L-groups for double covers of Chevalley-Steinberg groups

Martin H. Weissman
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Outline
Covering groups after Brylinski and Deligne

The dual group

The L-group

Evidences and questions
Covering groups after Brylinski and Deligne
Let $F$ be a field, and let $G$ be a reductive group over $F$.

**Definition (My working definition)**

A cover of $G$ over $F$ is a pair $\tilde{G} = (G', n)$, where

- $K_2 \hookrightarrow G' \twoheadrightarrow G$ is a central extension of $G$ by $K_2$;
- $1 \leq n$ (the degree) is such that $\#\mu_n(F) = n$. 

"A central extension of $G$ by $K_2$" was defined by Brylinski and Deligne (Pub. Math. IHES 94 (2001)). They classified such central extensions by root-theoretic data.
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Let $\tilde{G}$ be a degree 2 cover of a reductive group $G$ over $\mathbb{R}$. Taking $\mathbb{R}$-points and applying the Hilbert symbol yields a topological central extension,

$$\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow G,$$

where $G = G(\mathbb{R})$ and $\mu_2 = \{\pm 1\}$. 

Two questions:
1. What topological central extensions arise?
2. Why should one work with the Brylinksi-Deligne class of covers anyways?
What does a covering group over $\mathbb{R}$ give us?

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Proposition

Let $S$ be an algebraic torus over $\mathbb{R}$ such that $S = S(\mathbb{R})$ is compact. Then every topological central extension,

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N.B. the cover $\tilde{S}$ is not uniquely determined by the topological cover $\tilde{S}$.  

Compact tori
Definition

A Chevalley group (respectively Chevalley-Steinberg group) over a field $F$ is a split (resp., quasisplit), absolutely almost simple, simply-connected linear algebraic group $G$ over $F$. 
### Chevalley-Steinberg groups over $\mathbb{R}$

<table>
<thead>
<tr>
<th>Type</th>
<th>Group $G = G(\mathbb{R})$</th>
<th>$\pi_1(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_\ell$, $\ell \geq 2$</td>
<td>$SL_{\ell+1}(\mathbb{R})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$B_\ell$, $\ell \geq 3$</td>
<td>$Spin_{\ell,\ell+1}(\mathbb{R})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$C_\ell$, $\ell \geq 1$</td>
<td>$Sp_{2\ell}(\mathbb{R})$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$D_\ell$, $\ell \geq 4$</td>
<td>$Spin_{\ell,\ell}(\mathbb{R})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$E_6, E_7, E_8, F_4, G_2$</td>
<td>Exceptional groups</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$A_{2p}^{(2)}$, $p \geq 1$</td>
<td>$SU_{p,p+1}(\mathbb{R})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$A_{2p-1}^{(2)}$, $p \geq 2$</td>
<td>$SU_{p,p}(\mathbb{R})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
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Fix $G$ a Chevalley-Steinberg group over $\mathbb{R}$. There exists a unique, up to unique isomorphism, nonsplit topological central extension,

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**Theorem (Brylinski-Deligne)**

*There is a canonical central extension*

$$K_2 \hookrightarrow G' \twoheadrightarrow G.$$
Fix $G$ a Chevalley-Steinberg group over $\mathbb{R}$. There exists a unique, up to unique isomorphism, nonsplit topological central extension,

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**Theorem (Brylinski-Deligne)**

*There is a canonical central extension*

$$K_2 \hookrightarrow G' \twoheadrightarrow G.$$ 

**Theorem (Prasad-Rapinchuk, Prasad, Brylinski-Deligne)**

*The double cover arising from the canonical extension of Brylinski-Deligne is uniquely isomorphic to the nonsplit extension $\tilde{G}$.***
Nontrivial covers can yield (topologically) linear Lie groups.

