A NEW PARTITIONED STAGGERED SCHEME FOR FLEXIBLE MULTIBODY INTERACTIONS OF LOW MASS RATIOS

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- Introduction
- Existing partitioned staggered methods
- Present partitioned staggered method
- Applications
INTRODUCTION

- In generic flexible multibody interactions, the mass ratio of flexible body to rigid body can be very low
  - Small change in the higher mass structure triggers relatively large inertial effects on the lower mass structure

![Diagram showing coupled rigid-flexible body system: Body 1 (Floater) connected to Rigid, Body 2 (Mooring Line/System) connected to Flexible, forming a Coupled Rigid-Flexible Body System.]
INTRODUCTION

- Coupled system
  - Field elimination – integral transforms or model reduction
  - Monolithic – single entity and advanced simultaneously in time
  - Partitioned staggered – isolated entities that are sequentially marched in time

- Monolithic
  - Difficulties – mass ratio, added mass, nonlinear stiffness, etc.,

- Partitioned staggered
  - Strongly coupled and loosely coupled
  - Formulation and implementation of the interface conditions – coupling achieve the desired stability and accuracy
  - Separate solvers, matching or non-matching time stepping

- Low mass ratios (1/10 and below), inertia effects, nonlinear stiffness, etc.
  - Internal forces and velocities of a coupled system – significant high frequency content that can hamper the solution convergence process
  - Existing coupled time integrators – instability or less accuracy

Loosely coupled partitioned staggered time integration scheme for multibody system
CONVENTIONAL STAGGERED SCHEME

- Kinematic continuity

\[ \dot{u}_1 = \dot{u}_2 \quad (1) \]

- Traction force continuity

\[ f_1 = -f_2 \quad (2) \]

Body 1: Larger mass (Leader)
Body 2: Smaller mass (Follower)
FORCE PREDICTION AND AVERAGING SCHEME

- Kinematic continuity

\[ \dot{u}_1 = \dot{u}_2 \]  \hspace{1cm} (3)

- Traction force continuity

\[ f_{IP}^{n+1} = 2f^n - f^{n-1}; \]
\[ f^{n+1} = \beta_0 f_{IP}^{n+1} + (1 - \beta_0) f^{n+1} \]  \hspace{1cm} (4)

Body 1: Larger mass (Leader)
Body 2: Smaller mass (Follower)
PRESENT PARTITIONED STAGGERED SCHEME

- Many characteristics of a interacting multibody system – simple harmonic oscillators represent an intuitive model.

- Coupled oscillator model – coupling between the two domains and derivation of the stability, accuracy and convergence characteristics

\[ (1 - \alpha) \ddot{u}_1 + 2\xi_1 \omega_1 \dot{u}_1 + \omega_1^2 u_1 = f_1 \]  
(5)   

Body 1 : Larger mass (Leader)

\[ \alpha \ddot{u}_2 + 2\xi_2 \omega_2 \dot{u}_2 + \omega_2^2 u_2 = f_2 \]  
(6)   

Body 2 : Smaller mass (Follower)

- Solvers
  - Leader domain – Newmark method
  - Follower domain – Tamma & Namburu method
PRESENT PARTITIONED STAGGERED SCHEME

- Kinematic continuity

\[ \dot{u}_2^{n+1} = \dot{u}^*_n = \dot{u}_1^n + \square \]

- Traction force continuity

\[ f_1 = -f_2 \]

Body 1: Larger mass (Leader)
Body 2: Smaller mass (Follower)
PRESENT SCHEME – DISCRETISATION

- **Discretisation**
  - **Leader and follower domains**

\[
(1 - \alpha) \left( \frac{u_{1}^{n+1} - 2u_{1}^{n} + u_{1}^{n-1}}{\Delta t_{1}^{2}} \right) + 2\xi_1 \omega_1 \left( \frac{u_{1}^{n+1} - u_{1}^{n}}{\Delta t_{1}} \right) + \omega_1^2 u_{1}^{n+1} = f_{1}^{n+1}
\]

\[
\alpha \left( \frac{u_{2}^{n+1} - 2u_{2}^{n} + u_{2}^{n-1}}{\Delta t_{2}^{2}} \right) + 2\xi_2 \omega_2 \left( \frac{u_{2}^{n+1} - u_{2}^{n}}{\Delta t_{2}} \right) + \omega_2^2 u_{2}^{n+1} = f_{2}^{n+1}
\] (9)

