_d^{\text{exact}}(q_0, q_1) + O(h^{r+1})$$

then the discrete Euler–Lagrange equations yield an order $r$ accurate symplectic integrator.
Consider an alternative expression for the exact discrete Lagrangian,

\[
L_{d}^{\text{exact}}(q_0, q_1) \equiv \text{ext}_{q \in C^2([0, h], Q)} \int_{0}^{h} L(q(t), \dot{q}(t)) dt,\]

which is more amenable to discretization.

Replace the infinite-dimensional function space \( C^2([0, h], Q) \) with a \textit{finite-dimensional function space}.

Replace the integral with a \textit{numerical quadrature formula}.
A desirable property of a Ritz numerical method based on a finite-dimensional space $F_d \subset F$, is that it should exhibit **optimal rates of convergence**, which is to say that the numerical solution $q_d \in F_d$ and the exact solution $q \in F$ satisfies,

$$\|q - q_d\| \leq c \inf_{\tilde{q} \in F_d} \|q - \tilde{q}\|.$$

This means that the rate of convergence depends on the best approximation error of the finite-dimensional function space.
Ritz Variational Integrators

- Given a sequence of finite-dimensional function spaces $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset C^2([0, h], Q) \equiv \mathcal{C}_\infty$.
- For a correspondingly accurate sequence of quadrature formulas,
  \[
  L^i_d(q_0, q_1) \equiv \text{ext } \text{h } \sum_{j=1}^{s_i} b^i_j L(q(c^i_j h), \dot{q}(c^i_j h)),
  \]
  where $L^\infty_d(q_0, q_1) = L^\text{exact}_d(q_0, q_1)$.
- Proving $L^i_d(q_0, q_1) \rightarrow L^\infty_d(q_0, q_1)$, corresponds to $\Gamma$-convergence.
- For optimality, we require the bound,
  \[
  L^i_d(q_0, q_1) = L^\infty_d(q_0, q_1) + c \inf_{\tilde{q} \in \mathcal{C}_i} \| q - \tilde{q} \|,
  \]
  where we need to relate the rate of $\Gamma$-convergence with the best approximation properties of the family of approximation spaces.
Under suitable technical hypotheses:
- Regularity of $L$ in a closed and bounded neighborhood;
- The quadrature rule is sufficiently accurate;
- The discrete and continuous trajectories minimize their actions;

the Ritz discrete Lagrangian has the same approximation error as the best approximation error of the approximation space.

The critical assumption is action minimization. For Lagrangians $L = \dot{q}^T M \dot{q} - V(q)$, and sufficiently small $h$, this assumption holds.

Shows that Ritz variational integrators are order optimal; spectral variational integrators are geometrically convergent.
Order optimal convergence of the Kepler 2-body problem with eccentricity 0.6 over 100 steps of $h = 2.0$. 

Order Optimal Convergence of Ritz variational integrators

One Step Map Convergence with $h$-Refinement
Geometric Convergence of Spectral variational integrators

Geometric convergence of the Kepler 2-body problem with eccentricity 0.6 over 100 steps of $h = 2.0$. 
Comparison of inner solar system orbital diagrams from a spectral variational integrator and the JPL Solar System Dynamics Group.

- $h = 100$ days, $T = 27$ years, $25$ Chebyshev points per step.
Comparison of outer solar system orbital diagrams from a spectral variational integrator and the JPL Solar System Dynamics Group. Inner solar system was aggregated, and $h = 1825$ days.
Recall the implicit characterization of a symplectic map in terms of generating functions:

\[
\begin{aligned}
\rho_k &= -D_1 L_d(q_k, q_{k+1}) \\
\rho_{k+1} &= D_2 L_d(q_k, q_{k+1})
\end{aligned}
\quad \begin{aligned}
\rho_k &= D_1 H^+_d(q_k, \rho_{k+1}) \\
q_{k+1} &= D_2 H^+_d(q_k, \rho_{k+1})
\end{aligned}
\]

Symplecticity follows as a trivial consequence of these equations, together with \(d^2 = 0\), as the following calculation shows:

\[
d^2 L_d(q_k, q_{k+1}) = d(D_1 L_d(q_k, q_{k+1})dq_k + D_2 L_d(q_k, q_{k+1})dq_{k+1}) \\
= d(-\rho_k dq_k + \rho_{k+1} dq_{k+1}) \\
= -d\rho_k \wedge dq_k + d\rho_{k+1} \wedge dq_{k+1}
\]
We consider a multisymplectic analogue of Jacobi’s solution:

\[ L_d^{\text{exact}}(q_0, q_1) = \int_0^h L(q_{0,1}(t), \dot{q}_{0,1}(t)) \, dt, \]

where \( q_{0,1}(t) \) satisfies the Euler–Lagrange boundary-value problem.

