Kazhdan’s orthogonality conjecture for real reductive groups

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Let $G$ be a connected reductive algebraic group over $\mathbb{Q}_p$. Let $G \supset V_1, V_2$ be smooth representations. Assume that they have finite length.
Definition: The Euler-Poincaré pairing

\[ EP_G(V_1, V_2) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i_G(V_1, V_2). \]
Example 1
If $G$ has a non-compact center, then

$$\text{EP}_G(V_1, V_2) = 0.$$
Example 2

If $G$ has a compact center, and either $V_1$ or $V_2$ is supercuspidal, then

$$\text{EP}_G(V_1, V_2) = \dim \text{Hom}_G(V_1, V_2).$$
2. Global characters

The space of generalized functions:

$$\mathcal{C}^{-\infty}(G) := \text{Hom}_\mathbb{C}(\mathcal{C}_0^\infty(G) \, dg, \mathbb{C}).$$

The space of locally integrable functions:

$$L^1_{\text{loc}}(G) := \left\{ f : \int_G |f(g)\varphi(g)| \, dg < \infty \text{ for all } \varphi \right\} / \sim.$$
Put

\[ G_{rs} := \{ \text{regular semisimple element in } G \} . \]

**Inclusions and restrictions:**

\[
\begin{align*}
C^\infty(G) & \subset L^1_{\text{loc}}(G) & \subset & C^{-\infty}(G) \\
C^\infty(G_{rs}) & \subset L^1_{\text{loc}}(G_{rs}) & \subset & C^{-\infty}(G_{rs})
\end{align*}
\]
Let $G \curvearrowright V$ be a smooth representation of finite length. Its global character

$$\Theta_V \in C^{-\infty}(G)$$

is defined to be

$$\varphi \, dg \mapsto \left( \text{the trace of } V \rightarrow V, \, v \mapsto \int_G \varphi(g)g.v \, dg \right).$$
Clearly,

\[ \Theta_V \in C^{-\infty}(G)^G. \]

\textbf{Harish-Chandra’s regularity theorem:}

\[ \Theta_V \in L^1_{\text{loc}}(G) \quad \text{and} \quad \Theta_V|_{G_{rs}} \in C^\infty(G_{rs}). \]
3. Elliptic pairings

**Definition:** An element $x \in G$ is elliptic if its centralizer $Z_G(x)$ in $G$ has a compact center.

Put

$$G_{re} := \{\text{regular elliptic element in } G\}$$

and

$$C := G_{re}/G.$$
Remarks:

- elliptic $\Rightarrow$ semisimple.
- $C$ is non-empty $\iff G$ has a compact center.

We are concerned with

$$\Theta_V|_C \in C^\infty(C).$$
The Weyl measure $dc$ on $C$:

$$\int_G f(g) \, dg = \int_C \int_G f(gcg^{-1}) \, dg \, dc,$$

for all $f \in C_0^\infty(G_{re}) \subset C_0^\infty(G)$.

**Lemma** (Kazhdan, 1986)

$$\Theta_V|_C \in L^2(C; dc).$$
**Definition:** The elliptic pairing

$$(V_1, V_2)_{\text{ell}} := \int_C \Theta_{V_1}(c^{-1})\Theta_{V_2}(c) \, dc.$$ 

**Remarks:**

- $(V_1, V_2)_{\text{ell}} = (V_2, V_1)_{\text{ell}}.$
- $\Theta_V(c^{-1}) = \overline{\Theta_V(c)}.$
Example 1

If $G$ has a non-compact center, then

$$(V_1, V_2)_{\text{ell}} = 0.$$
Example 2
If either $V_1$ or $V_2$ is properly induced, then

$$(V_1, V_2)_{\text{ell}} = 0.$$
4. The elliptic Grothendieck group

Let $R(G)$ denote the Grothendieck group (with $\mathbb{C}$-coefficients) of the category of finite length smooth representations of $G$.

The elliptic Grothendieck group is

$$\overline{R}(G) := \frac{R(G)}{\text{the span of all properly induced representations}}.$$
Lemma (Harish-Chandra)

\[ R(G) \hookrightarrow C^{-\infty}(G), \quad V \mapsto \Theta_V. \]

Lemma (Kazhdan) Assume that \( G \) has a compact center.

\[ \overline{R}(G) \hookrightarrow L^2(C; dc), \quad V \mapsto \Theta_V|_C. \]

Remark: Assume that \( G \) has a compact center. The elliptic pairing induces an inner product on \( \overline{R}(G) \).
Theorem (Kazhdan’s orthogonality conjecture)  
(Schneider-Stuhler, 1997 and Bezrukavnikov, 1998) 

\[ EP_G(V_1, V_2) = (V_1, V_2)_{\text{ell}}. \]

- The same holds in the case of real groups.
Let $G$ be a real reductive group in Harish-Chandra’s class:

(a) it has only finitely many connected components, and its Lie algebra $\mathfrak{g}_0$ is reductive;

(b) it has a connected closed subgroup with finite center whose Lie algebra equals $[\mathfrak{g}_0, \mathfrak{g}_0]$;

(c) for every $g \in G$, the adjoint action $\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$ is an inner automorphism of $\mathfrak{g}$, where $\mathfrak{g}$ denotes the complexification of $\mathfrak{g}_0$. 
Let $K$ be a maximal compact subgroup of $G$.