- **Interface velocity**

\[
\left( \frac{u_{2}^{n+1} - u_{2}^{n}}{\Delta t_{2}} \right) = \left( \frac{u_{1}^{n} - u_{1}^{n-1}}{\Delta t_{1}} \right) + \frac{\Delta t_{1}}{1 - \alpha} \left[ -\alpha \left( \frac{u_{2}^{n} - 2u_{2}^{n-1} + u_{2}^{n-2}}{\Delta t_{2}^{2}} \right) - 2\xi_2 \omega_2 \left( \frac{u_{2}^{n} - u_{2}^{n-1}}{\Delta t_{2}} \right) \right] - \omega_2^2 u_{2}^{n} - 2\xi_1 \omega_1 \left( \frac{u_{1}^{n} - u_{1}^{n-1}}{\Delta t_{1}} \right) - \omega_1^2 u_{1}^{n} + \phi \left( \left( \frac{u_{1}^{n} - u_{1}^{n-1}}{\Delta t_{1}} \right) - \left( \frac{u_{2}^{n} - u_{2}^{n-1}}{\Delta t_{2}} \right) \right) \right]
\] (10)
PRESENT SCHEME – STABILITY

- Gudunov-Ryabenkii method

\[ u_{1,2}^n = \begin{cases} z^n s_1, & \text{in } \Omega_1 \\ z^n s_2, & \text{in } \Omega_2 \end{cases} \]  

(11)

- Matching time stepping

\[ \phi = \frac{(1 - \alpha)(A_1) + \alpha(A_2) + d_1(A_3) + r_1 s_1 + d_2(A_4) + r_2 s_2}{\Delta t(A_5)} \]  

(12)

\[ A_1 = (zs_2 - s_2 - s_1 + z^{-1}s_1) \quad A_3 = (s_1 - z^{-1}s_1) \]

\[ A_2 = (s_2 - 2z^{-1}s_2 + z^{-2}s_2) \quad A_4 = (s_2 - z^{-1}s_2) \]

\[ A_5 = (s_1 - z^{-1}s_1 - s_2 + z^{-1}s_2) \]

- Stability: \( |z| < 1 \)

\[ \left[ \frac{(1 - \alpha)(-16\alpha + 2d_1 + r_1 - 6d_2 - 3r_2 + 4)}{\Delta t(2d_1 + r_1 + 2d_2 + r_2 + 4)} \right] < \phi < \infty \]  

(13)

- Non-matching time stepping

\[ \left[ \frac{(1 - \alpha)(-12\alpha \Delta t + 4\alpha \Delta t_2 + 2d_1 \Delta t_2 + r_1 \Delta t_2 - 6d_2 \Delta t_1 - 3r_2 \Delta t_1 + 4\Delta t_2)}{(4\alpha \Delta t_1^2 - 4\alpha \Delta t_1 \Delta t_2 + 2d_1 \Delta t_1 \Delta t_2 + r_1 \Delta t_1 \Delta t_2 + 2d_2 \Delta t_1^2 + r_2 \Delta t_1^2 + 4\Delta t_1 \Delta t_2)} \right] < \phi < \infty \]  

(14)

Non-dimensional parameters

\[ d_1 = 2\xi_1 \omega_1 \Delta t \quad d_2 = 2\xi_2 \omega_2 \Delta t \]

\[ r_1 = \omega_1^2 \Delta t^2 \quad r_2 = \omega_2^2 \Delta t^2 \]
STABILITY – $\varphi$ ON RELATIVE DAMPING & STIFFNESS

- Stability range of the coupling parameter decreases for all the $\alpha$ as the damping and stiffness of the leader become relatively large.

- Stability range of the coupling parameter increases for all the $\alpha$ as the damping and stiffness of the follower become relatively large.

Dependence of $\varphi$ on the relative damping

$$r_1/r_2 = 1/0.01$$

Dependence of $\varphi$ on the relative stiffness

$$d_1/d_2 = 1/0.1$$
Stability range of the coupling parameter increases with $\Delta t_1$ for all the values of $\Delta t_2$.

Stability range of the coupling parameter increases with $\Delta t_2$ for all the values of $\Delta t_1$.

$\Delta t_1$ is more influential on the stability range of the coupling parameter $\varphi$.

$\alpha = 1/10^6$, $d_1/d_2 = 1/0.1$ and $r_1/r_2 = 1/0.01$. 
NUMERICAL EXPERIMENT ON INTERFACE STABILITY

- Moderate value of $\alpha$, the instability is mild in the CSS and FPA methods.
  - Do not blow up, yet they diverge from the mean interface solutions.

- Low value of $\alpha$, the instability is rapid.

- Present method remains stable for both values of $\alpha$ within the time interval of interest.