This is given by,

\[ L_d^{\text{exact}}(\varphi|_{\partial \Omega}) = \int_{\Omega} L(j^1 \tilde{\varphi}) \]

where \( \tilde{\varphi} \) satisfies the boundary conditions \( \tilde{\varphi}|_{\partial \Omega} = \varphi|_{\partial \Omega} \), and \( \tilde{\varphi} \) satisfies the Euler–Lagrange equation in the interior of \( \Omega \).
If one takes variations of the *multisymplectic exact discrete Lagrangian* with respect to the boundary conditions, we obtain,

\[ \partial \varphi(x,t) L^\text{exact}_{d}(\varphi|\partial \Omega) = p_\perp(x, t), \]

where \((x, t) \in \partial \Omega\), and \(p_\perp\) is a codimension-1 differential form, that by Hodge duality can be viewed as the normal component (to the boundary \(\partial \Omega\)) of the multimomentum at the point \((x, t)\).

These equations, taken at every point on \(\partial \Omega\) constitute a *multisymplectic relation*, which is the PDE analogue of,

\[
\begin{align*}
    p_k &= -D_1 L_d(q_k, q_{k+1}) \\
    p_{k+1} &= D_2 L_d(q_k, q_{k+1})
\end{align*}
\]

where the sign comes from the orientation of the boundary.
Theorem (Noether’s Theorem)
For every continuous symmetry of an action, there exists a quantity that is conserved in time.

Theorem (Noether’s Theorem for Gauge Field Theories)
For every differentiable, local symmetry of an action, there exists a Noether current obeying a continuity equation. Integrating this current over a spacelike surface yields a conserved quantity called a Noether charge.

Since gauge symmetries are associated with conserved quantities, we need finite-elements that are (approximately) group-equivariant.
Motivating Example: Lorentzian Metrics

Let $\mathcal{L}$ denote the space of **Lorentzian metric tensors**:

$$\mathcal{L} = \{ L \in \mathbb{R}^{4 \times 4} \mid L = L^T, \det L \neq 0, \ \text{signature}(L) = (3,1) \}.$$

**Problem**

Given $L^{(i)} \in \mathcal{L}$ at the vertices $x^{(i)}$ of a simplex $\Omega$, find a continuous function $\mathcal{I}L : \Omega \to \mathcal{L}$ such that:

1. $\mathcal{I}L(x^{(i)}) = L^{(i)}$ for each $i$.
2. $\mathcal{I}L(x) \in \mathcal{L}$ for every $x \in \Omega$.
3. *(Frame invariance)*: If $Q \in O(1,3)$ and $L^{(i)} \leftarrow QL^{(i)}Q^T$ for each $i$, then $\mathcal{I}L(x) \leftarrow Q\mathcal{I}L(x)Q^T$.

Here, $O(1,3)$ denotes the **indefinite orthogonal group**:

$$O(1,3) = \{ Q \in \mathbb{R}^{4 \times 4} \mid QJQ^T = J \},$$

where $J = \text{diag}(-1,1,1,1)$. 
Options:

1. Componentwise interpolation: Not signature-preserving, in general. For instance,

\[
\frac{1}{2} \begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 & -4 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\( \in \mathcal{L} \) since \( \lambda = -4,1,1,4 \) \( \in \mathcal{L} \) since \( \lambda = -2,1,1,6 \) \( \notin \mathcal{L} \) since \( \lambda = 1,1,1,1 \)
1. **Geodesic interpolation** [Grohs, Sander]:

\[ \mathcal{I} \mathcal{L}(x) = \arg \min_{L \in \mathcal{L}} \sum_{i=1}^{m} \phi_i(x) \text{dist}(L^{(i)}, L)^2, \]

where \( \{ \phi_i \}_{i=1}^{m} \) are scalar-valued shape functions satisfying \( \phi_i(x^{(j)}) = \delta_{ij} \). Also known as the weighted **Riemannian mean**.
Motivating Example: Lorentzian Metrics

Our approach:

- Idea: If $\mathcal{L}$ were a Lie group, one could use the exponential map and perform all calculations on its Lie algebra, a linear space.

