Simulation of fluid-structure interaction problems arising hemodynamics

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Main motivation

Computational fluid dynamics (CFD) is nowadays a tool of choice for the investigation of blood flow problems.

- Study the physiology and physiopathology of the cardiovascular system.
- Patient-specific planning of interventions for cardiovascular disease.
- Develop and/or analyze the performance of prosthetic heart valves, stents, etc.

Most of these problems involve the interaction of blood with a structure.
Two cases

We model blood as an *incompressible, viscous, and Newtonian fluid*. We distinguish between two cases:

- The elastic structure covers part of the fluid domain boundary and undergoes **small but non-negligible** displacement
  Ex.: blood interacting with the artery wall

- The elastic structure is immersed in the fluid and it features **large** displacement
  Ex.: blood interacting with a valve leaflet
The fluid model

**Fluid equations:** The fluid is governed by the incompressible Navier-Stokes equations

\[
\rho_f \frac{\partial \mathbf{u}}{\partial t} + \rho_f \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \sigma = \mathbf{f}_f \quad \text{in } \Omega_f(t) \times (0, T) \\
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_f(t) \times (0, T)
\]

\( \mathbf{u} \): fluid velocity \hspace{1cm} \sigma = -pI + 2\mu\varepsilon(\mathbf{u}) \): Cauchy stress tensor

\( p \): fluid pressure \hspace{1cm} \varepsilon(\mathbf{u}) = \frac{(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T}{2} \): strain rate tensor

The fluid domain changes in time → ALE formulation
The fluid model

**Fluid equations:** The fluid is governed by the *incompressible* Navier-Stokes equations

\[
\rho_f \left. \frac{\partial u}{\partial t} \right|_{x_0} + \rho_f (u - w) \cdot \nabla u - \nabla \cdot \sigma = f_f \quad \text{in } \Omega_f(t) \times (0, T)
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega_f(t) \times (0, T)
\]

- \( \mathbf{u} \): fluid velocity
- \( \sigma = -p \mathbf{I} + 2\mu \epsilon(\mathbf{u}) \): Cauchy stress tensor
- \( p \): fluid pressure
- \( \epsilon(\mathbf{u}) = \frac{(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T}{2} \): strain rate tensor
- \( \mathbf{w} \): ALE velocity
- \( \left. \frac{\partial \mathbf{u}}{\partial t} \right|_{x_0} \): ALE time derivative
Goal

Simulate the motion of blood in a compliant vessel. We assume the vessel undergoes small displacement.

Standard assumptions for modeling vascular wall mechanics:
- homogeneity
- isotropy

We propose a two-layer model: thin structure + thick structure
The structure model

**Structure equation:** linear elasticity in Lagrangian formulation

**Thick structure**

\[ \rho_s \partial_{tt} d - \nabla \cdot \Sigma(d) = 0 \quad \text{in } \hat{\Omega}_s \times (0, T) \]

\( d \): displacement

\( \Sigma(d) = 2\mu_s \varepsilon(d) + \lambda_s (\nabla \cdot d)I \): first Piola-Kirchhoff stress tensor

**Thin structure (membrane)**

\[ \rho_m h \partial_{tt} \eta - \mathcal{L}_E \eta = f_m \quad \text{in } \hat{\Gamma} \times (0, T) \]

\( \eta \): displacement

\( h \): thickness

\( \mathcal{L}_E \): linear differential operator derived from the elastic energy

[A. Figueroa et al., *CMAME* 2006]
The coupling conditions

- **Kinematic coupling condition**

  \[ u = \partial_t \eta = \partial_t d \quad \text{on } \Gamma(t) \times (0, T) \]

- **Dynamic coupling condition**

  \[ J \sigma n_f + \Sigma n_s + f_m = 0 \quad \text{on } \hat{\Gamma} \times (0, T) \]

  \( J \) denotes the Jacobian of the transformation from Eulerian to Lagrangian coordinates.
About the coupling

• A FSI algorithm is **weakly coupled** if the coupling conditions are not exactly satisfied at every time step. Otherwise it is called **strongly coupled**.

• A **partitioned** FSI method couples existing fluid and structure solvers.