(1) $$(1 - \alpha) m \ddot{u}_1 + k_1 (u_1 - u_2) = 0$$

(2) $$\alpha m \ddot{u}_2 + k_1 (u_1 - u_2) + k_2 u_2 = 0$$
PRESENT SCHEME – ACCURACY & CONVERGENCE

- Unforced linear structural system

\[ m\ddot{u} + c\dot{u} + ku = 0 \]  \hspace{1cm} (16)

\[ (1 - \alpha)m\ddot{u}_1 + c\dot{u}_1 = f_1 \] \hspace{1cm} (17)

\[ \alpha m\ddot{u}_2 + ku_2 = f_2 \]

\[ m = 1; c = 2\xi \omega = 2(0.2)(2\pi); k = (\omega)^2 = (2\pi)^2; u^0 = 1; \dot{u}^0 = 1 \]

- Duffing oscillator (nonlinear system)

\[ m\ddot{u} + c\dot{u} + ku + k_\ast u^3 = 0 \] \hspace{1cm} (18)

\[ (1 - \alpha)m\ddot{u}_1 + c\dot{u}_1 + k_\ast u_1^3 = f_1 \] \hspace{1cm} (19)

\[ \alpha m\ddot{u}_2 + ku_2 = f_2 \]

\[ m = 0.01; c = 0.1; k = 2; k_\ast = 0.5; u^0 = 7.5; \dot{u}^0 = 0. \]

- L2-norm error

\[ \epsilon_{nm} = \sqrt{\frac{\sum_{i=1}^{n} (\dot{u}_{\text{num}}^i - \dot{u}_{\text{mono}}^i)^2}{\sum_{i=1}^{n} (\dot{u}_{\text{mono}}^i)^2}} \] \hspace{1cm} (20)
It is observed that the L2-norm error increases with $\phi$ for all the $\alpha$.

There exists values of the coupling parameter beyond which the error for smaller $\alpha$ is less compared to larger $\alpha$. 
ACCURACY & CONVERGENCE – LINEAR SYSTEM

- L2-norm error decreases with \( \phi \) unlike matching time stepping. Expected increase in the error is produced as the ratio of the time steps in two domains, i.e., \( \Delta t_1/\Delta t_2 \) increases.

- There exists values of the coupling parameter beyond which the error produced using non-matching time stepping is less compared to matching time stepping.
Increasing the time step produces an expected increase in error, which for the present method is lower than the CSS and FPA methods.

- At $\Delta t = 0.001$, the coupling error of the present method is one order of magnitude lower.
0.5% of the amplitude and period errors.

- Improvement is factor of 10 for amplitude error
- Improvement is factor of 2 for period error
There exists an optimal range of the coupling parameter in which the L2-norm error is minimal.

- Therefore, it is important to choose the coupling parameter values from within this range while using matching time stepping for the coupled analysis.

- Within this range, the error increases with the coupling parameter.
There exists an optimal range of the coupling parameter in which the L2-norm error is minimal.

Therefore, it is important to choose the coupling parameter values from within this range while using matching time stepping for the coupled analysis.

Within this range, the error increases with the coupling parameter.
Increasing the time step produces an expected increase in error, which for the present method is lower than the CSS and FPA methods.

At $\Delta t = 0.001$, the coupling error of the present method is one order of magnitude lower.
2% of the amplitude and period errors.

- Improvement is factor of 1.1 for amplitude error
- Improvement is factor of 2 for period error
ENERGY CONSERVING

- Total energy of the coupled system is conserved for all the values of $\alpha$ using the present method.

- It can thus be stated that the total discrete energy of the coupled system is preserved by the proposed partitioned scheme for long time integration.
APPLICATION – WAVE ENERGY CONVERTER (WEC)

WEC : Concept model

WEC : Schematic model

\[
(m_a^\infty + m_b)\ddot{u}_b = F_e - F_h - F_{wp} - F_d \\
m_p\ddot{u}_p = F_{wb} - k\dot{u}_p - \gamma \dot{u}_p - F_{endstop}
\]
APPLICATION – WEC

**Input**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean water depth</td>
<td>25 m</td>
</tr>
<tr>
<td>Buoy radius</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Buoy height</td>
<td>0.8 m</td>
</tr>
<tr>
<td>Buoy mass</td>
<td>850 kg</td>
</tr>
<tr>
<td>Buoy draft</td>
<td>0.25 m</td>
</tr>
<tr>
<td>Spring constant</td>
<td>7064 N/m</td>
</tr>
<tr>
<td>Spring mass</td>
<td>400 kg</td>
</tr>
</tbody>
</table>

\[
\alpha = \frac{1}{3.125} \\
\phi = -5
\]