- In reality, $\mathcal{L}$ is not a Lie group (it is a **symmetric space**). Nonetheless, a similar construction is available.
Motivating Example: Lorentzian Metrics

1. Notice that $\mathcal{L}$ is diffeomorphic to $GL_4(\mathbb{R})/O(1,3)$: The map
   \[ \bar{\varphi} : GL_4(\mathbb{R})/O(1,3) \to \mathcal{L} \]
   \[ [A] \mapsto AJA^T, \]
   is a diffeomorphism, where $J = \text{diag}(-1,1,1,1)$.

2. Every coset $[A]$ has a canonical representative $Y$ by virtue of the \textit{generalized polar decomposition}:
   \[ A = YQ, \quad Y \in \text{Sym}_J(4), \quad Q \in O(1,3), \]
   where
   \[ \text{Sym}_J(4) = \{ Y \in GL_4(\mathbb{R}) \mid YJ = JY^T \}. \]

3. $\log(Y)$ lives in a linear space called a \textit{Lie triple system}:
   \[ \log(Y) \in \text{sym}_J(4) = \{ P \in \mathbb{R}^{4 \times 4} \mid PJ = JP^T \}. \]
Motivating Example: Lorentzian Metrics

To summarize:

1. $\mathcal{L}$ is locally diffeomorphic to the **Lie triple system**

   \[ \mathfrak{sym}_J(4) = \{ P \in \mathbb{R}^{4 \times 4} \mid PJ = JP^T \}, \]

   which is a **linear space**.

2. Interpolation on a linear space is easy.
Motivating Example: Lorentzian Metrics

The resulting interpolation formula reads

\[ \mathcal{I}L(x) = J \exp \left( \sum_{i=1}^{m} \phi_i(x) \log(JL^{(i)}) \right), \]

where \( J = \text{diag}(-1, 1, 1, 1) \), and \( \{\phi_i\}_{i=1}^{m} \) are scalar-valued shape functions satisfying \( \phi_i(x^{(j)}) = \delta_{ij} \).
Motivating Example: Lorentzian Metrics

The interpolant so defined enjoys the following properties:

**Signature preservation**

The interpolant $\mathcal{I}L$ is signature-preserving; that is,

$$\mathcal{I}L(x) \in \mathcal{L}$$

for every $x \in \Omega$.

**Frame invariance**

Let $Q \in O(1, 3)$. If $\tilde{L}^{(i)} = QL^{(i)}Q^T$, $i = 1, 2, \ldots, m$, and if $Q$ is sufficiently close to the identity matrix, then

$$\mathcal{I}\tilde{L}(x) = Q \mathcal{I}L(x) Q^T$$

for every $x \in \Omega$. 
Motivating Example: Lorentzian Metrics

**Symmetry under inversion**

If $\tilde{L}^{(i)} = (L^{(i)})^{-1}$, $i = 1, 2, \ldots, m$, then

$$\mathcal{I}\tilde{L}(x) = (\mathcal{I}L(x))^{-1}$$

for every $x \in \Omega$.

**Determinant averaging**

If $\sum_{i=1}^{m} \phi_i(x) = 1$ for every $x \in \Omega$, then

$$\det \mathcal{I}L(x) = \prod_{i=1}^{m} \left( \det L^{(i)} \right)^{\phi_i(x)}$$

for every $x \in \Omega$. 
Motivating Example: Lorentzian Metrics

Numerical example: Interpolating the *Schwarzschild metric*

\[-\left(1 - \frac{1}{r}\right) dt^2 + \left(1 - \frac{1}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\varphi^2\right).\]

<table>
<thead>
<tr>
<th>(N)</th>
<th>(L^2)-error</th>
<th>Order</th>
<th>(H^1)-error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.3 \cdot 10^{-3}</td>
<td></td>
<td>2.8 \cdot 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8.4 \cdot 10^{-4}</td>
<td>1.975</td>
<td>1.4 \cdot 10^{-2}</td>
<td>0.998</td>
</tr>
<tr>
<td>8</td>
<td>2.1 \cdot 10^{-4}</td>
<td>1.994</td>
<td>7.1 \cdot 10^{-3}</td>
<td>0.999</td>
</tr>
<tr>
<td>16</td>
<td>5.3 \cdot 10^{-5}</td>
<td>1.998</td>
<td>3.6 \cdot 10^{-3}</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Error incurred when interpolating the Schwarzschild metric over the region \(U = \{0\} \times [2, 3] \times [2, 3] \times [2, 3]\) on a uniform \(N \times N \times N\) grid of cubes, with shape functions \(\{\phi_i\}_i\) on each cube given by tensor products of Lagrange polynomials of degree 1.
Motivating Example: Lorentzian Metrics