**Example**

There exists a cover $\tilde{G} = (G', 2)$, where $G = \text{PGL}_2$, and the resulting extension

$$
\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow \text{PGL}_2(\mathbb{R})
$$

is isomorphic (nonuniquely) to the extension

$$
\mu_2 \hookrightarrow \begin{array}{c}
\text{GL}_2(\mathbb{R}) \\
\left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t > 0 \right\} \\
\end{array} \twoheadrightarrow \text{PGL}_2(\mathbb{R}).
$$

N.B. $\tilde{g} \mapsto g \cdot |\det(g)|^{-1/2}$ is a faithful continuous representation.
If $F$ is a local field, and $\tilde{G}$ is a degree $n$ cover of a reductive group $G$ over $F$, then one gets a topological central extension,

$$\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G,$$

where $G = G(F)$. Universal extensions of Chevalley-Steinberg groups arise from such a construction.
If $F$ is a global field and $\tilde{G}$ is a degree $n$ cover of a reductive group $G$ over $F$, then one gets a topological central extension,

$$\mu_n \hookrightarrow \tilde{G}_\mathbb{A} \xrightarrow{\sim} G_\mathbb{A} = G(\mathbb{A}),$$

as well as a splitting of this extension over $G(F)$. 

If $F$ is a global field and $\tilde{G}$ is a degree $n$ cover of a reductive group $G$ over $F$, then one gets a topological central extension,

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as well as a splitting of this extension over $G(F)$.

If $O$ is the ring of integers in a nonarchimedean local field $F$, and $\tilde{G}$ is a degree $n$ cover of a reductive group $G$ over $O$, then one gets a topological central extension

$$\mu_n \hookrightarrow \tilde{G}_F \twoheadrightarrow G_F = G(F),$$

as well as a splitting of this extension over $G(O)$. 
The class of covers described here has some nice properties:

- They include the most important topological central extensions, at least those that seem connected to arithmetic.
- Splitting properties allow one to study genuine unramified representations and genuine automorphic representations.
- They include some central extensions that can be studied using techniques for linear groups, but would not ordinarily get attention.
- They have a nice classification due to Brylinski and Deligne.
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- They include the most important topological central extensions, at least those that seem connected to arithmetic.
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I aim to extend the Langlands program to covers. For this purpose, I have defined an L-group associated to all covers of quasisplit groups over local and global fields.
The dual group
Let $\tilde{G}$ be a degree $n$ cover of a quasisplit reductive group $G$ over a field $F$. Let $T$ be a maximal torus in a Borel subgroup $B \subset G$. Define

$$Y = \text{Hom}(G_m, T), \quad X = \text{Hom}(T, G_m).$$
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To $\tilde{G}$, Brylinski and Deligne associate a Weyl- and Galois-invariant quadratic form

$$Q : Y \to \mathbb{Z}.$$
A quadratic form

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**Simplest case:** If $G$ a Chevalley-Steinberg group and $\tilde{G}$ is the canonical double cover, $Q: Y \to \mathbb{Z}$ is the unique Weyl-invariant quadratic form such that

$$Q(\alpha^\vee) = 1 \text{ for all short coroots } \alpha^\vee.$$
Modified root data

Let \( \Phi \subset X \) and \( \Phi^\vee \subset Y \) the subsets of roots and coroots. Let \( \Delta \subset \Phi \) and \( \Delta^\vee \subset \Phi^\vee \) be the subsets of simple roots and coroots.
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Let $\Phi \subset X$ and $\Phi^\vee \subset Y$ the subsets of roots and coroots. Let $\Delta \subset \Phi$ and $\Delta^\vee \subset \Phi^\vee$ be the subsets of simple roots and coroots.

Define

$$n_\alpha = \frac{n}{\text{GCD}(n, Q(\alpha^\vee))} \text{ for all } \alpha \in \Phi.$$
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Define modified roots and coroots

$$\tilde{\alpha} = n_\alpha^{-1} \alpha, \quad \tilde{\alpha}^\vee = n_\alpha \alpha^\vee \text{ for all } \alpha \in \Phi.$$ 

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Define a modified coweight lattice

$$Y_{Q,n} = \{y \in Y : Q(y + y') - Q(y) - Q(y') \in n\mathbb{Z} \text{ for all } y' \in Y\} \subset nY.$$  

Define a modified weight lattice

$$X_{Q,n} = \text{Hom}(Y_{Q,n}, \mathbb{Z}) \subset n^{-1}X.$$
Theorem (Lusztig, Finkelberg-Lysenko, McNamara, Reich, W.)