**Output**

- Spectral density (m²/Hz)
- Wave frequency (Hz)
- Acceleration heave (ms⁻² Hz⁻¹/₂)
- Wire tension (N⁻¹/₂)

Predictions within 12%
APPLICATION – FLOATER MOORING SYSTEM (FM)

FM : Concept model

FM : Schematic model

\[ M_s \ddot{u}_s + C_s \dot{u}_s + K_s u_s = F_s \]

\[-(EI\tau'')'' + [(EA\varepsilon + C_{ID} \dot{\varepsilon} - EI\kappa^2)r'] + [GJ\tau' \times r''']' + h + w = M_I \ddot{r} \] (21)
APPLICATION – FLOATER MOORING SYSTEM (FM)

- Input

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean water depth</td>
<td>1330.72 m</td>
</tr>
<tr>
<td>Total draft</td>
<td>131.15 m</td>
</tr>
<tr>
<td>Hard tank diameter</td>
<td>32.33 m</td>
</tr>
<tr>
<td>Hard tank draft</td>
<td>56.42 m</td>
</tr>
<tr>
<td>Heave plate</td>
<td>32.33 m</td>
</tr>
<tr>
<td>Heave plate thickness</td>
<td>0.50 m</td>
</tr>
<tr>
<td>Height of truss section</td>
<td>21.02 m</td>
</tr>
<tr>
<td>Soft tank depth</td>
<td>10.67 m</td>
</tr>
<tr>
<td>Distance of fairlead from hard tank bottom</td>
<td>5.49 m</td>
</tr>
<tr>
<td>Mass of the platform</td>
<td>47000 tonnes</td>
</tr>
<tr>
<td>CG from mean water level</td>
<td>40.13 m</td>
</tr>
<tr>
<td>Mass moment of inertia about pitch axis</td>
<td>1.165e11 km²</td>
</tr>
<tr>
<td>Natural time period for surge motion</td>
<td>106.2 s</td>
</tr>
<tr>
<td>Damping ratio for surge motion</td>
<td>0.0531</td>
</tr>
<tr>
<td>Natural time period for heave motion</td>
<td>20.7 s</td>
</tr>
<tr>
<td>Damping ratio for heave motion</td>
<td>0.0882</td>
</tr>
<tr>
<td>Natural time period for pitch motion</td>
<td>50.3 s</td>
</tr>
<tr>
<td>Damping ratio for pitch motion</td>
<td>0.0514</td>
</tr>
<tr>
<td>Mass of the mooring cable</td>
<td>121.67 tonnes</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Group</th>
<th>Mooring lines azimuth angles with respect to wave heading</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>45°, 50°</td>
</tr>
<tr>
<td>II</td>
<td>125°, 132.5°</td>
</tr>
<tr>
<td>III</td>
<td>222.3°, 229.7°, 235°</td>
</tr>
<tr>
<td>IV</td>
<td>308.5°, 314.1°, 320.3°</td>
</tr>
</tbody>
</table>

\[
\alpha = 1/39.682 \\
\phi = 100
\]
APPLICATION – FLOATER MOORING SYSTEM (FM)

- Output

Three graphs showing the comparison between numerical and experimental results for Surge RAO, Heave RAO, and Pitch RAO. The graphs indicate that predictions are within 9%.
CONCLUSIONS

- Novel loosely coupled staggered scheme – multibody interaction problems of low mass ratio – higher order interface corrections to counteract the artificial energy production caused by the staggering process

- Characteristics – mass ratios ranging from $10^{-6}$ to $10^{-1}$, relative damping and stiffness properties ranging from $10^{-3}$ to $10^3$ and matching and non-matching time stepping
  - Stability
    - Stability range of the coupling parameter reduces with increase in the relative properties of the leader and vice-versa of the follower domain
    - Time step of leader domain is more influential on the stability range of coupling parameter
  - Accuracy
    - Accuracy analysis for an unforced linear structural system is improved by choosing a lesser value for the coupling parameter
    - In non-matching time stepping, there exists a range for the coupling parameter in which the $L_2$-norm error is less than matching time stepping
    - Accuracy analysis for an unforced nonlinear system indicated that there exists an optimal range for the coupling parameter in which the error produced is minimal
  - $L_2$-norm error with uniform variation in the time step revealed that the present method is more accurate than the CSS and FPA methods

- Applicability of the present method
  - Wave energy converter – predictions within 12%
  - Floater mooring system – predictions within 9%
ACKNOWLEDGMENTS

- National Research Foundation, Singapore
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Q & A
THANK YOU