Integrability of \( p \)-adic matrix coefficients
joint work with Omer Offen

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New developments in representation theory
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1 Problem and motivation
2 Corollaries
3 Convergence of matrix coefficients
4 Results
**Setting**

\( G \) a reductive group defined over a \( p \)-adic field \( F \). \( G = G(F) \).

\((\pi, V)\) an admissible representation of \( G \).

For \( \nu \in V, \nu^* \in V^* \),

\[
m_{\nu, \nu^*}(g) = \nu^*(\pi(g)\nu), \quad g \in G
\]

is a **generalized matrix coefficient**.

(non-generalized when \( \nu^* \) is smooth.)
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Relative setting

\( H < G \) a closed subgroup. \( H = H(F) \).
Symmetric case: \( H = G^\theta \) for an \( F \)-involution \( \theta \) on \( G \).
Relative harmonic analysis is interested in possible embeddings

\[
\pi \quad \mapsto \quad C^\infty(H \setminus G)
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given by

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v \in V \mapsto m_{v,v^*}
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for \( 0 \neq v^* \in (V^*)^H \).
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For a smooth mod center function $f$ on $G$, we can try to define the integral

$$L_H(f) = \int_{(H \cap Z(G)) \setminus H} f(h) \, dh.$$  

$dh$ – Haar measure on $H$.

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Can be viewed as a distribution on $G/Z(G)$. 
Main Question - Are there local periods for $\pi$?

Given $0 \neq \nu^* \in (V^*)^H$, is there a \textit{smooth} $\tilde{\nu} \in \tilde{V}$, such that

$$\nu^*(\nu) = L_H(m_{\nu, \tilde{\nu}})$$

for all $\nu \in V$?

That is, which $H$-invariant functionals can be expressed as an integral over (smooth) matrix coefficients? In this case, we will say that $\nu^* = P(\tilde{\nu})$ is a \textit{local period}. 

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Sub-questions

1. Is the $H$-integral over $m_{v,\tilde{v}}$ absolutely convergent? If so, we say $\pi$ is $H$-integrable.

2. If convergent, are there non-zero local periods?

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Global motivation

A cuspidal automorphic representation $\Pi = \otimes'_v \pi_v$ of $G(\mathbb{A}_k)$ ($k$ a number field) has a canonical $H(\mathbb{A}_k)$-invariant functional - the period integral: $P(\phi) = \int_{H(k) \backslash H(\mathbb{A}_k)} \phi(h) \, dh$.

In certain cases (not symmetric), it is expected that when $\{\pi_v\}$ are tempered, the period integral will factorize as

$$|P(\phi)|^2 = P(\phi)P(\overline{\phi}) = \prod'_v L_H(m_{\phi_v}, \overline{\phi_v})$$

under *suitable normalizations* of measures, where $\phi = \otimes'_v \phi_v \in \Pi$.

- Ichino-Ikeda conjectures for the Gross-Prasad case.
- Lapid-Mao conjectures for the Whittaker case.
- Sakellaridis-Venkatesh - general framework.
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1. Problem and motivation

2. Corollaries

3. Convergence of matrix coefficients

4. Results
A representation \( \pi \) is called square-integrable if
\[
|m_{v, \tilde{v}}| \in L^2(G/Z(G))
\]
for all \( v \in V, \tilde{v} \in \tilde{V} \).

A representation \( \pi \) is called tempered if
\[
|m_{v, \tilde{v}}| \in L^{2+\epsilon}(G/Z(G))
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for all \( v \in V, \tilde{v} \in \tilde{V} \) and all \( \epsilon > 0 \).
Definitions

- A representation $\pi$ is called \textit{square-integrable} if
  \[ |m_{v,\tilde{v}}| \in L^2(G/Z(G)) \]
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Strongly tempered pair

A pair \((G, H)\) is called \textit{strongly tempered} if any tempered irreducible representation of \(G\) is \(H\)-integrable.

Tempered distributions

The distribution \(L_H\) on \(G/Z(G)\) is \textit{tempered} if it extends as a functional to the Harish-Chandra-Schwartz space of \(G/Z(G)\).

In particular, when \(L_H\) is tempered any square-integrable irreducible representation of \(G\) is \(H\)-integrable.
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In particular, when \(L_H\) is tempered any square-integrable irreducible representation of \(G\) is \(H\)-integrable.
The following families of pairs are strongly tempered:

\[(GL_n, O_J), (U_{n, E/F}, O(J)), (Sp_{2n}, U_{n, E/F})\]

for any orthogonal group \(O_J\) and any unitary group \(U_{n, E/F}\) relative to a quadratic extension \(E\) of \(F\).