The sextuple \((Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta})\) is a based root datum.

\[ \alpha^\vee \quad \beta^\vee \quad \tilde{\alpha}^\vee \quad \tilde{\beta}^\vee \]

**Figure 1:** Modifying the root datum for the double cover of \(Sp_4\). On the left, \(Y \supset \Phi^\vee \supset \Delta^\vee\). On the right, \(Y_{Q,2} \supset \tilde{\Phi}^\vee \supset \tilde{\Delta}^\vee\). In this case \(Y = Y_{Q,2}\) (I call such covers “sharp”).
Theorem (Lusztig, Finkelberg-Lysenko, McNamara, Reich, W.)

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Figure 2: Modifying the root datum for a double cover of \(GL_2\), with \(\alpha^\vee = (1, 1)\) and \(Q(u, v) = u^2 + uv + v^2\) in standard coordinates.
The dual group

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Definition

The dual group of the cover \(\tilde{G}\) is the pinned complex reductive group \(\tilde{G}^\vee\) associated to the based root datum above.

The pinning gives a Borel subgroup and maximal torus: 
\(\tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee\). Note \(\tilde{T}^\vee = \text{Hom}(Y_{Q,n}, \mathbb{C}^\times)\).
The dual group

**Theorem (Lusztig, Finkelberg-Lysenko, McNamara, Reich, W.)**

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The map (a homomorphism, in fact)

\[
Y_{Q,n} \to \mathbb{C}^\times, \quad y \mapsto e^{2\pi i Q(y)/n}
\]

defines a 2-torsion element

\[
\xi \in \tilde{\mathbb{Z}}^\vee := Z(\tilde{G}^\vee) = \text{Hom}\left(\frac{Y_{Q,n}}{\text{Span}_\mathbb{Z}(\check{\Phi}^\vee)}, \mathbb{C}^\times\right).
\]
<table>
<thead>
<tr>
<th>Double cover</th>
<th>$\tilde{G}^\vee$</th>
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<tbody>
<tr>
<td>$SL_2(\mathbb{R})$</td>
<td>*$SL_2(\mathbb{C})$</td>
</tr>
<tr>
<td>$SL_3(\mathbb{R})$</td>
<td>$PGL_3(\mathbb{C})$</td>
</tr>
<tr>
<td>$SL_4(\mathbb{R})$</td>
<td>$SL_4(\mathbb{C})/\mu_2$</td>
</tr>
<tr>
<td>$SL_5(\mathbb{R})$</td>
<td>$PGL_5(\mathbb{C})$</td>
</tr>
<tr>
<td>$SL_6(\mathbb{R})$</td>
<td>*$SL_6(\mathbb{C})/\mu_3$</td>
</tr>
<tr>
<td>$Spin_7(\mathbb{R})$</td>
<td>$SO_7(\mathbb{C})$</td>
</tr>
<tr>
<td>$Spin_9(\mathbb{R})$</td>
<td>$Spin_9(\mathbb{C})$</td>
</tr>
<tr>
<td>$Spin_{11}(\mathbb{R})$</td>
<td>$SO_{11}(\mathbb{C})$</td>
</tr>
<tr>
<td>$Spin_{13}(\mathbb{R})$</td>
<td>*$Spin_{13}(\mathbb{C})$</td>
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<td>$Spin_8(\mathbb{R})$</td>
<td>$Spin_8(\mathbb{C})$</td>
</tr>
<tr>
<td>$Spin_{10}(\mathbb{R})$</td>
<td>$SO_{10}(\mathbb{C})$</td>
</tr>
<tr>
<td>$Spin_{12}(\mathbb{R})$</td>
<td>*$Spin_{12}(\mathbb{C})$</td>
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</tr>
<tr>
<td>$G_2(\mathbb{R})$</td>
<td>$G_2(\mathbb{C})$</td>
</tr>
<tr>
<td>$F_4(\mathbb{R})$</td>
<td>$F_4(\mathbb{C})$</td>
</tr>
</tbody>
</table>