• A **monolithic** FSI method is an ad-hoc fluid-structure solver.

<table>
<thead>
<tr>
<th>Weakly Coupled</th>
<th>Strongly Coupled</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partitioned</td>
<td>UNSTABLE unless*</td>
</tr>
<tr>
<td>Monolithic</td>
<td>DOES NOT EXIST</td>
</tr>
</tbody>
</table>

* proper care is taken to deal with the added mass effect [Causin-Gerbeau-Nobile, *CMAME* 2005]

† Many strategies have been proposed: fixed point algorithms with acceleration techniques, Domain Decomposition strategies, etc. (see Gerbeau, Deparis, Quarteroni, Nobile, Badia, etc.)
The kinematically-coupled scheme

We propose an unconditionally stable weakly coupled scheme based on Lie operator splitting scheme. [Glowinski, *North-Holland* 2003]

Problem:

\[ \frac{\partial \phi}{\partial t} + A(\phi) = 0 \quad \text{in } (0, T) \]

\[ \phi(0) = \phi_0, \quad \text{with } A = \sum_{i=1}^{P} A_i \]

Lie’s splitting:

\[ \frac{\partial \phi_i}{\partial t} + A_i(\phi_i) = 0 \quad \text{in } (t^n, t^{n+1}) \]

\[ \phi_i(t^n) = \phi^{n+1} \frac{i-1}{P} \]

set \( \phi^{n+1} \frac{i}{P} = \phi_i(t^{n+1}) \)

Subproblems:

- **Fluid problem** with Robin type boundary conditions involving the thin structure inertia.
- **Structure problem** with Robin type boundary conditions and no computation of the fluid stress.

The numerical scheme

**Fluid**

\[
\rho_f \partial_t \mathbf{u} + \rho_f \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{\sigma} = \mathbf{f}_f \quad \text{in } \Omega_f(t) \times (0, T) \\
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_f(t) \times (0, T)
\]

**Thick structure**

\[
\rho_s \partial_{tt} \mathbf{d} - \nabla \cdot \mathbf{\Sigma}(\mathbf{d}) = 0 \quad \text{in } \hat{\Omega}_s \times (0, T)
\]

**Thin structure**

\[
\rho_m h \partial_{tt} \eta - \mathcal{L}_E \eta = -J \mathbf{\sigma}_f - \mathbf{\Sigma}_s \quad \text{in } \hat{\Gamma} \times (0, T)
\]

**Coupling conditions**

\[
\mathbf{u} = \partial_t \eta, \quad \eta = \mathbf{d} \quad \text{on } \Gamma(t) \times (0, T)
\]
The numerical scheme

Fluid

\[ \rho_f \partial_t u + \rho_f u \cdot \nabla u - \nabla \cdot \sigma = f_f \quad \text{in } \Omega_f(t) \times (0, T) \]
\[ \nabla \cdot u = 0 \quad \text{in } \Omega_f(t) \times (0, T) \]

Thick structure

\[ \rho_s \partial_{tt} d - \nabla \cdot \Sigma(d) = 0 \quad \text{in } \hat{\Omega}_s \times (0, T) \]

Thin structure

\[ \rho_m h \partial_{tt} \eta - \mathcal{L}_E \eta = -J \sigma_n_f - \Sigma n_s \quad \text{in } \hat{\Gamma} \times (0, T) \]

Coupling conditions

\[ u = \partial_t \eta, \quad \eta = d \quad \text{on } \Gamma(t) \times (0, T) \]
Step 1: fluid problem

\[ \begin{align*}
\rho_f \partial_t \mathbf{u} + \rho_f \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{\sigma} &= \mathbf{f}_f & \text{in } \Omega_f(t) \times (0, T) \\
\nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega_f(t) \times (0, T)
\end{align*} \]

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\rho_s \partial_{tt} \mathbf{d} - \nabla \cdot \mathbf{\Sigma}(\mathbf{d}) &= 0 & \text{in } \hat{\Omega}_s \times (0, T)
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Step 1: fluid problem