Let $(\mathfrak{g}, K) \acts V_1, V_2$ be two modules. Assume that they have finite length.

The Euler-Poincaré pairing

$$\text{EP}_{\mathfrak{g}, K}(V_1, V_2) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i_{\mathfrak{g}, K}(V_1, V_2).$$
Define

\[ C := G_{\text{re}}/G \quad \text{and} \quad dc \]

as in the \( p \)-adic case.

**Remark:**

\( C \) is non-empty \( \iff \) \( \text{rank} G = \text{rank} K \).
For every \((g, K) \triangleleft V\) of finite length,

- the global character \(\Theta_V \in C^{-\infty}(G)\) is defined;
- \(\Theta_V \in L^1_{\text{loc}}(G)\);
- \(\Theta_V|_{G_{\text{rs}}} \in C^\infty(G_{\text{rs}})\);
- \(\Theta_V|_C \in L^2(C)\).
The elliptic pairing:

$$(V_1, V_2)_{\text{ell}} := \int_{\mathcal{C}} \Theta_{V_1}(c^{-1}) \Theta_{V_2}(c) \, dc.$$ 

**Remark:** The elliptic pairing $(V_1, V_2)_{\text{ell}}$ only depends on $V_1|_K$ and $V_2|_K$. 

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Kazhdan's orthogonality conjecture for real reductive groups
Theorem (Huang-Sun, arXiv 2015)

\[ \text{EP}_{g,K}(V_1, V_2) = (V_1, V_2)_{\text{ell}}. \]
Corollary  If $\text{rank} G \neq \text{rank} K$, then

$$\text{EP}_{g,K}(V_1, V_2) = 0.$$ 

Example: $G = \text{SL}_n(\mathbb{R}), \ n \geq 3$. 
Corollary

- The Euler-Poincaré pairing $\text{EP}_{g,K}$ is symmetric and non-negative.
- Assume that $G$ has a compact center. It defines an inner product on

$$\mathcal{R}(g,K)_\mathbb{R} := \frac{\text{the K-group, with } \mathbb{R}\text{-coef., of f.l. } (g,K)\text{-modules}}{\text{the span of properly induced representations}}.$$
Remarks:

- If $G$ is connected and $\text{rank } G = \text{rank } K$, then the Dirac pairing

  \[(V_1, V_2)_{\text{Dir}} \in \mathbb{Z}\]

  is defined by using the Dirac cohomologies.

- It only depends on $V_1|_K$ and $V_2|_K$.

- It is easy to verify that $(V_1, V_2)_{\text{Dir}} = (V_1, V_2)_{\text{ell}}$. This is carried out by Huang and Renard.
7. The proof

**Step 1:** If $G$ has a non-compact center, then

$$\text{EP}_{g,K}(V_1, V_2) = (V_1, V_2)_{\text{ell}} = 0.$$
**Lemma** Let $\alpha$ be an abelian finite dimensional complex Lie algebra. If $\alpha \neq 0$, then

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(\alpha; V) = 0$$

for all finite dimensional representations $V$ of $\alpha$.

**Proof.**

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0, \quad \text{for } n \geq 1.$$
Step 2: If $V_2$ is properly induced, then

$$EP_{g,K}(V_1, V_2) = (V_1, V_2)_{\text{ell}} = 0.$$ 

Proof. Using Frobenius reciprocity, this is reduced to Step 1.

- Every proper Levi subgroup has a non-compact center.
Step 3: If $V_2$ is in discrete series or limit of discrete series, then

$$EP_{g,K}(V_1, V_2) = (V_1, V_2)_{\text{ell}}.$$ 

Proof. Assume that $\text{rank } G = \text{rank } K$, and let $T \subset K$ be a Cartan subgroup of $G$.

Then $V_2$ is cohomologically induced from $T \curvearrowleft V_0$, using a Borel subalgebra $\mathfrak{b} = \mathfrak{t} \ltimes \mathfrak{n}$. Then

$$(-1)^{\frac{\dim K/T}{2}} EP_{g,K}(V_1, V_2) = \sum_{j \in \mathbb{Z}} (-1)^j \dim \text{Hom}_T(H_j(n; V_1), V_0).$$
For all $t \in T \cap G_{rs}$,

Vogan:

$$\Theta_{V_1}(t) = \frac{\sum_{i \in \mathbb{Z}} (-1)^i \Theta_{H_i(n,V)}(t)}{\sum_{i \in \mathbb{Z}} (-1)^i \Theta_{\wedge^i n}(t)}.$$

Harish-Chandra:

$$(-1)^{\dim K/T} \Theta_{V_2}(t) = \sum_{w \in W(T,G)} \frac{\Theta_{V_0}(w.t)}{\sum_{i \in \mathbb{Z}} (-1)^i \Theta_{\wedge^i n}(w.t)};$$
Step 4: The general case.

Proof. Assume that $G$ has a compact center. Then $R(G)$ is generated by

- properly induced representations, when $\text{rank} G \neq \text{rank} K$;
- discrete series representations, limit of discrete series representations, and properly induced representations, when $\text{rank} G = \text{rank} K$. 
Thank you for attention!