Numerical example: Interpolating the Schwarzschild metric

\[- \left(1 - \frac{1}{r}\right) dt^2 + \left(1 - \frac{1}{r}\right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) . \]

<table>
<thead>
<tr>
<th>$N$</th>
<th>$L^2$-error</th>
<th>Order</th>
<th>$H^1$-error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1.7 \cdot 10^{-4}$</td>
<td></td>
<td>$2.5 \cdot 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$2.2 \cdot 10^{-5}$</td>
<td>3.001</td>
<td>$6.2 \cdot 10^{-4}$</td>
<td>1.993</td>
</tr>
<tr>
<td>8</td>
<td>$2.7 \cdot 10^{-6}$</td>
<td>3.000</td>
<td>$1.6 \cdot 10^{-4}$</td>
<td>1.998</td>
</tr>
<tr>
<td>16</td>
<td>$3.4 \cdot 10^{-7}$</td>
<td>3.000</td>
<td>$3.9 \cdot 10^{-5}$</td>
<td>1.999</td>
</tr>
</tbody>
</table>

Error incurred when interpolating the Schwarzschild metric over the region $U = \{0\} \times [2, 3] \times [2, 3] \times [2, 3]$ on a uniform $N \times N \times N$ grid of cubes, with shape functions $\{\phi_i\}_i$ on each cube given by tensor products of Lagrange polynomials of degree 2.
Motivating Example: Lorentzian Metrics

\[ \mathcal{I}L(x) = J \exp \left( \sum_{i=1}^{m} \phi_i(x) \log(JL^{(i)}) \right) \]

Remarks:

1. An alternative interpolant is obtained by defining \( \mathcal{I}L(x) \) implicitly via

\[ \mathcal{I}L(x) = \mathcal{I}L(x) \exp \left( \sum_{i=1}^{m} \phi_i(x) \log \left( \mathcal{I}L(x)^{-1}L^{(i)} \right) \right) . \]

This interpolant is equivalent to the \textit{geodesic interpolant}.

2. Replacing \( J = \text{diag}(-1, 1, 1, 1) \) with the identity matrix, one recovers the weighted \textit{Log-Euclidean mean} of symmetric positive-definite matrices [Arsigny et al.]:

\[ \mathcal{I}L(x) = \exp \left( \sum_{i=1}^{m} \phi_i(x) \log(L^{(i)}) \right) \]
This construction works if $\mathcal{L}$ is replaced by any **symmetric space** – a smooth manifold with an inversion symmetry (an involutive isometry) about every point. Examples include:

- Symmetric $n \times n$ matrices with signature $(p, n - p)$.
- Grassmannian $Gr(p, n)$ – space of $p$-dimensional linear subspaces of $\mathbb{R}^n$.

A key role in the construction is played by the **generalized polar decomposition**.
Let $G$ be a Lie group, and let $\sigma : G \to G$ be an involutive automorphism, i.e. $\sigma \neq \text{id}$, $\sigma^2 = \text{id}$, and $\sigma(gh) = \sigma(g)\sigma(h)$ for every $g, h \in G$. Then every $g \in G$ can be written as a product

$$g = pk, \quad p \in G_\sigma, \; k \in G^\sigma,$$

where

$$G^\sigma = \{ g \in G \mid \sigma(g) = g \},$$

$$G_\sigma = \{ g \in G \mid \sigma(g) = g^{-1} \}.$$

Moreover, this decomposition is locally unique.

Examples:

- $G = \text{GL}_n(\mathbb{R}), \; \sigma(A) = A^{-T} \implies G^\sigma = O(n), \; G_\sigma = \text{Sym}(n)$.
- $G = \text{GL}_4(\mathbb{R}), \; \sigma(A) = JA^{-T}J \implies G^\sigma = O(1, 3), \; G_\sigma = \text{Sym}_J(4)$. 
Introduction
Gauge Field Theories
Dirac Mechanics
Lorentzian Metrics
Symmetric Spaces

Abstraction to Symmetric Spaces

Abstract setting:

- \( S \) – smooth manifold
- \( \eta \) – distinguished element of \( S \)
- \( G \) – Lie group that acts transitively on \( S \)
- \( \sigma : G \to G \) – involutive automorphism
- \( G^\sigma = \{ g \in G \mid \sigma(g) = g \} \)
- \( G_\sigma = \{ g \in G \mid \sigma(g) = g^{-1} \} \)

Key assumption: Isotropy subgroup of \( \eta \) coincides with the fixed set \( G^\sigma \), i.e.