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\rho_f \partial_t u + \rho_f u \cdot \nabla u - \nabla \cdot \sigma = f_f \quad \text{in } \Omega_f(t) \times (0, T)
\]
\[
\nabla \cdot u = 0 \quad \text{in } \Omega_f(t) \times (0, T)
\]

\[
\rho_s \partial_{tt} d - \nabla \cdot \Sigma(d) = 0 \quad \text{in } \hat{\Omega}_s \times (0, T)
\]

\[
\rho_m h \partial_t u - \mathcal{L}_E d = -J \sigma_n f - \Sigma n_s \quad \text{in } \hat{\Gamma} \times (0, T)
\]

Fluid problem with Robin type boundary conditions involving the thin structure inertia.
Step 2: structure problem

\[ \rho_f \partial_t u + \rho_f u \cdot \nabla u - \nabla \cdot \sigma = f_f \quad \text{in } \Omega_f(t) \times (0, T) \]

\[ \nabla \cdot u = 0 \quad \text{in } \Omega_f(t) \times (0, T) \]

\[ \rho_s \partial_{tt} d - \nabla \cdot \Sigma(d) = 0 \quad \text{in } \hat{\Omega}_s \times (0, T) \]

\[ \rho_m h \partial_{tt} d - \mathcal{L}_E d = -J \sigma n_f - \Sigma n_s \quad \text{in } \hat{\Gamma} \times (0, T) \]
Step 2: structure problem

\[ \rho_f \partial_t u + \rho_f u \cdot \nabla u - \nabla \cdot \sigma = f_f \quad \text{in } \Omega_f(t) \times (0, T) \]
\[ \nabla \cdot u = 0 \quad \text{in } \Omega_f(t) \times (0, T) \]

\[ \rho_s \partial_{tt} d - \nabla \cdot \Sigma(d) = 0 \quad \text{in } \hat{\Omega}_s \times (0, T) \]

\[ \rho_m h \partial_{tt} d - L_E d = -J \sigma_{nf} - \Sigma n_s \quad \text{in } \hat{\Gamma} \times (0, T) \]

Structure problem with Robin type boundary conditions and no computation of the fluid stress.
Straight cylinder

We consider a cylinder of radius 0.5 cm, length 5 cm, and different values of the thin structure thickness $h$ and thick structure thickness $H$ such that $h + H = h_{tot} = 0.12$ cm.

- We impose a pressure pulse at the inlet and let the pressure wave propagate over time interval $[0, 0.02]$ s.
- We fix the structure at the inlet and outlet sections.
• As $h \to 0$, the solution of the problem converges to the solution of the fluid-thick structure interaction problem.
• The thin layer has a smoothing effect on the interface displacement.
Diseased artery with implanted stent

We consider a stented straight artery affected by atheromatous disease. We simulate the presence of a stent by increasing the Young’s modulus of the thin layer.

We impose non-homogeneous Neumann conditions at the inflow and outflow boundaries, using physiologic pressures taken from measurements in coronary arteries.

Simulating the stent as a 3D elastic body is computationally very expensive.

The thin structure provides a computationally inexpensive way of simulating slender stent struts without sacrificing accuracy.
Goal

Simulate the motion of a beam $\Gamma$ immersed in incompressible fluid. We assume the beam undergoes large displacement.

We would like to use a method that has the following advantages:

- Problem specific information on the interface (e.g. hydrodynamic force) can be computed very accurately $\leftarrow$ typical of interface tracking methods such as ALE methods.
- Flexibility in handling large displacements of $\Gamma$ $\leftarrow$ typical of interface capturing methods such as level set methods and immersed boundary methods.
The structure model

**Structure equation**: beam with negligible torsional effects

\[ \rho_s \ddot{x} + EI \dddot{x} = f_\Gamma, \]  
\[ \text{on } (0, L) \times (0, T). \]

- \( \rho_s \): linear density
- \( x \): position
- \( \dot{x} = \frac{\partial x}{\partial t} \): time derivative
- \( x' = \frac{\partial x}{\partial s} \): arc length derivative
- \( EI \): flexural stiffness
- \( L \): beam length

[Diniz dos Santos-Gerbeau-Bourgat, *CMAME* 2008]  
The coupling conditions

The beam ideally separates $\Omega_f(t)$ into two subdomains $\Omega^1_f(t)$ and $\Omega^2_f(t)$ and it deforms due to the contact force exerted by the fluid.