**Table 1:** Table of double covers of real Chevalley groups and their dual groups. Asterisks denote the cases where the 2-torsion element $\xi$ is nontrivial. (For quasisplit groups, Gal acts by outer automorphisms preserving $\xi$)
The L-group
The naïve L-group

Others (e.g., Savin, Adams-Barbasch-Paul-Trapa-Vogan, Crofts, Finkelberg-Lysenko, McNamara, Reich) related the dual $\tilde{G}^\vee$ to the parameterization of genuine representations of $\tilde{G}$. 
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I suggest a more elaborate L-group is the natural L-group. It will be an extension,

$$\tilde{G}^\vee \leftrightarrow L\tilde{G} \rightarrow \text{Gal},$$

but **without a distinguished splitting** in general.
The first step in constructing the L-group is the “first twist”.
Let $\sigma$ denote complex conjugation, so $\text{Gal} = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. Define a cocycle, $\text{Gal} \times \text{Gal} \to \tilde{\mathbb{Z}}^\vee = Z(\tilde{G}^\vee)$ by

$$(\sigma, \sigma) \mapsto \xi.$$ 

$(1, 1)$ and $(1, \sigma)$ and $(\sigma, 1) \mapsto 1$. 


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$$(\sigma, \sigma) \mapsto \xi.$$  

$(1, 1)$ and $(1, \sigma)$ and $(\sigma, 1) \mapsto 1$. This yields a central extension,

$$\tilde{Z}^\vee \hookrightarrow E_1 \twoheadrightarrow \text{Gal}.$$
But this isn’t enough – a second twist is needed, which requires another invariant of covers due to Brylinski-Deligne.

To a cover $\tilde{\mathcal{G}}$ of a reductive group $\mathcal{G}$ over a field $F$, Brylinski and Deligne associate a $\text{Gal}(\bar{F}/F)$-equivariant extension of groups,

$$\bar{F}^\times \to D \to Y,$$

where $Y$ is the coweight lattice and $\bar{F}$ is a separable closure of $F$. 

Consider $\tilde{G}$ the canonical double cover of a Chevalley-Steinberg group $G$ over $\mathbb{R}$. We can describe $D$ by generators and relations.

**Generators:** all elements of $\mathbb{C}^\times$, and elements $d_\alpha$ for each simple root $\alpha \in \Delta$.

**Relations:** $\mathbb{C}^\times$ is contained in the center of $D$, and for any $\alpha, \beta \in \Delta$, $[d_\alpha, d_\beta] = (-1)^Q(\alpha+\beta)-Q(\alpha)-Q(\beta)$.

Inclusion of $\mathbb{C}^\times \hookrightarrow D$ and projection $d_\alpha \mapsto \alpha^\vee$ yields a central extension,

$$\mathbb{C}^\times \hookrightarrow D \twoheadrightarrow Y.$$ 

Galois-invariance of $Q$ gives a $\text{Gal}(\mathbb{C}/\mathbb{R})$-action on $D$. 
Chevalley-Steinberg groups

From a double cover \( \tilde{G} \), we have a central extension,

\[
\mathbb{C}^\times \twoheadrightarrow D \twoheadrightarrow Y,
\]

endowed with a \( \text{Gal}(\mathbb{C}/\mathbb{R}) \)-action.
Chevalley-Steinberg groups

From a double cover $\tilde{G}$, we have a central extension,

$$\mathbb{C}^\times \hookrightarrow D \twoheadrightarrow Y,$$

endowed with a $\text{Gal}(\mathbb{C}/\mathbb{R})$-action.

Pull back to the sublattice $Y_{Q,2} \subset Y$.

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This is an abelian extension.
Chevalley-Steinberg groups

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This is an abelian extension.

Take $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ invariants.

$$\mathbb{R}^\times \hookrightarrow D_{Q,2}^\sigma \twoheadrightarrow Y_{Q,2}^\sigma.$$
In the split case, we have an abelian extension,

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**Definition**

The flipped extension $E_2$ is the set of homomorphisms $f : D_{Q,2}^\sigma \rightarrow \mathbb{C}^\times$ such that

- $f(t) = 1$ for all $t \in \mathbb{R}_{>0}^\times$.
- $f(d_{\alpha}^{n_\alpha} \cdot r_{\alpha}) = 1$ for all $\alpha \in \Delta$. Here $r_{\alpha} = (-1)^{Q(\alpha \vee \cdot (n_\alpha^2)}$.