\[
g \cdot \eta = \eta \iff \sigma(g) = g.
\]

\[
AJA^T = J \iff JA^{-T}J = A
\]

Then \( S \) is diffeomorphic to \( G/G^\sigma \) (a symmetric space), and every \([g] \in G/G^\sigma\) has a canonical representative \( p \in G_\sigma \) by the generalized polar decomposition \( g = pk \), \( p \in G_\sigma \), \( k \in G^\sigma \).
Abstract setting, continued:

1. $\mathfrak{g}$ – Lie algebra of $G$

2. $\exp: \mathfrak{g} \to G$ – exponential map

3. The preimage of $G_\sigma$ under $\exp$ is the linear space

$$
\mathfrak{p} = \{ P \in \mathfrak{g} \mid d\sigma(P) = -P \} \subset \mathfrak{g}
= \{ P \in \mathbb{R}^{4\times4} \mid -JP^TJ = -P \}
$$

This space is a **Lie triple system** – it is closed under the double commutator $[\cdot, [\cdot, \cdot]]$, but not under $[\cdot, \cdot]$. 
Abstraction to Symmetric Spaces

\[ \exp: p \rightarrow G_\sigma \rightarrow G/G^\sigma \rightarrow S \]

\[ GL_4(\mathbb{R}) \]

\[ \text{Sym}_J(4) \rightarrow \text{Sym}_J(4) \rightarrow GL_4(\mathbb{R})/O(1, 3) \rightarrow \mathcal{L} \]
To summarize:

1. $S$ is locally diffeomorphic to the Lie triple system $\mathfrak{p}$, which is a linear space.
2. Interpolation on a linear space is easy.
3. The resulting formula for interpolating $\{u^{(i)}\}_{i=1}^{m} \subset S$ reads

$$\mathcal{I}u(x) = F \left( \sum_{i=1}^{m} \phi_i(x) F^{-1}(u^{(i)}) \right),$$

where $\phi_i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are scalar-valued shape functions satisfying $\phi_i(x^{(j)}) = \delta_{ij}$, and

$$F : \mathfrak{p} \to S$$
$$P \mapsto \exp(P) \cdot \eta.$$
Abstraction to Symmetric Spaces

**\(G^\sigma\)-equivariance**

Let \( g \in G^\sigma \). If \( \tilde{u}^{(i)} = g \cdot u^{(i)}, \ i = 1, 2, \ldots, m \), and if \( g \) is sufficiently close to the identity, then

\[
I\tilde{u}(x) = g \cdot Iu(x)
\]

for every \( x \in \Omega \).

**Symmetry under geodesic reflection**

If \( \tilde{u}^{(i)} = s_\eta(u^{(i)}), \ i = 1, 2, \ldots, m \), then

\[
I\tilde{u}(x) = s_\eta(Iu(x))
\]

for every \( x \in \Omega \), where \( s_\eta : S \rightarrow S \) denotes the geodesic reflection about \( \eta \).
Connection with Geodesic Interpolation

Interpolation formula:

\[ \mathcal{I} u(x) = F \left( \sum_{i=1}^{m} \phi_i(x) F^{-1}(u^{(i)}) \right), \]

where \( F(P) = \exp(P) \cdot \eta \).

Interpolation formula (generalized):

\[ \mathcal{I}_{\bar{g}} u(x) = F_{\bar{g}} \left( \sum_{i=1}^{m} \phi_i(x) F_{\bar{g}}^{-1}(u^{(i)}) \right), \]

where \( F_{\bar{g}}(P) = \bar{g} \exp(P) \cdot \eta \).

1. By allowing \( \bar{g} \) to vary with \( x \), we may define \( \bar{g}(x) \) implicitly via

\[ \mathcal{I}_{\bar{g}(x)} u(x) = \bar{g}(x) \cdot \eta. \]

2. The resulting interpolant coincides with the geodesic interpolant [Grohs, Sander].

3. The geodesic interpolant has the advantage of being \( G \)-equivariant rather than being merely \( G^\sigma \)-equivariant.
Summary

- Gauge field theories exhibit gauge symmetries that impose Cauchy initial value constraints, and are also underdetermined.
- These result in degenerate field theories that can be described using multi-Dirac mechanics and multi-Dirac structures.
- Described a systematic framework for constructing and analyzing Ritz variational integrators, and the extension to Hamiltonian PDEs.
- Presented a local isomorphism between a Lie triple system and the associated symmetric space, which can be used to construct group-equivariant finite-element spaces that take values in a symmetric space.
Thank you!