- Adherence $\Rightarrow$ Continuity of velocities

\[ u = \dot{x} \quad \text{on} \quad \Gamma(t); \]

- Action-Reaction principle $\Rightarrow$ Continuity of stresses

\[ f_\Gamma = -\sigma^1 n^1 - \sigma^2 n^2 \quad \text{on} \quad \Gamma(t). \]
An extended ALE method

We are interested in having a triangulation that is at every time:

- aligned with \( \Gamma \)
- of "optimal" quality

We use a mesh optimization technique with an additional constraint to enforce the alignment of the edges of the resulting triangulation with the interface.

[Standard ALE][Extended ALE]

[Basting-Weismann, *JCP* 2013] [Basting-Weismann, *CMAME* 2013]
Level set aligned

Let $\phi : [0, T] \times \Omega \to \mathbb{R}$ be a continuous level set function:

$$
\Omega_f^{1/2}(t) = \{ x \in \Omega : \phi(t, x) \geq 0 \}, \\
\Gamma(t) = \{ x \in \Omega : \phi(t, x) = 0 \}.
$$

A triangulation $\mathcal{T}$ is called linearly aligned with $\Gamma(t)$ if for all edges $e$ we have:

$$
\phi(x_{e,1}) \phi(x_{e,2}) \geq 0
$$

where $x_{e,1}$ and $x_{e,2}$ are the endpoints of $e$. 
Level set aligned

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A triangulation $\mathcal{T}$ is called **linearly aligned** with $\Gamma(t)$ if for all edges $e$ we have:

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$$

where $x_{e,1}$ and $x_{e,2}$ are the endpoints of $e$.
Enforcing level set alignment

We can characterize aligned triangulations by using a single scalar constraint:

\[ 0 = c = \sum_{e \in T} H(\varphi(x_{e,1})\varphi(x_{e,2})) , \]

where

\[ H(z) : \begin{cases} > 0 & \text{for } z < 0 \\ = 0 & \text{for } z \geq 0. \end{cases} \]
Optimal triangulation

Starting from an initial triangulation $\mathcal{T}$ of $\Omega$, we want to find an optimal triangulation $\mathcal{T}_{opt}$ resulting from a mesh deformation $\chi_{opt}$:

$$\mathcal{T}_{opt} = \chi_{opt}(\mathcal{T}).$$

Deformation $\chi_{opt}$ is:

- piecewise affine
- orientation preserving
- globally continuous
- optimal in the sense it is the argument for which a certain functional $\mathcal{F}$ attains its minimum value:

$$\mathcal{F}(\chi_{opt}) = \min \mathcal{F}(\chi).$$
A measure for the quality of triangulations

**Assumption:** $F$ can be represented by a sum of weighted, element-wise contributions $F_T$:

$$F(\chi) = \sum_{T \in T} \mu_T F_T(\chi)$$

Let $R_T$ denote the linear reference mapping from the optimally deformed simplex $T_{opt}$ to $T$.

A classical example of function $F_T$ is given by

$$F_T(\chi) = (\|\nabla R_T(\chi)\|^2 - 2)^2 + \det(\nabla R_T(\chi)) + \frac{1}{\det(\nabla R_T(\chi))}$$

- $\|\nabla R_T(\chi)\|^2$ measures the change of edge lengths
- the second term measures the change in area
- the third term rules out deformations with vanishing determinant

An optimal, level set aligned triangulation is obtained from the nonlinear constrained minimization problem

$$\min \mathcal{F}(\chi) \quad \text{s.t.} \quad c(\chi) = 0.$$ 

With this technique, the optimal triangulations are guaranteed to be non-degenerate (i.e., no self intersection occurs) and we have local mesh quality control.