This gives an extension (not so obviously)

\[ \tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \text{Gal}(\mathbb{C}/\mathbb{R}). \]
Flipping the extension (quasisplit case)

We have an abelian extension with $\text{Gal}(\mathbb{C}/\mathbb{R})$-action.

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Choose a splitting $s : Y_{Q,2} \to D_{Q,2}$ which satisfies 

$$s(\tilde{\alpha}^\vee) = d_{\alpha}^{n_{\alpha}} \cdot r_{\alpha} \text{ for all } \alpha \in \Delta.$$ 

Then $\sigma s/s \in \text{Hom}(Y_{Q,2}, \mathbb{C}^\times)$, and has a square root $\sqrt{\sigma s/s}$. 
Flipping the extension (quasisplit case)

We have an abelian extension with $\text{Gal}(\mathbb{C}/\mathbb{R})$-action.

\[ \mathbb{C}^\times \hookrightarrow D_{Q,2} \twoheadrightarrow Y_{Q,2}. \]

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Then $\sigma s/s \in \text{Hom}(Y_{Q,2}, \mathbb{C}^\times)$, and has a square root $\sqrt{\sigma s/s}$.

There are two $\tilde{Z}^\vee$-torsors:

\[ E_{2,1} = \tilde{Z}^\vee = \{ f \in \text{Hom}(Y_{Q,2}, \mathbb{C}^\times) : f(\tilde{\alpha}^\vee) = 1 \text{ for all } \alpha \in \Delta \}. \]

\[ E_{2,\sigma} = \{ f \in \text{Hom}(Y_{Q,2}, \mathbb{C}^\times) : \left[ f \cdot \sqrt{\frac{\sigma s}{s}} \right](\tilde{\alpha}^\vee) = 1 \text{ for all } \alpha \in \Delta \}. \]
Define $E_2 = E_{2,1} \sqcup E_{2,\sigma}$. Then we find a short exact sequence,

$$\tilde{Z}^\vee \hookrightarrow E_2 \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}).$$

The cocycle $(\sigma, \sigma) \mapsto \xi$ gave another short exact sequence,

$$\tilde{Z}^\vee \hookrightarrow E_1 \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}).$$

Take the Baer sum,

$$\tilde{Z}^\vee \hookrightarrow E_1 \dot{+} E_2 \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}).$$

Push out via the $\text{Gal}(\mathbb{C}/\mathbb{R})$-equivariant inclusion $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$ to get a short exact sequence.

$$\tilde{G}^\vee \hookrightarrow ^L \tilde{G} \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}).$$
Evidences and questions
Theorem (W. (also see Gan-Gao))

Let $T$ be a split torus over a local or global field. Then there is a natural one-to-one parameterization:

\[
\{ \text{Irreducible genuine admissible/automorphic reps of } \tilde{T} \} 
\leftrightarrow \{ \text{Weil parameters valued in } L\tilde{T} \}
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Theorem (W. (also see Gan-Gao))

Let $\mathbf{T}$ be a split torus over a local or global field. Then there is a natural one-to-one parameterization:

$\left\{ \text{Irreducible genuine admissible/automorphic reps of } \tilde{T} \right\} \leftrightarrow \left\{ \text{Weil parameters valued in } L\tilde{T} \right\}$

Theorem (W.)

Let $\mathbf{T}$ be a torus over $\mathbb{R}$ with $T = \mathbf{T}(\mathbb{R})$ compact. Then there is a natural one-to-one parameterization:

$\left\{ \text{Irreducible genuine characters of } \tilde{T} \right\} \leftrightarrow \left\{ \text{Weil parameters valued in } L\tilde{T} \right\}$
Let $\tilde{G}$ be a cover of a reductive group $G$, defined over the ring of integers in a nonarchimedean local field.