For the following families of pairs \((G, H)\), the distribution \(L_H\) is tempered\(^a\):

\[(G(E), G(F)), (GL_n, GL_{[n/2]} \times GL_{[n/2]})(GL_{2n}, GL_n(E))\]

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\(^a\)as interpreted by Chong Zhang
Corollaries

Theorems (G.-Offen)

1. The following families of pairs are strongly tempered:

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Sakellaridis-Venkatesh: For a strongly tempered \((G, H)\), every (tempered) \(H\)-distinguished irreducible representation of \(G\) which is parabolically induced from a square-integrable representation has non-zero local periods.

C. Zhang: When \(L_H\) is a tempered distribution, every \(H\)-distinguished square-integrable representation of \(G\) has non-zero local periods.
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Casselman’s criterion

A representation is square-integrable (tempered), if and only if, its exponents are (weakly) positive.
Some structure

- Fix a maximal $F$-split torus $A < G$, which is $\theta$-stable and such that $A_0 := (A^\theta)^\circ$ is a maximal $F$-split torus of $H$.
- Fix a minimal $\theta$-stable parabolic $A < P_0 < G$ and a minimal parabolic $B < P_0$.

\[ \Delta_G \subset \Sigma^G = \Sigma(A, \text{Lie}(G)) \subset X^*(A) \]
\[ \Delta_H \subset \Sigma^H \subset \Sigma^G_H = \Sigma(A_0, \text{Lie}(G)) = \Sigma^G|_{A_0} \subset X^*(A_0) \]

- $\Sigma^G_H$ is a root system with basis $\Delta^G_H = \Delta_G|_{A_0}$. $W_H < W^G_H$
- Cartan decomposition: $H^\circ = \bigcup_{c \in C, a \in A_0^{+,\Delta_H}} KcaK$, where $K < G$ is a maximal compact subgroup, $C$ is finite and

\[ A_0^{+,\Delta_H} = \{ x \in A_0 : |\alpha(x)|_F \leq 1, \forall \alpha \in \Delta_H \} \]
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Convergence of matrix coefficients

- Convergence of $L_H(m,\tilde{v})$ reduces to summability on $A^{+,\Delta_H}$.
- Yet, the asymptotics of $m(v,\tilde{v})$ (matrix coefficient of $G!$) can be effectively measured only on the subcone
  \[ A_0 \cap A^{+,\Delta_G} = \{ x \in A_0 : |\alpha(x)|_F \leq 1, \forall \alpha \in \Delta^G_H \} \]
- This can be solved by choosing coset representatives 
  \[ D = [W^G_H/W_H] \subset W^G_H \text{ for which} \]
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Exponents

- For a parabolic $B < P = MN$ with $A < M$, let $A_M < Z(M)$ be the maximal $F$-split torus.
- For irreducible $\pi$, $\text{Exp}(\pi, P) \subset \text{Hom}(A_M, \mathbb{C}^\times)$ is the collection of central characters appearing in subquotients of the Jacquet module $J_P(\pi)$.
- For $\chi \in \text{Exp}(\pi, P)$,
  \[
  |\chi| \in \text{Hom}(A_M, \mathbb{R}_+^\times) \cong a_{A_M}^* := X^*(A_M) \otimes \mathbb{R}
  \]
- For $\theta$-stable $P$ we say that $\lambda \in a_{A_M}^*$ is relatively positive if $\lambda|_{(A_M^\theta)^\circ}$ is in the cone spanned by $\Delta_G|_{(A_M^\theta)^\circ}$. 

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- For $\theta$-stable $P$ we say that $\lambda \in a_{A_M}^*$ is \textit{relatively positive} if $\lambda|_{(A_M^\theta)^\circ}$ is in the cone spanned by $\Delta_G|_{(A_M^\theta)^\circ}$.
Main theorem

Convergence criterion (G.-Offen)

An admissible representation $\pi$ of $G$ is $H$-integrable, iff, for every $\theta$-stable standard parabolic $P$, every $\chi \in \text{Exp}(\pi, P)$ and every $w \in D$,

$$|\chi| + \rho_{w}^{G/H}$$

is relatively positive. Here,

$$\rho_{w}^{G/H} := \delta_{P_{0}}^{1/2}|a_{0}^{*}| - w(\delta_{P_{0}}^{1/2}|a_{0}^{*}|),$$

with $\delta_{P_{0}}, \delta_{P_{0}}^{\theta}$ being the modular characters of the groups.
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with $\delta_{P_{0}}$, $\delta_{P_{0}^{\theta}}$ being the modular characters of the groups.

In particular, combining with Casselman’s criterion, $(G, H)$ is strongly tempered when all $\rho_{w}^{G/H}$ are relatively positive, and $L_{H}$ is tempered when all $\rho_{w}^{G/H}$ are weakly relatively positive.
Thank you!