**Price to pay:** $\mathcal{F}$ is highly non-linear, non-convex, and global minimizers may be non-unique.
Details of the fluid solver we use

To solve the fluid-structure problem, we use the **Dirichlet-Neumann algorithm** (with an Aitken acceleration technique) or **Robin-Neumann algorithm**, which at every time step iterate over the fluid and structure subproblems until convergence.

- For the time discretization we use **BDF1** or **BDF2**.
- The **convective term is linearized** at each time step (extrapolation).
- For the space discretization we use inf-sup stable FE pair $P_2 - P_1$.
- We allow for **discontinuities across the interface**, since the pressure discontinuity is needed for the correct evaluation of the hydrodynamic force.
- The **Subspace Projection Method** is used to enforce the continuity of the velocity across the interface. [Bäumler-Bänsch, *JCP* 2013]
- The linear system yielded by discretization and linearization is solved by **PARDISO** (automatic combination of iterative and direct solver algorithms). [Schenk-Wchter-Hagemann, *Comput Optim Appl* 2007]
Details of the structure solver we use

- For the time discretization we use a generalized Crank-Nicolson scheme. [Glowinski-Le Tallec, *SIAM* 1988]
- After time discretization, at every time step we have to solve a quasi-static problem which is equivalent to minimization problem:

\[
\mathbf{x}_{k+1} = \arg\min_{\mathbf{y} \in K} J(\mathbf{y}), \quad \text{with } K = \{ \mathbf{y} \in (H^2(0, L))^2, |\mathbf{y}'| = 1, B.C. \},
\]

where the total energy of the beam can be written as:

\[
J(\mathbf{y}) = \frac{1}{2} \int_0^L \frac{\rho_s}{\Delta t^2} |\mathbf{y}|^2 \, ds + \frac{1}{2} \int_0^L E I \alpha |\mathbf{y}''|^2 \, ds - \int_0^L \tilde{f}_{k+1} \cdot \mathbf{y} \, ds.
\]
Setting

We consider a periodic (closed) beam and an open beam clamped at the midpoint of the base.

- Inlet: the flow is driven by a time dependent velocity profile or a time periodic normal stress.
- Outlet: stress free condition.
- Bottom wall: no slip condition.
- Top wall: no slip condition for the closed beam and symmetry condition for the open beam.

[Basting-Q-Canic-Glowinski, Submitted]
Closed beam

At the inlet we impose Dirichlet condition:

\[
\begin{align*}
  u(t, x) &= \begin{cases} 
    \left[ -2t^3 + 3t^2, 0 \right]^T & \text{if } 0 \leq t \leq 1, \\
    \left[ 1, 0 \right]^T & \text{if } 1 \leq t \leq 20, \\
    \left[ 0, 0 \right]^T & \text{if } t > 20.
  \end{cases}
\end{align*}
\]

Excellent agreement between the results obtained with standard and extended ALE methods until the standard ALE method breaks down.
Closed beam: maximum angle of the elements

- The maximum angle in the mesh given by the standard ALE method increases up to nearly 170 degrees, right before the simulation crashes.
- The maximum angle for the mesh given by the extended ALE method never exceeds 132 degrees.
Open beam

At the inlet we impose nonhomogeneous Neumann condition:

\[ \sigma_n(t, x) = \begin{cases} 
[1.5, 0]^T & \text{if } 0 \leq \tilde{t} \leq 3, \\
[0, 0]^T & \text{if } 3 < \tilde{t} < 10 
\end{cases} \text{ for } x \in \Gamma_{in} \]

where \( \tilde{t} \in [0, 10) \) is \( \tilde{t} = \text{mod}(t, 10) \).

Our method does not introduce spurious energy over time and no instabilities arise over a rather long time interval.
Conclusions

- We proposed an **unconditionally stable weakly coupled ALE method** for FSI problems with small structural displacement.
- We applied it to the interaction of an **incompressible fluid** with a **two-layer structure**.
- We proposed an **extended ALE method** for the simulation of FSI problems with large structural displacement.
- Our extended ALE method relies on a **mesh optimization technique** with an additional constraint to enforce the alignment of the interface with the edges of the resulting triangulation.
- We applied it to the interaction of an **incompressible fluid** with a **beam**.

THANK YOU FOR YOUR ATTENTION!