**Theorem (W. (also see Gan-Gao))**

There is a natural bijective parameterization:

\[
\{ \text{Irreducible genuine spherical reps of } \tilde{G} \} \pmod{\text{equiv}} \rightarrow \{ \text{Unramified Weil parameters valued in } \mathbb{L}^1\tilde{G} \} \pmod{\text{Ad}(\tilde{G}^\vee)}.
\]

Proof: Satake isomorphism (McNamara, WenWei Li, Gan-Gao, W.) + Parameterization for split tori + carefully tracing through Weyl-group action.
Let $\tilde{G}$ be a cover of a quasisplit semisimple group $G$ over $\mathbb{R}$, such that $G$ contains a compact maximal torus $T$ over $\mathbb{R}$.

**Theorem (W.)**

*There is a natural one-to-one parameterization:*

\[
\{ \text{Discrete series reps of } \tilde{G} \} \pmod{\text{equivalence}} \hookrightarrow \{ \text{Discrete series Weil parameters valued in } \left. \right|^L \tilde{G} \} \pmod{\text{Ad}(\tilde{G}^\vee)}.
\]
Let $\tilde{G}$ be a degree $n$ cover of a simple Chevalley group over $\mathbb{Z}_p$, type A,D,E, with $p = 3 \mod 4$. Let $G_{\text{lin}}$ be the split reductive group whose Langlands dual group is $\tilde{G}^\vee$.

**Theorem (Savin, 2004)**

For each Satake painting $S$ of the Dynkin diagram, choose a square root of $(-1)^{\# S}$ in $\mathbb{C}$. This set of choices determines an isomorphism from the Iwahori Hecke algebra of $\tilde{G}$ to the Iwahori Hecke algebra of $G_{\text{lin}}$. 

**Theorem (Gao? W.?)**

For each Satake painting $S$ of the Dynkin diagram, choose a square root of $(-1)^{\# S}$ in $\mathbb{C}$. This set of choices determines an isomorphism of $L$-groups from the $L$-group $L_{\tilde{G}}$ of the cover to the $L$-group $L_{G_{\text{lin}}} = \text{Gal} \ltimes \tilde{G}^\vee$. 

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For each Satake painting $S$ of the Dynkin diagram, choose a square root of $(-1)^{#S}$ in $\mathbb{C}$. This set of choices determines an isomorphism of $L$-groups from the $L$-group $L\tilde{G}$ of the cover to the $L$-group $L^{\text{Gal}}G_{\text{lin}} = \text{Gal} \ltimes \tilde{G}^\vee$. 
Consider a “linearish cover” $\tilde{G}$. Then genuine irreps of $\tilde{G}$ correspond to irreps of a linear group $H$ with a specific central character.

**Example**

$(G = PGL_2$ and $H = GL_2)$

There’s a cover $\tilde{G}$ for which $\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow PGL_2(\mathbb{R})$ in which

$$\tilde{G} \cong GL_2(\mathbb{R})/\left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t > 0 \right\}.$$  

**Exercise:** Pullback from $\tilde{G}$ to $H$ should be functorial, reflected in a homomorphism of L-groups $L\tilde{G} \rightarrow LH$. This has been considered for $H = GL_2$ by Gan and Gao.
Parameterization has been accomplished for covers of split tori and covers of compact tori over \( \mathbb{R} \).

**Exercise:** Complete the parameterization for all covers of tori over \( \mathbb{R} \).

**Problem:** Complete the parameterization for all covers of tori over local fields.
The parameterizations are often one-to-one. Some Weil parameters do not correspond to irreducible genuine irreps.

**Question:** Which Weil parameters are “relevant”? I.e., what is the image of the parameterization map? This seems related to endoscopy for covering groups.
If $\tilde{G}$ is a cover, there is an “opposite cover” $\tilde{G}^{\text{op}}$. (For double covers, they can be taken to be the same).

If $\pi$ is a genuine irreducible representation of $\tilde{G}$ (work over a local field), its contragredient is a genuine irreducible representation of $\tilde{G}^{\text{op}}$.

**Question:** (Adams-Vogan?) What is the corresponding map on L-groups?
Some references:


(2012) McNamara, “Principal series representations of metaplectic groups over local fields”, in Prog. Math. 300


